Chapter 1
Economic Growth and Business Cycles

Thomas F. Cooley and Edward C. Prescott

1. Introduction

The intent of this book is to describe the methods and problems of modern business cycle research, using the neoclassical growth framework to study the economic fluctuations associated with the business cycle. Advances in dynamic economic theory and progress in computational methods over the past two decades have provided economists with a new set of tools for the study of important economic issues. These tools have enhanced our ability to construct and study artificial economies that serve as laboratories for economic research. The construction and analysis of equilibrium paths for simple artificial economies based on the neoclassical growth model has proven a very fruitful approach to studying and better understanding the business cycle. In this volume, we present the primary methods and the most important applications of this research.

Research on economic fluctuations has progressed rapidly since Robert Lucas revived the profession's interest in business cycle theory. Prior to the publication of Keynes's *General Theory of Employment, Interest, and Money*, business cycle theory was a well established part of twentieth-century economics. Business cycle theory evolved in the early twentieth century because the empirical study of fluctuations across periods of prosperity and decline had highlighted a remarkable degree of regularity in the characteristics of these cycles. Such economists as Wesley Mitchell, Simon Kuznets, and Frederick Mills carefully documented the characteristics of business cycle fluctuations on the basis of the available data for the United States and other countries. Mitchell was primarily concerned with documenting the simultaneity of movement (comovement) of variables over the cycle, with the view that this could be helpful in learning to predict economic upturns and downturns. Frederick Mills was concerned with documenting the behavior of prices, and in particular the comovement of prices and quantities, over economic expansions and contractions, because he believed that this would yield important clues to the origins of cycles. His view was that if prices appeared to be procyclical, it would be evidence of what he called demand-driven fluctuations, while if they turned out to be countercyclical, it would be indicative of supply-driven fluctuations. Simon Kuznets studied the patterns of both growth and fluctuations. The empirical investigations of these researchers suggested that the defining features of economic fluctuations were surprisingly similar over time and across countries, suggesting that business cycles may be candidates for explanation by "economic laws."

The 1930s was a very active period for business cycle research. The National Bureau of Economic Research (NBER) continued its program (begun by Mills, Mitchell and Kuznets) of empirically documenting the features of cycles. What emerged from the data was the finding that business cycle fluctuations are a recurrent event with many similarities over time and across countries. This finding prompted many attempts to explain the cycle as a natural property or outcome of an economic system. Different views emerged about why cycles occur with regularity in economic systems. One idea, associated mainly with Ragnar Frisch (1933), regarded the cycle as the set of damped oscillations that result from the propagation of random shocks to the economy. The business cycle model of Frisch showed how cycles could arise in the solution of the second-order–difference equations that characterize an economic system. Frisch was heavily influenced by Wicksell in the development of his views. Eugen Slutsky (1937), in an important paper, put forth an alternative theory. He showed how fluctuations resembling business cycles could result from the sum of random shocks to the economy if the economy were characterized by a stable stochastic difference equation with large positive real roots. Additional theories of the cycle were put forth by Kalecki, Schumpeter (who argued that technological innovations can lead to both long-term growth and cycles, and Metzler (who focused on the inventory cycle, among others. This was a period of proliferation of business cycle models without much real progress at resolving important questions.

The first attempts to construct economywide econometric models were motivated by the desire to test competing business cycle theories. Although business cycles are inherently dynamic phenomena, economists did not really have the theoretical tools to deal with rigorous models of the cycle. The search for empirical versions of economic laws appeared to be a more fruitful path to follow. Interest in business cycles per se also waned in the aftermath of the publication of Keynes's *General Theory*. The so called Keynesian revolution that followed the publication of the *General Theory* turned attention away from thinking about cycles. Instead, the intellectual problem for macroeconomists became to explain the forces that determine the level of economic output at a point in time, conditional on the prior history of the economy. This was the research agenda suggested by the *General Theory* and given empirical content by the important contributions of Tinbergen and Klein. The overriding concern with the question of output determination is easy to understand: at the time of these developments, the United States and Britain were emerging from World War II. Many economists were concerned that the prolonged downturn of the Great Depression might recur after the war. An understanding of the determinants of the level of output at a point in time suggested the possibility of designing policies ("stabilization policies") that would influence the level of output, attenuate the business cycle, and possibly prevent the recurrence of large-scale economic crises, such as the Great Depression. One result of this change in focus was that later empirical research was directed almost entirely at the question of output determination at a point in time rather than the study of the whole shape and characteristics of the business cycle. This research agenda, combined with the increased availability of aggregate economic data led to the creation of fully specified artificial economies that were designed to capture the process of output determination. Interest in understanding the business cycle as a recurrent event waned. This remained the state of affairs until Lucas (1972, 1975) and Kydland and Prescott (1982) rekindled interest in the theoretical and empirical investigation of business cycles.[1]

Concurrent with the emergence of Keynesian macroeconomics was a renewed interest in the problem of understanding the long-term laws of motion of modern economies. This was the research agenda of modern growth theory that was initiated by Harrod and Domar and given its preeminent expression in the work of Robert Solow. The modern theory of economic growth evolved from the observation of empirical regularities, as had business cycle theory. As economic data became more available in the Twentieth century, it grew apparent that economic growth displayed striking empirical regularities both over time and across countries. These observations, labeled by Nicholas Kaldor (1957) the "stylized facts" of economic growth, became the benchmarks of the theory of economic growth. These observed regularities suggested economic laws at work that could be captured in formal models. Kaldor's "stylized facts" of growth (as characterized by Solow [1970]) are as follows:

1) Real output grows at a more or less constant rate.

2) The stock of real capital grows at a more or less constant rate greater than the rate of growth of the labor input.

3) The growth rates of real output and the stock of capital tend to be about the same.

4) The rate of profit on capital has a horizontal trend.

5) The rate of growth of output per-capita varies greatly from one country to another.

6) Economies with a high share of profits in income tend to have a high ratio of investment to output.

The third and fourth of these stylized facts imply that capital's share in total income will be constant, while the second and third imply that the investment-output ratio is constant.[2] The first four together describe an economy experiencing "balanced" growth. The scale of an economy experiencing balanced growth will change over time, but the composition of output will not. When Nicholas Kaldor summarized the main observations about economic growth, as they were known in the 1950s, the task of developing a coherent theoretical model of growth became a primary focus of interest for economists in the United States and England. The growth theory that evolved from these observations was concerned primarily with exploring the properties of model economies that exhibit balanced growth or have well-defined steady-state paths and with analyzing whether artificial economies not initially in a steady state would tend to converge to one. The elements of that theory and its evolution are documented in many places. The fifth and sixth stylized facts have posed more difficulty for neoclassical growth theory, and much of the modern endogenous growth literature has been concerned with these features.

The neoclassical model of capital accumulation reproduces many of the stylized facts about economic growth and is consistent with many features of actual growing economies. We also observe that in most industrialized economies, output per capita grows over time, capital per worker grows over time, and productivity grows over time. Robert Solow used the neoclassical growth model as the basis for decomposing the growth in output per capita into portions accounted for by increased inputs and the portion attributable to increases in productivity. Solow's findings prompted much additional research in productivity measurement and considerable development of mathematical growth models with different ways of incorporating technical change.

What is surprising about the development of growth theory is that for a very long time, the theory evolved in an empirical vacuum. It did not much influence, nor was it greatly influenced by, the corresponding developments in empirical macroeconomics. Study of short-term economic behavior or fluctuations and study of long-term growth were divorced. The generally accepted view was that we needed one theory to explain long-term growth and a completely different one to explain short-term fluctuations in output. Several important developments in growth theory established the foundation that made it possible to think about growth theory and business cycles within the same theoretical framework. One of the most important of these developments from the standpoint of the issues addressed in this book was Brock and Mirman's (1972) characterization of optimal growth in an economy with stochastic productivity shocks. A second was the introduction of the labor-leisure choice into the basic neoclassical model. The most thorough and up-to-date treatment of the important theoretical issues in the theory of economic growth is contained in Stokey and Lucas, with Prescott (1989). That theory is the fundamental building block of the modern approach to studying the business cycle. This volume describes both the methods and the findings of this modern approach.

Modern business cycle theory starts with the view that *growth* and *fluctuations* are not distinct phenomena to be studied with separate data and different analytical tools.[3] This theory adheres to the notion, familiar from modern growth theory, that simple artificial economies are useful vehicles for assessing those features of actual economies that are important for business cycles. A distinguishing feature of these model economies is that economic outcomes do not occur arbitrarily, but instead arise as the equilibrium outcomes of fully rational economic agents. The artificial economies described are also in the spirit of the postwar macroeconometric tradition of Tinbergen and Klein in that they are fully specified *empirical* model economies; that is, they are constructed to mimic important aspects of the behavior through time of actual economies. We will show that these artificial economies are useful laboratories for studying the business cycle and for studying economic policy. The goal of this research is to better understand the behavior of actual economies by studying the equilibria of these synthetic economies. The realization of this goal requires a careful marriage of economic theory and empirical observation.

In this chapter, we describe some of the common features of the approach to studying business cycles that unify the chapters in this volume. The most common element is the neoclassical model of economic growth. In the next section, we describe a deterministic economic environment that is designed to capture Kaldor's stylized facts described above. We also describe in some detail a particular competitive equilibrium concept that many of the later chapters and many of the papers in the literature use. Section 3 introduces the stochastic growth environment, and we present the recursive competitive equilibrium concept for this environment. Section 4 describes how the basic neoclassical growth framework can be calibrated to the U.S. economy to yield quantitative statements about the evolution of the model economy. We devote Section 5 to an overview of the solution techniques that are used for computing the equilibria of business cycle models. The known facts about business cycles are described in Section 6; we discuss some of the issues and choices one makes in representing cyclical and growth components in the data. In Section 7, we describe the results of simulating the stochastic growth model.

2. Deterministic Growth

Our purpose in this section is to describe a neoclassical model economy that was designed to capture the features of economies experiencing balanced growth. This environment is the most basic underpinning of all that follows in this book. We are going to be concerned here and throughout the book with *general equilibrium* descriptions of economic growth, so we begin with a deterministic dynamic general equilibrium model.

The Environment

The economy we examine is populated with a large number of identical households, each of which will live forever and each with identical preferences defined over consumption at every date. We assume that preferences are additively separable, with the form

$$u(c_0, c_1, c_2, \ldots) = \sum_{t=0}^{\infty} \beta^t U(c_t), \quad 0 < \beta < 1. \tag{1}$$

The period utility function $U: R_+ \to R$ has the properties that U is continuously differentiable in its arguments, is increasing, is strictly concave, and $\lim_{c\to 0} U'(c) = \infty$. The parameter β is the discount factor that households apply to future consumption.

The households in this economy do not value leisure. We assume that the population size is constant, and we normalize the total time endowment of labor available for production to unity. There is an initial endowment of capital, K_0. Each period, households supply labor, H_t, and capital, K_t, to firms. The latter have access to a technology for producing the single good, Y_t. The aggregate production function is

$$Y_t = F(K_t, H_t), \tag{2}$$

where $F: R_+^2 \to R_+$ has the properties that F is increasing in K and H, is concave in K and H separately, is continuously differentiable in K and H, and is homogeneous of degree one. Moreover,

$$F(0,0) = F(0,H) = F(K,0) = 0;$$
$$F_K(K,H) > 0,\ F_H(K,H) > 0,\ \forall K, H, > 0;$$
$$\lim_{K\to 0} F_K(K,1) = \infty, \quad \text{and} \quad \lim_{K\to\infty} F_K(K,1) = 0.$$

We assume further that capital depreciates at a constant rate $0 < \delta \leq 1$. The aggregate resource constraint implies that consumption, C_t, and gross investment, $K_{t+1} - (1-\delta)K_t$, have to satisfy the condition

$$C_t + K_{t+1} - (1-\delta)K_t \leq F(K_t, H_t),\ \forall t. \tag{3}$$

The Planner's Problem

If we imagine that this economy is governed by a benevolent social planner, the problem faced by the planner is to choose sequences for consumption, labor supply, and the capital stock, $\{C_t, H_t, K_{t+1}\}_{t=0}^{\infty}$, that maximize (1), given K_0, subject to the aggregate resource constraint. The solution to this problem requires that no output be wasted, which in turn implies that equation (3) holds with equality, and that $H_t = 1$ for all t. Accordingly, K_t and Y_t represent capital and output per worker as well as aggregate capital and output.

We can rewrite the aggregate resource constraint as

$$C_t + K_{t+1} = F(K_t, 1) + (1-\delta)K_t = f(K_t),\ \forall t. \tag{4}$$

Equation (4) allows us to rewrite the planner's problem as

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U[f(K_t) - K_{t+1}]$$
$$\text{s.t.} \quad 0 \leq K_{t+1} \leq f(K_t),\ t = 0, \ldots,$$
$$\text{given } K_0 > 0. \tag{5}$$

One technique for solving dynamic optimization problems of the form (5) is to rewrite them in a recursive form that can be solved by dynamic programming. Let $V(K_0)$ denote the maximum value of the function (5) that could be obtained for any $K_0 > 0$. The planner's problem can then be represented as

$$V(K_0) = \max_{0\leq K_1 \leq f(K_0)} \{U[f(K_0) - K_1] + \beta V(K_1)\}. \tag{6}$$

The theory of dynamic programming necessary to solve problems like this is discussed in great detail in Stokey and Lucas, with Prescott (1989). The reader should have at least some familiarity with those techniques.

Supporting the Solution to the Planner's Problem as an Arrow-Debreu Competitive Equilibrium

Our interest in the planner's problem is motivated by the fact that for this model economy (and under fairly general assumptions), the solution to the planner's problem is the competitive equilibrium allocation. This result can be established by using the two fundamental theorems of welfare economics. The First Welfare Theorem can be used to conclude that any competitive equilibrium allocation for this economy is a Pareto-optimal allocation. As there is only one Pareto-optimal allocation, if a competitive equilibrium exists it is the solution to the social planner's problem.

One way to establish the existence of a competitive equilibrium is to use the Second Welfare Theorem to support the solution to the social planner's problem as a competitive equilibrium. If there is discounting, the optimum can be supported as a decentralized competitive equilibrium with a price system that has an inner product representation. Stated differently, the value of a commodity bundle is the price of each commodity in the commodity vector times the quantity of that commodity summed over the infinity of the Arrow-Debreu date and event contingent commodities (see Stokey and Lucas, with Prescott 1989, 466, Theorem 15.6; or Harris 1987). This is the valuation equilibrium concept used by Prescott and Lucas (1972) among others.

For our simple economy, the price system just referred to can be found as follows: We assumed that the households own the capital and labor and rent them to firms (if there are constant returns to scale, there need be only one firm). Since the firm needs only to hire capital and labor each period, we can describe the firm as solving a series of static, one-period profit maximization problems:

$$\max_{K_t, H_t} p_t \cdot [F(K_t, H_t) - r_t K_t - w_t H_t], \ \forall t. \tag{7}$$

From the necessary and sufficient marginal conditions for maximization, the real wage rate, w_t, and the real rental price of capital, r_t, in terms of output must be

$$w_t = F_2(K_t, H_t) \tag{8}$$

and

$$r_t = F_1(K_t, H_t), \tag{9}$$

for all t. If we assume constant returns to scale, in equilibrium there are no profits or dividends to distribute to households, and we can ignore issues involving the ownership of firms. Households solve the problem:

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t U(c_t) \\ &\text{s.t.} \quad \sum_{t=0}^{\infty} p_t[c_t + K_{t+1}] \leq \sum_{t=0}^{\infty} p_t[w_t + (r_t + 1 - \delta)K_t], \\ &\qquad c_t \leq 0, \ K_{t+1} \geq 0. \end{aligned} \tag{10}$$

The first-order conditions for the household's problem imply that prices must make households indifferent between consumption at different dates:

$$p_t / p_{t+1} = U'(c_t)/[\beta \cdot U'(c_{t+1})]. \tag{11}$$

From this we determine the interest rate as $i_t = p_t/p_{t+1} - 1$.

This approach to finding the equilibrium process for the economy is limited. The principal class of economies to which these methods apply is economies with a single type of household and no distortions. This rules out economies with externalities, cash-in-advance constraints, limited contracting technology, monopolistic elements or non–lump sum taxes, among other things. These are all factors that result in nonoptimality of the equilibrium allocation.[4]

Supporting the Solution to the Planner's Problem as a Recursive Competitive Equilibrium

One approach that has proven very useful in the study of business cycle models is to support the allocation given by the planner's problem by using the decentralized stationary recursive competitive equilibrium (RCE) concept suggested by Prescott and Mehra (1980). This equilibrium concept is the one most widely used in this book.[5] Given a set of securities that is sufficient to realize all gains from allocating risks, recursive competitive equilibrium allocations support a state-contingent equilibrium.[6] The recursive competitive equilibrium is particularly convenient for the problem considered here and for the majority of those considered in this book because it fits naturally into the dynamic programming approach to solving optimization problems. It is also very easily applied in a wide variety of settings, including those with distortions, as Chapter 2 and many of the other chapters in this volume will make clear. Another great advantage of the RCE approach is that for an increasingly rich class of model economies, the equilibrium process can be computed and can be simulated to generate equilibrium paths for the economy. These paths can be studied to see whether model economies mimic the behavior of actual economies and can be used to provide quantitative answers to questions of economic welfare.

For the problem at hand, we illustrate the competitive solution, using the recursive competitive equilibrium concept just described. In this approach, we view households and firms as decisionmaking units, and we view individual households as solving dynamic programming problems. We distinguish between the economywide per capita capital stock, K, and the household's own capital stock, k, over which it has control. We further distinguish variables over which the household has control from their aggregate counterparts by using lower-case and upper-case letters, respectively. In equilibrium it will be true that $K = k$, but the problem the household solves makes a distinction between these. The state variables for the household are (k, K). Let $v(k, K)$ be the household's optimum value function and let primes denote next-period values. The household's decision problem is to choose a path for investment, x, and consumption, c, that solves the problem:

$$\begin{aligned} v(k, K) &= \max_{c,x,\geq 0} \{u(c) + \beta v[k', K']\} \\ \text{s.t.} \quad & c + x \leq r(K)k + w(K), \\ & k' = (1 - \delta)k + x, \\ & K' = (1 - \delta)K + X(K). \end{aligned} \tag{12}$$

Let $d(k, K)$ be the policy function that gives the optimal decisions for this problem. Because all households are identical it must be the case that in equilibrium $d(k, K) = D(K)$. This leads us to the following definition:

A *recursive competitive equilibrium* is a value function, $v(k, K): \mathbf{R}_+^2 \to \mathbf{R}$; a policy function, $d(k, K): \mathbf{R}_+^2 \to \mathbf{R}_+$, which gives decisions on $c(k, K)$, $x(k, K)$ for the representative household; an aggregate per capita policy function, $D(K): \mathbf{R}_+ \to \mathbf{R}_+$, which gives aggregate decisions $C(K)$ and $X(K)$; and factor price functions, $r(K): \mathbf{R}_+ \to \mathbf{R}_+ w(K): \mathbf{R}_+ \to \mathbf{R}_+$, such that these functions satisfy

1) the household's problem (12);

2) the necessary and sufficient conditions for profit maximization, (8) and (9);

3) the consistency of individual and aggregate decisions, i.e., the condition $d(K, K) = D(K)$, $\forall K$; and

4) the aggregate resource constraint, $C(K) + X(K) = Y(K)$.

If (v, d, D, r, w) is a recursive competitive equilibrium, then the statement that competitive equilibrium allocations are Pareto optimal implies that $v(K, K)$ coincides with the value function $V(K)$ for the social planner's problem discussed earlier and $D(K)$ coincides with the optimal policy function for that problem. This equivalence will be exploited in many of the problems discussed in the ensuing chapters. In the next chapter we show how to solve for a competitive equilibrium by solving the social planner's problem. Equivalence between the competitive equilibrium allocation and the solution to the social planner's problem will not always hold. In some applications, an economy is subject to distortions (because of taxes, money, or other variations in the arrangements), so the competitive equilibrium allocation will not be Pareto optimal. In such cases, we cannot obtain the competitive equilibrium allocation by solving the social planner's problem. In the next chapter we show how to solve a related problem that yields the recursive competitive equilibrium allocation in these situations.

Using the Growth Model

The previous section describes a "model economy" that is explicitly designed to depict how an economy might grow over time in a way that is consistent with the growth facts described at the beginning of this section. It is a competitive general equilibrium economy because we can represent the paths of consumption, investment, output, and hours worked as the market-clearing outcome of individual households and firms responding to prices. An important feature of this economy, from our point of view, is that we can compute the equilibrium and use it to generate data.

To generate time series of the variables of the equilibrium for this economy, we would first assume functional forms for the preferences and technology, assign values to the parameters of those functional forms and assign a value to the initial condition K_0. Given those, we would first use equation (6) to compute K_1. Then equation (4) can be used to compute C_0. Because households do not value leisure and the marginal product of labor is strictly positive, $H_0 = 1$. Equations (7) and (8) are then used to determine the rental prices of the factors r_0 and w_0, and equation (9) is used to determine the interest rate i_t. This process can be repeated to determine the Date 1 values of the variables, along with the Date 2 capital stock. Similarly, the value of each variable can be determined for every subsequent date. This is the sense in which we can operate this artificial economy to study its equilibrium path.

The behavior of economies with these features is well understood. We know that for suitably parametrized preferences and technology, this economy will converge to a balanced growth path that is consistent with the stylized facts described earlier. Producing a balanced growth path is not the main objective of the research and methods described in this book. We view the growth model as a basic building block—a platform that can be extended and elaborated to address richer and more interesting questions.

Robert Solow (1957) addressed one such question when he applied the neoclassical growth model to the U.S. economy to calculate the sources of long-term growth. Using data for the period from 1909 to 1949, Solow estimated that changes in productivity accounted for 87.5 percent of the growth in real output per worker over this period while only about 12.5 percent was accounted for by increased capital per worker.[7] If we were to revisit Solow's calculation of the sources of secular growth but use more recent data we would obtain a breakdown something like that shown in the accompanying table. If we impute the value of the flow of services from consumer durables and add it to measured output, then capital's share in output will be about one-third (we will make these shares more precise in Section 4). Thus approximately one-third of the growth in output per worker is attributable to changes in capital per worker. We also know that there is no trend in the average hours of work per worker in the post–World War II period, so variations in the labor input do not contribute to secular growth. That suggests that the remaining two-thirds of secular growth in output is attributable to improvements in productivity. These are features we would like a model economy to reproduce.

Changes in Output per Worker	*Secular Growth*	*Business Cycle*
Due to changes in capital	1/3	0
Due to changes in labor	0	2/3
Due to changes in productivity	2/3	1/3

The primary focus of this book is on business cycles. This leads us to ask how we should modify the basic growth model to see if it can generate cycles. Business cycle accounting is more difficult and the breakdown is more approximate. The decomposition of output fluctuations for the business cycle (again based on the neoclassical growth model) reveals that the sources of business cycle fluctuations are quite different. We know that the capital input fluctuates very little over the cycle and that variations in capital are largely uncorrelated with the cycle. About two-thirds of the fluctuations in aggregate output are attributable to fluctuations in the labor input. Since capital does not vary much, the remaining one-third of fluctuations is attributable to fluctuations in productivity.

This accounting exercise tells us that a model that *integrates* both growth and fluctuations is going to require some features not present in the deterministic growth model. First, we must introduce some mechanism that causes productivity to change at the business cycle frequency. Second, we must introduce labor supply in a way that is consistent with large movements in the labor input over the cycle. Simply stating the problem this way does not make it immediately obvious that there is only one way or a correct way to modify the growth model to produce business cycle–type fluctuations. Kydland and Prescott (1982) and Long and Plosser (1983) explored a number of alternatives, and not all of them were successful. Many alternatives will be explored in the following chapters. In the next section, we describe a modification of the basic growth model that can in principle reproduce these fluctuations. The economy we describe is a stochastic version of the model we have been analyzing here. It also incorporates variations in employment, so it has the potential to explain the features of business cycle fluctuations as well as secular growth. After we have described the model, we will discuss how to use it empirically, describing how we calibrate its parameters from observed features of the U.S. economy, compute an equilibrium, and generate data from that equilibrium.

3. A Stochastic Growth Economy with Labor-Leisure Choice

The Environment

In this section we describe an economic environment that is richer than the one portrayed above. In the basic neoclassical growth model, neither employment nor savings varies over time once the economy has entered its long-run steady state. Here, we modify that economy so that households may vary their consumption and labor supply over time. Their reason for so doing is that in this economy, the agents face uncertainty about their future productivity. The economy is populated by infinitely many identical households that will exist forever. Each of these households has an endowment of time for each period, which it must divide between leisure, ℓ_t, and work, h_t. We normalize the households' time endowment to unity, that is, we set $h_t + \ell_t = 1$. In addition, the households own an initial stock of capital, k_0, which they rent to firms and may augment through investment.

Households' utility for each period is defined over stochastic sequences of consumption and leisure:

$$U[c(\cdot), h(\cdot)] = E\{\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)\} \quad 0 < \beta < 1, \tag{13}$$

where $c(\cdot)$, $h(\cdot)$ represent the sequences of Arrow-Debreu event-contingent consumptions and labor supplies. We assume that u is continuously differentiable in both arguments, that u is increasing in both arguments, and that u is strictly concave. The households in this economy supply capital and labor to firms, which have access to a technology described by the aggregate production function $F(K_t, H_t) : \mathbf{R}_+ \rightarrow \mathbf{R}$. We assume that F is continuously differentiable in K and

H, that F is monotonic in K and H, that F is concave in K and H separately, and that $F(0, 0) = 0$.

Aggregate output is determined by the production function:

$$Y_t = e^{z_t} F(K_t, H_t), \tag{14}$$

where z_t is a random productivity parameter. This productivity shock is the source of uncertainty in the economy. We will make the very specific assumption that z_t evolves according to the law of motion:

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \quad 0 < \rho < 1. \tag{15}$$

where ϵ is distributed normally, with mean zero and standard deviation σ_ϵ. Brock and Mirman (1972) showed that if the $\{z_t\}$ are identically distributed random variables then there exists a solution to the social planner's problem for this economy.[8]

We assume that the capital stock depreciates exponentially at the rate δ and that consumers add to the stock of capital by investing some amount of the real output each period. Investment in period t produces productive capital in period $t + 1$ so that the law of motion for the aggregate capital stock is

$$K_{t+1} = (1 - \delta)K_t + X_t. \tag{16}$$

As in the previous example, the firms rent capital and hire labor in each period. We can treat this as a single firm that solves a period-by-period profit maximization problem. All relative prices are in terms of output, and we can write the firm's period t problem as

$$\max_{K_t, H_t} p_t \cdot [e^{z_t} F(K_t, H_t) - r_t K_t - w_t H_t], \ \forall t. \tag{17}$$

This optimization problem yields factor prices (stated in terms of the price of output):

$$r_t = e^{z_t} F_k(K_t, H_t), \tag{18}$$

and

$$w_t = e^{z_t} F_H(K_t, H_t). \tag{19}$$

As in the previous example, given constant returns to scale, in equilibrium profits will be equal to zero.

The households in this economy face a very difficult problem because they must form expectations over future prices. Households will choose consumption, investment, and hours of work at each date to maximize the expected discounted value of utility, given their expectations over future prices subject to sequences of budget constraints and the law of motion for the household's capital stock:

$$\max_{c,x,h} E\{\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)\} 0 < \beta < 1,$$

$$\text{s.t.} \quad c_t + x_t \leq w_t h_t + r_t k_t,$$

$$k_{t+1} = (1 - \delta)k_t + x_t. \tag{20}$$

Note that for this problem, the prices, w and r, depend on the economywide state variables (z, K). Moreover, the decisions on quantities depend on the individual-level state variables (z, k, K). The equilibrium definition set forth in the next section makes this dependence more precise.

The Recursive Competitive Equilibrium

Again, we will use the recursive competitive equilibrium concept. The state variables for the households in this economy are $s_t = (z_t, k_t, K_t)$, and the aggregate state variables are $S_t = (z_t, K_t)$. The optimality equation for the household's problem can then be written as

$$v(z, k, K) = \max_{c,x,h}\{u(c, 1 - h) + \beta E[v(z', k', K')|z]\}$$

$$\text{s.t.} \quad c + x \leq r(z, K)k + w(z, K)h,$$

$$k' = (1 - \delta)k + x,$$

$$K' = (1 - \delta)K + X(z, K),$$

$$z' = \rho z + \epsilon,$$

$$c \geq 0, 0 \leq h \leq 1. \tag{21}$$

A *recursive competitive equilibrium* for this economy consists of a value function, $v(z, k, K)$; a set of decision rules, $c(z, k, K)$, $h(z, k, K)$, and $x(z, k, K)$, for the household; a corresponding set of aggregate per capita decision rules, $C(z, K)$, $H(z, K)$, and $X(z, K)$; and factor price functions, $w(z, K)$ and $r(z, K)$, such that these functions satisfy:

1) the household's problem (21);

2) The condition that firms maximize and satisfy (18) and (19), that is, $r = r(z, K)$ and $w = w(z, K)$;

3) the consistency of individual and aggregate decisions, that is, the conditions $c(z, K, K) = C(z, K)$, $h(z, K, K) = H(z, K)$, and $x(z, K, K) = X(z, K)$, $\forall (z, K)$; and

4) the aggregate resource constraint, $C(z, K) + X(z, K) = Y(z, K)$, $\forall (z, K)$.

This completes the description of the environment and the equilibrium concept that we will use. This basic framework is consistent with many different model economies, depending on the further restrictions or different arrangements that are added. In order to use this framework to make quantitative statements about business cycles, we need a more explicit structure. In the next section we describe how parameter values are assigned to provide that structure.

4. Calibration

The description of the economic environment and the equilibrium concept together provide a framework that we can use to study business cycles. The environment described in the previous section has features that are motivated by the questions we want to address. In this chapter, the question we address is quite simple: does a model designed to be consistent with long-term economic growth produce the sort of fluctuations that we associate with the business cycle? In subsequent chapters many more complicated questions are addressed, and the basic structures studied in those sections are correspondingly altered. These alterations can take the form of changes in the environment and/or changes in the equilibrium concept.

The framework described in the previous section is consistent with many different equilibrium processes for the variables of interest—output, employment, investment, and so on. To go from that general framework to quantitative statements about the issues of interest is a three-step process. The first step is to restrict these processes to a parametric class. We stress the idea of using a model that is consistent with growth observations to study fluctuations. This requires the use of more economic theory and some observations. The second step in this process is to construct a set of measurements that are consistent with the parametric class of models. With enough theory and observations to define a parametric class of models, we can establish the correspondence between this class and the observed data for the U.S. (or some other) economy. As we will demonstrate, establishing this correspondence may well require that we reorganize the data for the U.S. economy in ways that make them consistent with our class of model economies. The third step is to assign values to the parameters of our models. This involves setting parameter values so that the behavior of the model economy matches features of the measured data in as many dimensions as there are unknown parameters. We observe over time that certain ratios in actual economies are more or less constant. We choose parameters for our model economy so that it mimics the actual economy on the dimensions associated with long term growth. Once this is accomplished we will be in a position to study the quantitative behavior of fluctuations in the particular model economy.

The process just described is called calibration. This approach has a long tradition in economics. This strategy for finding numerical values for parameters uses economic theory extensively as the basis for restricting the general framework and mapping that framework onto the data. As we will see in the chapters that follow, the kinds of restrictions that are used depend very much on the kinds of questions being asked of these artificial economies. If one is interested in studying the behavior of an economy with more than one sector, with distortions due to taxes or money, and/or with different arrangements due to contracting or non-Walrasian elements, the mapping between the theory and the data will be different. The common thread in all of these studies is that they all preserve the neoclassical growth framework.

Restricting the Growth Economy

The distinguishing features of the stochastic growth environment described in Section 3 are the household's labor-leisure choice and the presence of the shocks to technology. These features were added, as we noted earlier, to see if the model designed to explain long-term growth might also be capable of explaining fluctuations. In order to address this question, we are going to restrict our attention to artificial economies that display balanced growth. In balanced-growth consumption, investment and capital all grow at a constant rate while hours stay constant. This behavior is consistent with the growth observations described earlier.

The basic observations about economic growth suggest that capital and labor shares of output have been approximately constant over time even while the relative prices of these inputs have changed. This suggests a Cobb-Douglas production function, which has the form

$$Y_t = e^{z_t} K_t^{\theta} H_t^{1-\theta}. \tag{22}$$

The parameter θ is referred to as capital's share because if capital is paid its marginal product, it will earn that fraction of output. The Cobb-Douglas assumption defines a parametric class of technologies for this economy.

As with the technology, certain features of the specification of preferences are tied to basic growth observations for the U.S. economy. There is evidence that per capita leisure increased steadily until the 1930s. Since that time, and certainly for the postwar period, it has been approximately constant. We also know that real wages (defined as real average hourly total compensation, including benefits and contributions for social insurance) have increased steadily in the postwar period. Taken together, these two observations imply that the elasticity of substitution between consumption and leisure should be near unity. We consider the general parametric class of preferences of the form

$$u(c_t, \ell_t) = \frac{(c_t^{1-\alpha} \ell_t^{\alpha})^{1-\sigma} - 1}{1-\sigma}, \tag{23}$$

where $1/\sigma$ is the intertemporal elasticity of substitution and α is the share parameter for leisure in the composite commodity. The parameter σ is among the most difficult to pin down because variations in the intertemporal elasticity of substitution affect transitions to balanced growth paths but not the paths themselves. In the sections below, we will further restrict this to the limiting case where $\sigma = 1$, which is $u(c_t, 1-h_t) = (1-\alpha)\log c_t + \alpha \log(1-h_t)$.[9]

Defining Consistent Measurements

Calibrating the parametric class of economies chosen requires that we consider the correspondence between the model economy and the measurements that are taken for the U.S. economy. The neoclassical growth framework emphasizes the central role of capital in determining long-term growth in output. Consequently, the first thing we have to consider is the match between the capital as it is conceived in our class of model economies and capital as it is measured and as it is conceptualized in the U.S. National Income and Product Accounts (NIPA).[10]

Our model economy is very abstract: it contains no government sector, no household production sector, no foreign sector and no explicit treatment of inventories. Accordingly, the model economy's capital stock, K, includes capital used in all of these sectors plus the stock of inventories. Similarly, output, Y, includes the output produced by all of this capital. The NIPA are somewhat inconsistent in their treatment of these issues in that the output of some important parts of the capital stock are not included in measured output.[11] For example, the NIPA does not provide a consistent treatment of the household sector. The accounts do include the imputed flow of services from owner-occupied housing as part of GNP. But they do not attempt to impute the flow of services from the stock of consumer durables. The NIPA lump additions to the stock of consumer durables with consumption rather than treating them as investment. Because our model economy does not treat the household sector separately, when we deal with the measured data, we will add the household's capital stock—the stock of residential structures and the stock of consumer durables—to producers' equipment and structures. To be consistent, we will also have to impute the flow of services from durables and add that to measured output. In Chapter 6, where the household production sector is explicitly modeled, the distinction between household capital and the business sector capital will be explicitly preserved, as will the distinction between household output and business sector output.

In a similar vein, although there are estimates of the stock of government capital and estimates of the portion of government consumption that represents additions to that stock of capital, the NIPA make no attempt to impute the flow of services from the government's capital stock and include it as part of output. Nor do the NIPA include government investment as part of measured investment. Because our model economy does not have a government sector, we will add the government capital stock to the private capital stock and the capital stock in the household sector. We will also impute the flow of services from this capital stock and add it to measured output.

Finally, our technology makes no distinction among the roles of reproducible capital, land, and inventories. Some of the later chapters will assign a different role to inventories, but here they are treated as identical to the other forms of capital. When we consider the mapping between the model economy and measured data, it will be important to include the value of land and the value of the stock of inventories as part of the capital stock. The Flow of Funds Accounts, Balance Sheets for the U.S. Economy are the source for estimates of the value of land, and the stock of inventories is reported in the NIPA.[12]

The measurement issues discussed above are central to the task of calibrating any model economy because a consistent set of measurements is necessary to align the model economy with the data. For example, in order to estimate the crucial share parameter in the production function, θ, for our model economy, it is important to measure all forms of capital and to augment measured GNP to include measures of all forms of output. Similarly, when we treat aggregate investment it will be necessary to include in investment additions to all forms of capital stock. For this model economy, the concept of investment that corresponds to the aggregate capital stock includes government investment, "consumption" of consumer durables, changes in inventories, gross fixed investment, and net exports.[13] Making sure that the conceptual framework of the model economy and the conceptual framework of the measured data are consistent, is a crucial step in the process of calibration.

To impute the flow of services from government capital and consumer durables, we will use more economic theory. We know that the income from capital is related to the stock of capital, as follows:

$$Y_{KP} = (i + \delta_{KP})K_P, \tag{24}$$

where Y_{KP} is the income on fixed private capital, K_P is the fixed private capital stock, and δ_{KP} is the depreciation rate of that capital stock. Given measured values of the capital stock, measured values for capital income, and a measured value for depreciation, we can obtain an estimate of i, the return on capital. Measured K_P includes the net stock of Fixed Reproducible Private Capital (*not* including the stock of consumer durables), from Musgrave (1992); the stock of inventories, from the NIPA; and the stock of land, from the Flow of Funds Accounts.

The measured value of income from fixed private capital is taken from the NIPA. There is some judgment involved in defining this because of ambiguity about how much of Proprietors' income and some other smaller categories (specifically, the difference between Net National Product and National Income) should be treated as capital income. We define the measured income in the following way. Let unambiguous capital income be defined as follows:

$$\begin{aligned}\text{Unambiguous Capital Income} &= \text{Rental Income} + \text{Corporate Profits}\\ &\quad + \text{Net Interest},\end{aligned}$$

with Rental Income, Corporate Profits, and Net Interest from the NIPA (see table 1.14).

Our strategy is to allocate the ambiguous components of income according to the share of capital income in measured GNP. Let θ_P denote the share of capital in measured GNP. Further, note that the measured value of $\delta_P K_P$ is "Consumption of Fixed Capital" (GNP − NNP) in the NIPA (Table 1.9). We denote this variable as DEP. Define Y_{KP} as follows:

$$\begin{aligned}Y_{KP} &= \text{Unambiguous Capital Income} + \theta_P(\text{Proprietors Income}\\ &\quad + \text{Net National Product} - \text{National Income}) + \text{DEP}\\ &= \theta_P \cdot \text{GNP}.\end{aligned}$$

This equation can be solved for θ_P:

$$\theta_P = \frac{(\text{Unambiguous Capital Income} + \text{DEP})}{(\text{GNP} - \text{Ambiguous Capital Income})}.$$

Multiplying θ_P by GNP, gives us the measured value of Y_{KP}. Given measured Y_{KP}, we use the equation above to determine the interest rate, i:

$$i = (Y_{KP} - \text{DEP})/K_P. \tag{25}$$

Over the sample period, 1954–1992, this yields an average interest rate of 6.9 percent.

To estimate the flow of services from the stock of consumer durables and the stock of government capital we need estimates of the depreciation rates for those portions of the capital stock. These are obtained from the laws of motion for these capital stocks:

$$K_{t+1} = (1 - \delta)K_t + X_t, \tag{26}$$

where X represents investment. Normalizing by output, Y_t, and multiplying by Y_{t+1}/Y_{t+1} yields

$$\frac{Y_{t+1}}{Y_t} \cdot \frac{K_{t+1}}{Y_{t+1}} = (1 - \delta)\frac{K_t}{Y_t} + \frac{X_t}{Y_t}. \tag{27}$$

On a balanced-growth path, $K_{t+1}/Y_{t+1} = K_t/Y_t$, and equation (27) provides the basis for measuring the depreciation rates for consumer durables and government capital. For consumer durables, investment is simply consumption of consumer durables as reported in the NIPA. For government investment the figures also come from the NIPA (table 3.7b). The stock of durables and the government capital stock are taken from Musgrave (1992). If we use the data for 1954–1992, the depreciation rate implied by equation (27) is 0.21 per annum for durables and 0.05 for government capital.[14] The service flows are then estimated as

$$Y_D = (i + \delta_D)K_D.$$

and

$$Y_G = (i + \delta_G)K_G,$$

where the interest rate and the depreciation rates are those estimated above.

The process just described used economic theory to help define a consistent set of measurements. We rearranged and augmented the measured data to correspond to the structure of the model economy. The parameters we used to do this, however, depend only on information in the NIPA and are not specific to the model economy being studied.

Calibrating a Specific Model Economy

The model economy described in Section 3, displays no population growth or long-term productivity growth. The economy we want to match it to is characterized by both. If we let η denote the rate of population growth and let λ denote the long-term real growth rate, then we can rewrite our model economy to take account of these features. If we further restrict the parametric class of preferences so that $\sigma = 1$, we can rewrite the problem in (20) as

$$\max E\left[\sum_{t=0}^{\infty} \beta^t (1+\eta)^t [(1-\alpha)\log c_t + \alpha \log(1-h_t)]\right]$$

$$\begin{aligned} \text{s.t.}\quad & c_t + x_t = e^{z_t}(1-\gamma)^{t(1-\theta)} k_t^{\theta} h_t^{1-\theta}, \\ & (1+\gamma)(1+\eta)k_{t+1} = (1-\delta)k_t + x_t, \\ & z_{t+1} = \rho z_t + \epsilon_t. \end{aligned} \tag{28}$$

All variables are in per capita terms. The parameters η and γ can be measured from the data as the rate of population growth and the rate of growth of real per capita output, respectively. Similarly, given estimates of the missing components of output as described above, capital's share in output, θ, is calibrated as follows:

$$\theta = (Y_{KP} + Y_D + Y_G)/(\text{GNP} + Y_D + Y_G) = 0.40. \tag{29}$$

Labor's share is $1 - \theta = 0.60$. These estimates are somewhat different than those that appear elsewhere in the literature because they include the imputed income from government capital.[15]

We calibrate the remaining parameters by choosing them so that the balanced-growth path of our model economy matches certain long-term features of the measured economy. Substituting the constraints into the objectives and deriving the first-order condition for k yields

$$\frac{(1+\gamma)(1+\eta)}{c_t} = \frac{\beta(1+\eta)[\theta k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + 1 - \delta]}{c_{t+1}}. \tag{30}$$

In balanced growth, this implies that:

$$\frac{(1+\gamma)}{\beta} + \delta - 1 = \theta \cdot \frac{y}{k}. \tag{31}$$

The first-order condition for hours, h, on a balanced-growth path implies that:

$$(1-\theta)\cdot\frac{y}{c} = \frac{\alpha}{1-\alpha}\cdot\frac{h}{1-h}. \tag{32}$$

Finally, the law of motion for the capital stock in steady state implies that

$$\begin{aligned} (1+\gamma)(1+\eta)\frac{k}{y} &= (1-\delta)\frac{k}{y} + \frac{x}{y} \\ \delta &= \frac{x}{k} + 1 - (1+\gamma)(1+\eta). \end{aligned} \tag{33}$$

Equation (33) is the basis for calibrating the aggregate depreciation rate, δ, for this economy, which is seen to depend on the aggregate investment/capital ratio. The steady-state investment/capital ratio for this economy is 0.076. Given the values of γ and η, the parameter δ is calibrated to match this ratio. This yields an annual depreciation rate of 0.048, or a quarterly rate of 0.012. This number depends on the real growth rate, γ, and the population growth rate for the economy. In an economy that does not explicitly include growth this number must be larger to match investment.

Once δ is calibrated, equation (31) provides the basis for determining β. Given values for γ, δ, and θ, β is chosen to match the steady-state output/capital ratio. Under the broad definitions of output and capital consistent with our model economy, the capital/output ratio is 3.32. This yields an annual value for β of 0.947, which implies a quarterly value of about 0.987.

Given an estimate of h, the fraction of time devoted to market activities, equation (32) provides the basis for calibrating the preference parameter, α, based on the steady-state output consumption ratio. The value of h is determined by microeconomic evidence from time allocation studies. Ghez and Becker (1975) and Juster and Stafford (1991) have found that households allocate about one-third of their discretionary time—i.e., time not spent sleeping or in personal maintenance—to market activities. The specific value we use for h is 0.31. Given the broad definition of consumption and output appropriate for this model economy, the steady-state ratio of output to consumption is 1.33. This implies a value of $\alpha/(1-\alpha) = 1.78$.

Finally, completion of our calibration of this model economy requires parameters of the process that generates the shocks to technology. One approach to calibrating this process would be to do as Robert Solow did and calculate technological change as the difference between changes in output and the changes in measured inputs (labor and capital) times their shares. Using equation (22), we obtain:

$$z_t - z_{t-1} = (\ln Y_t - \ln Y_{t-1}) - [\theta \cdot (\ln K_t - \ln K_{t-1}) + (1-\theta) \cdot (\ln H_t - \ln H_{t-1})] \tag{34}$$

These are the Solow residuals for this economy. Using our estimate of $\theta = 0.4$ and observations on measured output, given a measure of the labor input, we can generate a series for the z_t and their difference. We use a quarterly hours series based on the Establishment Survey for the labor input. An alternative would be to use the hours series based on the Household Survey. The other decision we face is whether to use the broad definition of capital stock consistent with our model economy, and whether to use the broad definition of output, including the imputed service flows from consumer durables and government capital. We elect to use simply measured output (real GNP) and measured labor input, assuming quarterly variations in the capital stock to be approximately zero. We choose this alternative because the capital stock series is only reported annually. Consequently, the imputed service flows that we described above are also annual. One can interpolate quarterly versions of these, but any procedure for doing so is essentially arbitrary and may add to the variability of both output and the residuals. The residuals computed using measured real GNP are highly persistent, and the autocorrelations are quite consistent with a technology process that is a random walk. We assume a value of $\rho = 0.95$, in the law of motion for the technology and use this to define a set of innovations to technology. These innovations have a standard deviation of about 0.007, which is similar to the value calibrated in Prescott (1986).[16]

We summarize our calibrated parameters in the accompanying table.

Technology					*Preferences*			
θ	δ	ρ	σ_ϵ	γ	β	σ	α	η
0.40	0.012	0.95	0.007	0.0156	0.987	1	0.64	0.012

One of the standard meanings of the word calibration is "to standardize as a measuring instrument." This definition applies to our calibration of the stochastic growth model. Since the underlying structure is the neoclassical growth framework, the choice of parameters and functional forms ensures that this model economy will display balanced growth. This is the standard that we insist on preserving in our study of business cycles. It remains to be seen whether this model economy reproduces anything that looks like business cycles. Before we can address that, we need to describe how the model can be solved for an equilibrium path.

5. Computing the Recursive Competitive Equilibrium

We have motivated our interest in the recursive competitive equilibrium construct in part by arguing that it fits naturally into the dynamic programming language in a convenient way. This connection is also exploited in solving the model. Many methods for solving for the competitive equilibria of these types of economies have been proposed and used in recent business cycle literature. Indeed, this area of research—the study of techniques for solving for the equilibria of dynamic rational expectations economies—has been one of the most active in macroeconomics. This book does not attempt to provide an exhaustive discussion of solution methods. To do so would occupy a whole volume in itself. In the next three chapters we describe a variety of methods for solving such models. Chapter 2, by Gary Hansen and Edward Prescott, describes a set of methods that are among the most important for solving straightforward representative agent economies and overlapping-generations economies. It also describes how those methods may be applied to economies that are subject to distortions where the competitive equilibrium is not a Pareto optimum (i.e., where the Second Welfare Theorem does not apply.) In Chapter 3, Jean-Pierre Danthine and John Donaldson describe a different set of methods, which have been used fruitfully in economies that are subject to distortions and which have computational advantages in a variety of other situations. Chapter 4, by José-Víctor Ríos-Rull describes a set of techniques that are used to solve economies with heterogeneous agents and also shows how to structure economies with overlapping generations so that they can be solved using the methods described in Chapter 2. We do not intend to discuss solution techniques in this chapter, but we do want to present a brief overview of what different solution techniques try to do and provide a taxonomy for understanding where the techniques discussed in the next few chapters fit into the literature.[17]

It is easy to see from the development in the preceding sections that most applications based on the neoclassical growth model result in a stationary dynamic programming problem. The general form of the problem that must be solved in order to obtain an equilibrium path when there is only one type of agent in the economy is:[18]

$$\begin{aligned} v(z, s) &= \max\{r(z, s, d) + \beta E[v(z', s')|z]\} \\ \text{s.t.}\quad z' &= A(z) + \epsilon' \\ s' &= B(z, s, d), \end{aligned} \tag{35}$$

where $v(z, s)$ is the optimal value function, z is a vector of exogenous state variables (for example, technology shocks), s is a vector of endogenous state variables (for example, the capital stock), d is a vector of decision variables, and $r(z, s, d)$ is the return function for the problem. The two constraints describe the evolution of the state variables.

The object of interest is a function mapping a state space into decisions. Existing solution methods for finding a decision rule fall into two categories. Each seeks to define an operator that will find the fixed point of a particular functional equation. In the first approach the operator maps the space of continuous bounded functions into itself in such a manner that the value function is the unique fixed point of the operator. Since, in addition, this operator can be shown to be a contraction, successive applications of the operator applied to any continuous bounded function will generate a sequence of functions which converges uniformly to the true $v(.)$. Since this process cannot be literally replicated on a computer, computational techniques have been developed that solve for an approximation to the operator's true fixed point. Given this approximation to the true $v(.)$, an approximation to the economy's optimal policy functions or decision rules follows indirectly.

The second approach, the Euler method, goes directly for the decision rules by using the information contained in the necessary and sufficient first-order conditions for an optimum. This approach uses the first-order conditions to define, implicitly, a monotone operator—again, with domain and range the space of continuous bounded functions—for which the unique fixed point is the economy's optimal policy function. These two methods are more precisely contrasted below in the context of problem (35).

Operating Directly with the Value Function

For any $g \in x$, the space of continuous bounded functions defined on an appropriately chosen compact subset of the economy's state space, define the operator, T, as follows:

$$Tg(z, s) = \max_{s' \in B(z,s,d)} [r(z, s, d) + \beta E(g(s', z')|z)]. \tag{36}$$

We summarize the economic environment—i.e., β; the return function, $r(z, s, d)$; and the constraint set, $B(x, s, d)$—by E (not to be confused with the expectation operator). As noted earlier, the search for a $v(.)$ that solves problem (35) is equivalent to searching for a fixed point of the operator, $T : x \to x$. Since the operator acts on an (infinite-dimensional) function space it is impossible to replicate it computationally. Two approaches have been followed in the literature to deal with this issue. Both involve replacing the economy E by an approximate economy $\tilde{E}$ for which a solution is feasible.

The first approach is to simplify the space x by restricting the domain of definition to be only a finite subset, or "grid," of the state space. This amounts to finding an operator T that maps a finite-dimensional subspace. Under suitable regularity conditions, this modified map will also be a contraction with a fixed point approximating the true $v(.)$, provided the approximating grid is chosen finely enough. This approach to computing equilibria has been taken by Danthine and Donaldson (1981), Christiano (1990), and Greenwood, Hercowitz, and Huffman (1988). This technique is not discussed in any great detail in this book.

The second approach to simplifying (35) is discussed in detail in the next chapter, by Hansen and Prescott, and, in a different context, in Chapter 4, by José-Víctor Ríos-Rull. The primary strategy for finding a solution by these methods involves forming a linear quadratic approximation around the steady-state equilibrium path of the original economy and looking for a solution for this approximate linear quadratic economy. This technique was introduced in the literature by Kydland and Prescott (1982) and has been widely explored and applied in subsequent research.

Operating Directly on the Euler Equations

Approaches that operate directly on the Euler equations also involve an operator, T, that maps a function space into itself. The difference here is that the operator is defined by the structure of the necessary and sufficient conditions for an optimum (on which market-clearing conditions have been imposed) that characterize the economy's equilibrium. Nowhere is the value function for the economy, $v(.)$, computed. Indeed, such a $v(.)$ may not exist if the equilibrium being characterized is not optimal because of the presence of taxes or other distortions. One advantage of this approach is that in recursive competitive equilibrium problems where there is a distinction between individual and aggregate state variables, it permits one to operate simply with the aggregate state variables. All implications of the distinction between aggregate and individual variables are lost in the way derivatives are taken to get the Euler equations.

For an economy with one intertemporal decision—say, how much to invest—equilibrium can be described as an aggregate function, $I(z, s)$, that satisfies an integral equation of the form

$$0 = -h[I(z, s), z, s] + \beta \int J[I(z', s'), z', s']dg(z', z). \tag{37}$$

The endogenous state variable, s, for this simple economy is k, and $k' = (1 - \delta)k + I(z, k)$.

Under quite general conditions we can define an operator $T : T(z, k) \to TI(z, k)$ where $TI(z, k)$ satisfies the conditions:

$$\arg\min | - h[TI(z, k), z, k] + \beta \int J[I(z', k')z', k']dg(z', z)|, \tag{38}$$

and $k' = (1 - \delta)k + I(z, k)$. For the studies in this volume the operator defined above will usually be monotone: successive application of the operator will generate a monotone increasing sequence of functions, which is bounded above and which converges to the fixed point of T.

For any function $I(z, k)$, $TI(z, k)$ is also a function, and the above operator cannot be exactly replicated numerically. As before, two avenues are open. One is to replace the function space by a space of finite-dimensional functions. In this approach, one solves the countable infinity of equations, using functions defined by a finite number of parameters. Clearly, the functional equation cannot be solved exactly. What these methods do is to solve the equation as closely as possible. Solution strategies based on this approach include the method of parametrized expectations of Marcet (1989), the algorithms developed by Danthine and Donaldson (1990) and Coleman (1991), and the minimum weighted residuals technique of Judd (1991). These computational techniques, along with a few others, are discussed in considerable detail in Chapter 3, by Danthine and Donaldson.

Again, as before, we may also substantially simplify the economy's primitive to allow for a simple representation of equation (37). One example of this approach is the strategy adopted by King, Plosser, and Rebelo (1988a) of finding a linear approximation to the first-order conditions characterizing equilibrium. This is related to solution methods proposed much earlier by Blanchard and Kahn (1980). It can also be seen to be related to the "backsolving" method proposed by Sims (1989). The King, Plosser and Rebelo approach is discussed in Chapter 3, by Danthine and Donaldson.

The intent of the next three chapters is to provide a firm grounding in these solution techniques, which can then be used to solve a wide variety of model economies. Throughout the book, special problems arise that require slight modifications to make these methods for computing equilibria work. In Chapter 5, by Finn Kydland, some new wrinkles are needed to solve the models he considers. In Chapter 7, by Cooley and Hansen, special information structures arise that require some modifications to the solution techniques. Where such problems do arise or where the solution procedure is somehow special, we provide further details about how to compute the equilibria.

6. Business Cycle Facts

What features of economic life do we think the stochastic growth economy just described might explain? Lucas, following on the work of Burns and Mitchell, argued that the business cycle should be thought of as apparent deviations from a trend in which variables move together. An examination of the time path of output for any modern industrialized economy quickly reveals that output tends to fluctuate about a long-term growth path. These fluctuations about trend are what we most often think of as the business cycle. The fluctuations are typically irregularly spaced and of varying amplitude and duration. Nevertheless, the one very regular feature of these fluctuations is the way variables move together. It is these comovements of variables that Burns and Mitchell worked so hard to document and that Robert Lucas emphasized as the defining features of the business cycle. These are the features of fluctuations that we would like an artificial economy—a business cycle model—to replicate.

Representing the Business Cycle

Every researcher who has studied growth and/or business cycle fluctuations has faced the problem of how to represent those features of economic data that are associated with long-term growth and those that are associated with the business cycle—the deviations from the growth path. Kuznets, Mitchell, and Burns and Mitchell all employed techniques (moving averages, piecewise trends, etc.) that define the growth component of the data in order to study the fluctuations of variables around the long-term path defined by the growth component. Whatever choice one makes about this is somewhat arbitrary. There is no single correct way to represent these components. They are simply different features of the same observed data.

For a long time, the prevailing view was that we need two different types of theory to explain the different features of the observed data: growth theory to explain the long-term movements; something else to explain the rest. Modern business cycle theory is based on a different premise: the same theory should be capable of explaining both features of the data. It is important to recognize that it was not obvious ahead of time that this premise would be borne out; in choosing a representation, one does not necessarily take a stand on whether a separate theory is needed for both components.

In the chapters that follow and in the modern business cycle literature in general, the authors typically follow one of two procedures to represent the cyclical component in the data. Some assume a stochastic trend and first difference the (logarithms of the) data to remove it. This procedure is well understood by economists. Most of the authors use a technique for representing growth and business cycle components known as the Hodrick-Prescott filter (the H-P filter) a procedure that is less well understood. We describe this filter, explain briefly what it does, and illustrate its effect on the time series of real GNP.

We characterize an observed time series, y_t, as the sum of a cyclical component, y_t^c, and a growth component, y_t^g. Let λ be a parameter that reflects the relative variance of the growth component to the cyclical component. Then, given a value for λ, the H-P filtering problem is to choose the growth component, y_t^g, to minimize the loss function:

$$\sum_{t=1}^{T}(y_t^c)^2 + \lambda \sum_{t=1}^{T}[(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)]^2. \tag{39}$$

The nature of this optimization problem is to trade off the extent to which the growth component tracks the actual series (which yields a smaller cyclical component, y_t^c against the smoothness of the trend.[19] For $\lambda = 0$ the growth component is simply the series. As $\lambda \to \infty$, the growth component approaches a linear trend. For quarterly data it is customary to choose $\lambda = 1,600$. The motivation behind this choice is that if the original series were stationary, then the H-P filter with this choice of λ would eliminate fluctuations at frequencies lower than about thirty-two quarters, or eight years. We normally think of the business cycle as fluctuations about the growth path that occur with a frequency of three to five years. This is what Burns and Mitchell (1946) characterized as the usual business cycle frequency. Hence the H-P filter suppresses the really low-frequency fluctuations and emphasizes those in this range. In contrast, the first-difference filter suppresses only the zero frequency and tends to emphasize really high-frequency movements in the data.

Perhaps the easiest way to see what the H-P filter does is to look at the representation of the cyclical component and the growth component that it gives for a typical time series. Figure 1.1 shows a plot of real GNP and its H-P–filtered growth component. The parameter λ was set at 1,600 for this exercise. One can see that the growth component tracks the series reasonably closely but produces a "trend" that highlights cyclical movements, as intended. Figure 1.2 shows the H-P–filtered cyclical component of real GNP, as well as the cyclical component derived from the first-difference filter. It appears from these figures that the first-difference filter leads to more short-term fluctuation than does the H-P filter. This is to be expected since the latter filter emphasizes the high-frequency movements more.

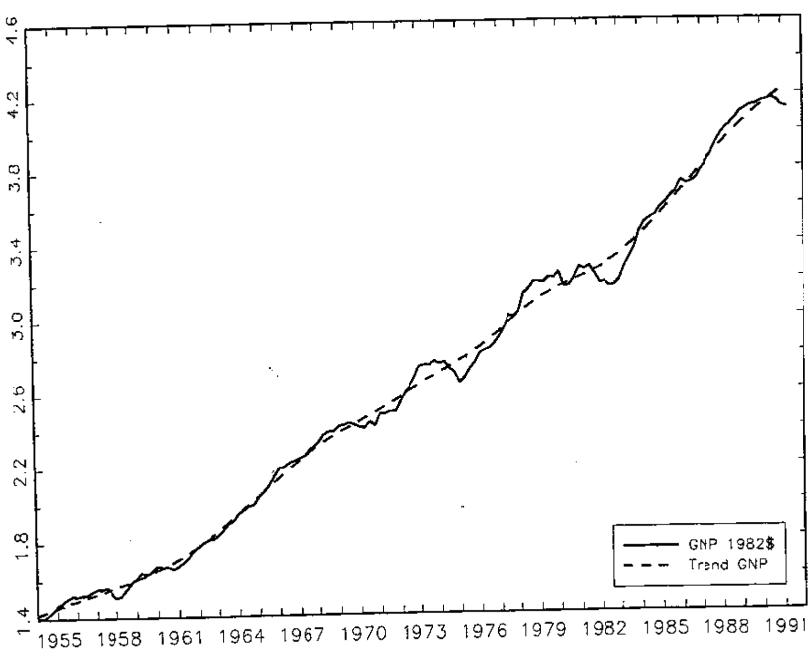

Figure 1.1 Log of Real GNP and Its Growth Component
Trillions of 1982 dollars (log scale).

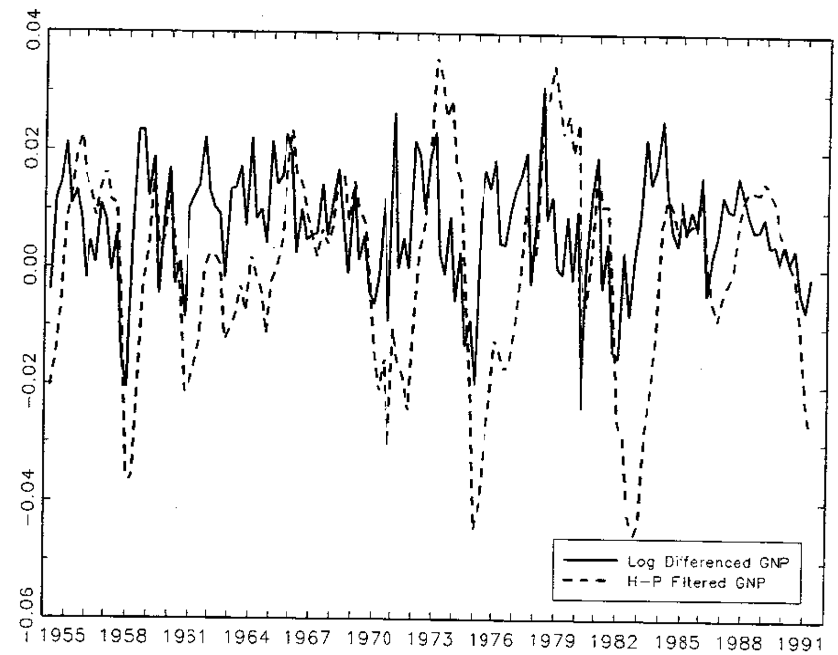

Figure 1.2 H-P–Filtered Cyclical Component and Log-Differenced Real GNP

Correspondingly, it can also be seen that the H-P–filtered data display more serial correlation.

Another alternative for removing fluctuations other than those that occur at the business cycle frequencies is to use a band-pass filter that would eliminate fluctuations at frequencies higher than, say, three years and lower than eight years. Such a filter has been used by some authors, notably Englund, Persson, and Svensson (1992). To implement the filter requires first applying the H-P, or differencing, filter to remove the lowest frequencies, transforming to the frequency domain, and then removing the remaining high-frequency elements. This is fully feasible only with very long data series, and in practice it seems to create no substantive difference in the properties of business cycles.

The Facts

We represent the business cycle facts by calculating several statistics from the H-P filtered time series data for the U.S. economy. We report the amplitude of the fluctuations in aggregate variables in order to assess their relative magnitudes. We also measure the correlation of aggregate variables with real output to capture the extent to which variables are procyclical (positively correlated) or countercyclical ble 1.1

yclical Behavior of the U.S. Economy: Deviations from Trend of Key Variables, 1954:I–1991:II

		Cross-Correlation of Output with:										
Variable	*SD%*	$x(-5)$	$x(-4)$	$x(-3)$	$x(-2)$	$x(-1)$	x	$x(+1)$	$x(+2)$	$x(+3)$	$x(+4)$	$x(+5)$
ıtput component												
GNP	1.72	.02	.16	.38	.63	.85	1.0	.85	.63	.38	.16	−.02
onsumption expenditures												
CONS	1.27	.25	.42	.57	.72	.82	.83	.67	.46	.22	−.01	−.20
CNDS	0.86	.22	.40	.55	.68	.78	.77	.64	.47	.27	.06	−.11
CD	4.96	.24	.37	.49	.65	.75	.78	.61	.38	.11	−.13	−.31
vestment												
INV	8.24	.04	.19	.38	.59	.79	.91	.76	.50	.22	−.04	−.24
INVF	5.34	.08	.25	.43	.63	.82	.90	.81	.60	.35	.09	−.12
INVN	5.11	−.26	−.12	.05	.30	.57	.79	.88	.83	.60	.46	.24
INVR	10.7	.42	.55	.65	.72	.74	.63	.39	.11	−.14	−.33	−.43
Ch. INV	17.3	−.03	.07	.22	.38	.53	.67	.51	.27	.04	−.15	−.30
overnment purchases												
GOVT	2.04	.03	−.01	−.03	−.01	−.01	.04	.08	.11	.16	.25	.32
xports and imports												
EXP	5.53	−.48	−.42	−.29	−.10	.15	.37	.50	.54	.54	.52	.44
IMP	4.88	.11	.19	.31	.45	.62	.72	.71	.52	.28	.04	−.18
abor input based on household survey												
HSHOURS	1.59	−.06	.09	.30	.53	.74	.86	.82	.69	.52	.32	.11
HSAVGHRS	0.63	.04	.16	.34	.48	.63	.62	.52	.37	.23	.09	−.05
HSEMPLMT	1.14	−.10	.04	.23	.46	.69	.85	.86	.76	.59	.40	.18
GNP/HSHOURS	0.90	.06	.14	.20	.30	.33	.41	.19	.00	−.18	−.25	−.24
Labor input based on establishment survey												
ESHOURS	1.69	−.12	.07	.38	.54	.78	.92	.90	.78	.63	.42	.21
ESAVGHRS	0.48	.14	.26	.42	.58	.68	.62	.45	.22	.05	.15	−.30
ESMPLMT	1.41	−.19	−.01	.22	.47	.72	.89	.92	.86	.73	.55	.34
GNP/ESHOURS	0.73	.35	.44	.44	.45	.34	.34	.10	−.09	−.30	−.38	−.42
Average hourly earnings based on establishment survey												
WAGE	0.757	.20	.35	.47	.58	.66	.68	.59	.46	.29	.12	−.03
Average hourly compensation based on national income accounts												
COMP	0.55	.24	.25	.21	.14	.09	.03	−.07	−.09	.09	−.09	−.10

Notes: GNP—real GNP, 1982\$; CONS—personal consumption expenditure, 1982\$; CNDS—consumption of nondurables and services, 1982\$; CD—consumption of durables, 1982\$; INV—gross private domestic investment, 1982\$; INVF—fixed investment, 1982\$; INVN—nonresidential fixed investment, 1982\$; INVR—residential fixed investment, 1982\$; Ch. INV–change in inventories, 1982\$; GOVT—government purchases of goods and services, 1982\$; EXP—exports of goods and services, 1982\$; IMP—imports of goods and services, 1982\$; HSHOURS—total hours of work (Household Survey); HSAVGHRS—average weekly hours of work (Household Survey); HSEMPLMT—employment (Household Survey); ESHOURS—total hours of works (Establishment Survey); ESAVGHRS—average weekly hours of work (Establishment Survey); ESEMPLMT—employment (Establishment Survey); WAGE—average hourly earning, 1982\$ (Establishment Survey); COMP—average total compensation per hour, 1982\$ (National Income Accounts). The Establishment Survey sample is for 1964:I–1991:II.

(negatively correlated). Finally, we measure the cross-correlation over time to indicate whether there is any evidence of phaseshift, i.e., evidence that variables lead or lag one another. Table 1.1 shows the standard deviations and correlations with output of the aggregate time series that characterize the *real* U.S. economy. The model economy articulated above contains no monetary elements. These, along with the monetary business cycle facts are discussed in Chapter 7. Here, we focus only on the real economy. The framework of the neoclassical growth model directs our attention to the behavior of output and the inputs that go to produce it. Moreover, it leads us to examine how much factor inputs fluctuate relative to their shadow prices. Several features of the data are worth noting:

1) The magnitude of fluctuations in output and aggregate hours of work are nearly equal. It is well known that the business cycle is most clearly manifested in the labor market and this observation confirms that.

2) Employment fluctuates almost as much as output and total hours of work, while average weekly hours fluctuate considerably less. This suggests that most fluctuations in total hours represent movements into and out of the work force rather than adjustments in average hours of work.

3) Consumption of nondurables and services (CNDS) is smooth, fluctuating much less than output.

4) Investment in both producers' and consumers' durables fluctuates much more than output.

5) The capital stock fluctuates much less than output and is largely uncorrelated with output.

6) Productivity is slightly procyclical but varies considerably less than output.

7) Wages vary less than productivity.

8) The correlation between average hourly compensation and output is essentially zero.[20]

9) Government expenditures are essentially uncorrelated with output.

10) Imports are more strongly procyclical than exports.

These are the some, but clearly not all, of the salient features of the business cycle based on U.S. time series.

To assess whether the stochastic growth economy is useful as a model of the business cycle, we can compute the same statistics for the data generated by our artificial economy and see how they compare. Of course, there are many features we cannot possibly capture because our model economy does not include those elements. It does not allow for a separate choice of hours and employment, it does not include a government sector, and it has no foreign trade sector. These are all elements that will be added and studied in later chapters.

7. The Findings from the Stochastic Growth Economy

Table 1.2 presents the results of simulating the stochastic growth economy using the parameter values discussed above. The model has been simulated 100 times, with each simulation being 150 periods long, to match the number of observations underlying the statistics reported in Table 1.1. The simulated data were filtered by using the H-P filter just as the original data were to give us the same representation of the business cycle. Table 1.2 presents the standard deviations of the key variables from the model economy (with the standard deviations of these statistics across simulations in parentheses below). In addition we present the cross-correlation of each of the variables with output.

What do we learn from this exercise? One question that is answered is: How much of the variation in output can be accounted for by technology shocks? In this artificial economy, output fluctuates less than in the U.S. economy, suggesting that much, but not all, of the variation in output is accounted for by technology shocks. The labor input in this model economy fluctuates only about half as much as in the U.S. economy suggesting that some important feature of the labor market is not captured here. Hours and productivity in the model economy go up and down together, whereas in the data they do not. This also suggests an important missing element in this artificial economy as a business cycle model.

Investment in the model economy fluctuates much more than does output, just as it does in the U.S. economy; in relative magnitudes, it fluctuates about as much as does Fixed Investment but less than does Gross Private Domestic investment. Consumption in the model economy fluctuates much less than does output and less than consumption of nondurables and services do in the U.S. economy. Consumption, investment, and hours in the model economy are all strongly procyclical, as they are in the U.S. economy. Indeed, one of the striking features of the model economy is that all the variables are highly correlated with output. This is an inevitable consequence of the fact that there is only one shock in this economy—only one source of uncertainty. Many of the explorations in the following chapters will explore the consequences of adding other sources of uncertainty.

Judged on the dimension of the composition of output and its comovements, the match between the model economy and the observed data for the U.S. economy is pretty good but clearly not perfect. It does display a business cycle. Our assessment is that this exercise is a success. The broad features of the model economy suggest that it makes sense to think of fluctuations as caused by shocks to productivity. The failures of the model economy tell us there are important margins along which

Table 1.2
Cyclical Behavior of the Artificial Economy: Deviations from Trend of Key Variables, 150 Observations

		Cross-Correlation of Output with:										
Variable	*SD%*	$x(-5)$	$x(-4)$	$x(-3)$	$x(-2)$	$x(-1)$	x	$x(+1)$	$x(+2)$	$x(+3)$	$x(+4)$	$x(+5)$
Output	1.351 (0.148)	−.049	.071	.232	.441	.698	1.0	.6983	.441	.232	.071	−.049
Consumption	0.329 (0.041)	.232	.340	.460	.592	.725	.843	.502	.229	.022	−.128	−.234
Investment	5.954 (0.646)	−.112	−.007	.171	.389	.664	.992	.713	.470	.270	.115	−.003
Hours	0.769 (0.083)	−.130	.012	.152	.373	.652	.986	.715	.478	.281	.127	.010
Productivity	0.606 (0.068)	.055	.175	.325	.512	.732	.978	.649	.376	.160	−.002	−.122

Note: Values in parentheses are standard deviations across simulations.

decisions are made that have not been captured in this simple world. They also tell us that in comparing these simple artificial economies to the U.S. economy, we must think about how closely the constructs we measure in this model economy compare to those that are measured in the U.S. economy.

This may seem a very informal approach to assessing the performance of our model economy. Some authors have sought ways to summarize more concretely how well a business cycle model captures the features of the U.S. economy. This involves imposing a metric to measure the distance between summary statistics for the model economy and corresponding statistics for the actual economy. One such procedure is described in Christiano and Eichenbaum (1992). They describe a generalized method of moments interpretation of the calibration exercise that provides them with a metric for assessing the difference between their model and the data. An alternative procedure that aims to come up with something like an R_2 measure for a model economy is described in Watson (1993). Some other techniques are described in Kim and Pagan (1993). Developing formal measures of model fit for general equilibrium model economies is a difficult problem with many dimensions to it. Creators of traditional econometric models struggled for many years to develop formal summary measures of fit for simultaneous equations systems, without a lot of success. In the chapters that follow you will find a variety of ways in which the performance of a model economy is assessed.

8. Conclusion and Summary

This model economy gives us only a glimpse of what it is possible to learn and accomplish by taking the basic neoclassical growth model seriously as a description of how actual economies behave. In the following chapters, we show how this basic framework can be implemented and employed to address many of the substantive questions confronting macroeconomics and business cycle theory today. This book is not intended as a comprehensive survey of the field. The coverage of topics and the treatment of individual contributions is necessarily incomplete because the research program is so active.

The next three chapters are concerned primarily with methodology. In Chapter 2, Gary Hansen and Edward Prescott explain recursive methods for computing the equilibria of business cycle models. They illustrate these methods with a number of examples, which cover many of the applications described later in the book. In Chapter 3, Jean-Pierre Danthine and John Donaldson describe a variety of methods that take a different approach to computing the equilibria of dynamic economies. The methods they describe are widely used for problems where the equilibria are suboptimal and where the dimensionality of the state space would be very large if one used the methods described in Chapter 2. In Chapter 4, José-Víctor Ríos-Rull explains how to structure economies with heterogeneous agents and how to compute their equilibria. He also describes how to compute the equilibria of overlapping-generations economies.

For the most part the remaining chapters deal with applications to substantive issues. Chapter 5, by Finn Kydland, discusses the labor market. As we showed in this chapter, much of the fluctuations at business cycle frequencies are characterized by changes in the labor input. That being the case, it is apparent that an understanding of how that market functions is central to understanding the cycle. In Chapter 6, Jeremy Greenwood, Richard Rogerson, and Randall Wright describe how decisions made by the household sector about the allocation of labor and capital between the household sector and the market sector affects what we observe about fluctuations in the labor market.

Beginning with the important work of Robert Lucas, much of the modern general equilibrium analysis of the business cycle has treated the business cycle as a monetary phenomenon. Real business cycle models like the one described above have been a significant departure from that tradition because they assign a very small role to money. Chapter 7, by Thomas Cooley and Gary Hansen, examines the role of money in business cycle fluctuations. First, using the basic real business cycle model as a vehicle, they examine the quantitative importance of the kinds of information problems (implicitly caused by monetary fluctuations) that Lucas postulated as the source of business cycle fluctuations. Then they look at the potential for rigid wages, resulting from contracting behavior, to act as a propagation mechanism for monetary shocks.

Chapter 8, by Jean-Pierre Danthine and John Donaldson, is an analysis of various types of non-Walrasian economies based on the neoclassical growth model. Danthine and Donaldson construct models designed to assess the quantitative importance of such phenomena as efficiency wages and labor hoarding. More important, they show how the basic framework can be modified to incorporate different arrangements in the economy and provide a quantitative assessment of their importance. Chapter 9, by Julio Rotemberg and Michael Woodford, considers the relation between business cycle fluctuations and noncompetitive elements in the economy. Their findings show that imperfect competition can be important because it significantly affects the way the economy responds to different shocks.

Chapter 10, by K. Geert Rouwenhorst, is devoted to asset-pricing issues. Rouwenhorst shows how the basic real business cycle model can be used to study various contemporary issues in finance. In Chapter 11, David Backus, Patrick Kehoe, and Finn Kydland describe some of the puzzles that arise when we consider the international features of business cycles. They describe a two-country version of the basic real business cycle model that can used to study international business cycles and discuss its ability to explain the international puzzles. Chapter 12, by V. V. Chari, Lawrence Christiano, and Patrick Kehoe shows how the kinds of model developed in this book can be modified to analyze economic policy issues in a rigorous way.

Notes

We thank Larry Christiano, John Donaldson, Ed Green, Jeremy Greenwood, Gary Hansen, Finn Kydland, Glenn MacDonald, and Kevin Salyer for advice and comments and the National Science Foundation for research support.

1. An important exception to this general lack of concern with the business cycle was the famous paper by Adelman and Adelman (1959), who showed that a small econometric model developed by Lawrence Klein displayed business cycle fluctuations when subjected to random shocks. This was consistent with the Slutsky view of how business cycles might get started.

2. To see this, let Y, K, and R_k denote output, capital, and the income from capital, respectively, all of which are functions of time. The third stylized fact implies that $K'/K = Y'/Y$. We could write this equivalently as $[\ln(K)]' = [\ln(Y)]'$, so the fundamental theorem of calculus implies that $\ln(K) = \alpha \ln(Y)$, where α is a time-invariant constant. Equivalently, $K = \exp(\alpha)Y$ so that the ratio K/Y remains constant over time. The fourth stylized fact asserts that R_k/K is constant over time, which in conjunction with the constancy of K/Y implies that capitals' share of income is constant over time.

3. The view that growth and cycles might be integrated phenomena is not unique to this literature. There were earlier theoretical models by Hicks (1949), Goodwin (1955), and Smithies (1957), among others, that treated these together. There were also efforts in the econometric literature of the 1960s to develop economywide econometric models that were consistent with the growth facts.

4. Sometimes there is a "fake" social planner's problem, which is concave and whose first-order conditions coincide with the first-order conditions of the competitive equilibrium. Assuming such a function can be identified, these methods still apply even though there are distortions. An example of such a model is described in Chapter 3.

5. Other decentralized equilibrium concepts have appeared in the literature. Brock (1982) proposed a decentralized equilibrium concept that is very similar to the RCE except that the firm's technology need not be constant returns to scale and the capital stock sequence can be unbounded.

6. One set that is always sufficient are the Arrow securities.

7. Solow also argued that productivity improvements, *not* increasing returns, were the likely explanation for this finding.

8. Stokey and Lucas with Prescott (1989) develop the stochastic growth model under much more general mathematical assumptions about the nature of the shocks. We make these very specific assumptions to facilitate the exposition here.

9. The general specifications of preferences that are consistent with balanced growth are described in King, Plosser, and Rebelo (1988b) and by Caballe and Santos (1993).

10. The most recent measurement of the U.S. fixed reproducible capital stock is reported by Musgrave (1992) in *The Survey of Current Business*. Musgrave distinguishes three major components of the reproducible capital stock: Fixed Private Capital, which includes producers' durables, producers' structures, and residential structures; Government Capital, which includes both equipment and structures at both the federal level and the state & local government level; and Consumer Durables. Another component of the capital stock that we will incorporate is land. A measure of the value of land is reported in the Flow of Funds Accounts, in the Balance Sheets for the U.S. Economy.

11. The complete National Income and Product Accounts for the United States are published in selected issues of *Survey of Current Business*, published monthly by the U.S. Department of Commerce. The complete historical accounts from 1929 through 1988 are contained in the two volumes of *National Income and Product Accounts of the United States*, published by the U.S. Department of Commerce.

12. The value of land reported in the Flow of Funds accounts is an estimate based on current market prices. The measured value of land as a fraction of GNP is quite volatile over time, which suggests that this part of aggregate capital may be poorly measured.

13. Since there is no foreign sector in this economy, net exports are viewed as representing additions to or claims on the domestic capital stock, depending on whether they are positive or negative.

14. These estimates account for the growth in real output and population through the term Y_{t+1}/Y_t.

15. Kydland and Prescott (1982), Prescott (1986), and Hansen (1985) all use a smaller value of θ, .36, because they do not impute the output of government capital.

16. Prescott (1986) argues that the standard deviations of these innovations may be affected by measurement error in the measured labor input. Fortunately, there are two independent estimates of this input, one based on the establishment survey and one based on the household survey. Under the assumption that the measurement errors in these two series are orthogonal, the covariance between the two series is an estimate of the variance of the change in hours. Taking account of this measurement error would actually increase very slightly the standard deviation of the innovations to technology. We chose to ignore it here.

17. The taxonomy in section 5 was suggested by Larry Christiano.

18. Chapters 2 and 4 show how to generalize this to more than one type of agent.

19. One can show that the H-P filter is an optimal signal extractor for y_t^* in the model

$$y_t = y_t^* + \nu_t(1 - L)^2 y_t^* = \eta_t,$$

where $\lambda = \mathrm{var}(\eta)/\,\mathrm{var}(\eta)$. This is optimal for a process that is integrated of order 2. Since most macroeconomic time series are not $I(2)$, this has led some researchers to question the use of the H-P filter and advocate a filter that assumes the series are $I(1)$. See King and Rebelo (1993) and Kim and Pagan (1993).

19. The absence of correlation between average hourly compensation and output is related to the lack of correlation between capital stock and output above and to the observation that the correlation between productivity and hours is very small, being slightly positive in the establishment survey data and slightly negative in the household survey data. The latter correlation is not reported in Table 1.1, but it has been stressed by Christiano and Eichenbaum (1992) as an important puzzle. The observation is at odds with the Keynesian view that observed labor market fluctuations take place along a labor demand curve, and it is also at odds with the classical view that they take place along a labor supply curve. For a thorough discussion of this, see Gomme and Greenwood (1993).