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STOCHASTIC MONOTONICITY AND STATIONARY DISTRIBUTIONS FOR DYNAMIC ECONOMIES

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The existence and stability of invariant distributions for stochastically monotone processes is studied. The Knaster-Tarski fixed point theorem is applied to establish existence of fixed points of mappings on compact sets of measures that are increasing with respect to a stochastic ordering. Global convergence of a monotone Markov process to its unique invariant distribution is established under an easily verified assumption. Topkis' theory of supermodular functions is applied to stochastic dynamic optimization, providing conditions under which optimal stationary decisions are monotone functions of the state and induce a monotone Markov process. Applications of these results to investment theory, stochastic growth, and industry equilibrium dynamics are given.

KEYWORDS: Stationary distributions, fixed points, monotone functions, stochastic dynamic programming, stochastic growth theory, investment theory.

1. INTRODUCTION

A PROBLEM THAT is arising with increasing frequency in dynamic economic analyses is the study of time invariant distributions. These arise in at least two classes of problems. The first is when the object of the research is an equilibrium distribution of agents indexed by some economic characteristics such as income, asset holdings, information or beliefs, employment status, or the capital stocks of firms.[2] The second is when the object of the research is the long run behavior of a stationary stochastic process, as occurs in capital theory for the process induced by the optimal accumulation policy. Invariant distributions for such processes provide information on their long run behavior. In particular, the problem of uniqueness of an invariant distribution is closely related to the independence of this behavior from the initial data.

Existence arguments based on continuity conditions have been well studied.[3] Recently, interesting economic models have been developed where nonconvexities or switching costs give rise to discontinuous stochastic behavior,[4] for which those arguments are not applicable. In some of these cases, however, the existence of stationary equilibria can still be established using different methods based on stochastic monotonicity conditions. We systematically develop this

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[2]Some examples of this category are Lucas and Prescott (1974), Lucas (1978), Jovanovic (1982), Marimon (1989), and Manuelli (1985).

[3]For a general formulation see Duffie, Genakoplos, Mas Colell, and McLennan (1989). Green and Majumdar (1975) indicated the possibility of looking at an invariant distribution as an equilibrium. Futia (1982) provides an early comprehensive analysis of stationary equilibria for stochastic processes using fixed point theory in the economic literature.

[4]Examples of these are growth models with nonconvex production functions, as in Majumdar, Mitra, and Nyarko (1989) and models with switching costs, such as in Dixit (1989). Discontinuities can also arise in optimal contracts from the nonconvexity of incentive constraints.

fixed point theory based on some recent results in probability. Equilibrium existence arguments based on monotonicity have been developed in other areas of economics (see Vives (1990)). The emphasis of this paper is on stochastic dynamics.

Stochastic monotonicity arises in economic models from the monotonicity of decision rules or equilibrium mappings that result from the optimizing behavior of agents. The question of when does optimization lead to monotone rules becomes thus relevant. Topkis (1978) developed the necessary mathematical structure. His work has recently seen much applicability in economic problems.[5] We apply this structure to dynamic stochastic theory and provide general conditions under which optimal stationary policies for dynamic stochastic problems will be monotone.

Beyond the issue of existence of stationary distributions is the question of whether the sequence of predictive probability distributions of future states has a limit and whether this limit is independent of the initial data. This has been the motivation of turnpike theory in stochastic growth models. The methods currently used have not proven easy, often requiring considerable investment in specialized mathematics or considerable ingenuity in verifying conditions of the available theorems. We provide a simple and easily verified condition for the global stability of monotone stochastic processes.[6]

Section 2 develops the basic framework and presents the existence theorem. This argument applies more generally to the existence of fixed points for monotone mappings of a compact set of measures into itself. The mapping need not be linear and norm preserving as is the case for the Markov process. Section 3 specializes these results to the case of a Markov process. Section 4 provides conditions to obtain monotonic decision rules in dynamic stochastic problems. Section 5 presents the uniqueness (and global stability) condition for monotone Markov processes. Finally, in Section 6 we discuss several economic applications of the theorems.

2. A FIXED POINT RESULT

In this section we present a fixed point result and some useful corollaries. Loosely speaking, we will show that monotone maps defined on compact sets of measures have fixed points. The following definitions will allow us to make the above statement precise.

Preliminaries. Let $(S, \geqslant)$ be a compact metric space ordered with a reflexive, transitive, antisymmetric and closed relation $\geqslant$. (The order $\geqslant$ is *closed* if the graph of $\geqslant$ is a closed subset of $S \times S$.) An *upper* (*lower*) *bound* for $M \subset S$ is an element $s \in S$ with $s' \leqslant s$ $(s \geqslant s')$ for all $s' \in M$. The *supremum* of M, if it exists, is an upper bound for M which is a lower bound for the set of all upper

[5] For an extensive list of applications, see Milgrom and Shannon (1991). Vives (1990) presents an interesting application to Bayesian games.

[6] A result very close to ours appeared in Battacharya and Lee (1988), which extends a result of Dubins and Freedman (1966) to $\mathbb{R}^n$.

bounds of M. A *chain* C on S is a subset of S for which all pairs of elements are comparable; i.e., for all s and s' in C, $s \geqslant s'$ or $s' \geqslant s$.

For $A \subset S$, let $d(A)$ be the set of all elements in S that are smaller than some element in A, i.e. $d(A) = \{s' \in S\colon s' \leqslant s \text{ for } s \in A\}$ and let $i(A)$ denote the set of all elements of S that are larger than some element in A. We will say that A is a *decreasing* [*increasing*] *set* if $A = d(A)$ [$A = i(A)$]. The set A is said to be *monotone* if it is either decreasing or increasing. A mapping f from an ordered space $(S, \geqslant)$ to an ordered space $(T, \succcurlyeq)$ is said to be an *increasing function* if for any two elements s, s' in S, $s \geqslant s'$ implies $f(s) \succcurlyeq f(s')$.

The measure space considered in this paper will be $(S, \mathscr{S})$, where $\mathscr{S}$ is the Borel σ-algebra of subsets of S. Let $\mathscr{M}(S)$ be the space of finite measures on $(S, \mathscr{S})$ endowed with the weak$*$topology.

Stochastic order. For any pair of elements μ and μ' in $\mathscr{M}(S)$, we will say that $\mu \succcurlyeq \mu'$ if $\int f\mu(ds) \geqslant \int f\mu'(ds)$ for every increasing, measurable, and bounded function $f\colon (S, \geqslant) \to \mathbb{R}_+$. Whenever $\mu \succcurlyeq \mu'$ we will say that μ *stochastically dominates* or is *stochastically greater* than μ'.[7]

Note that when μ and μ' are probability measures the order considered coincides with the familiar notion of stochastic dominance used in economics and finance. In particular, when S is a subset of the real line it is simple to show that $\mu \succcurlyeq \mu'$ if and only if $F(s) \leqslant G(s)$ for every $s \in S$, where F and G are the distribution functions of μ and μ', respectively. (If S is a subset of $\mathbb{R}^n$ where $n > 1$, however, $G(s) \leqslant F(s)$ does not imply $\mu \succcurlyeq \mu'$. Only the converse is true.)

Kamae, Krengel, and O'Brien (1977, 1978) establish that when S is a Polish (complete, separable, and metric) space, $\succcurlyeq$ is a closed order on $\mathscr{M}(S)$ and that $\mu \succcurlyeq \nu$ if and only if $\mu(A) \geqslant \nu(A)$ for every increasing $A \subset S$. Whitt (1980) established that if in addition $(S, \geqslant)$ is normally ordered (see Nachbin (1965) for definitions) the functions used to test for stochastic ordering can be restricted to be continuous.[8] Since compact metric ordered spaces are normally ordered, we will make use of these results. Denote by $\boldsymbol{M}$ the set of nonnegative, increasing, and continuous real valued functions on S.

For the fixed point theorems we will restrict the set of measures to a compact subset of $\mathscr{M}(S)$ which we will denote by Λ. Some examples of compact subsets of $\mathscr{M}(S)$ encountered in economic problems are the following:

(a) The space of uniformly bounded measures, i.e. $\{\mu \in \mathscr{M}(S)$ such that $\mu(S) \leqslant m\}$ where m is an upper bound fixed for all μ.[9]

(b) Any closed subset of the above, e.g. the space of probability measures.

[7]This ordering does not give $\mathscr{M}(S)$ a lattice structure. For an example see Kamae, Krengel, and O'Brien (1977).

[8]Torres (1988) extends these results to a preordered space and also develops a condition for compactness of ordered intervals of measures.

[9]If S is compact, by the Riesz representations theorem (see Dunford and Schwartz (1958, IV.6.3) the dual space of the set of continuous and bounded functions on S, $C(S)$, can be identified with the space of measures on $(S, \mathscr{S})$. The space of all measures uniformly bounded by m is closed in the weak* topology and bounded in the strong topology. By the Banach-Alaoglu theoem (see Corollary V.4.3 in Dunford and Schwartz) this set is compact.

(c) Intervals of measures in the stochastic order; i.e., given two measures $\mu_a \succcurlyeq \mu_b$, the set $[\mu_a, \mu_b] = \{\mu \in \mathscr{M}(S)\colon \mu_a \succcurlyeq \mu \succcurlyeq \mu_b\}$, so it is a closed subset of a uniformly bounded set of measures.

(d) Given a continuous function $p\colon S \to R$ and two real numbers a and b, the set of measures that satisfy $a \leqslant \int p(s)\mu(ds) \leqslant b$ is closed. Furthermore, if $p(s) \geqslant \varepsilon > 0$ then $\mu(S) \leqslant m = b/\varepsilon$, and the above set is compact.

Our main theorem establishes the existence of a fixed point for an increasing mapping from a compact subset of $\mathscr{M}(S)$ into itself. The proof is based on the following fixed point theorem (see Dugundji and Granas (1982)).

THEOREM (Knaster-Tarski): *Let $(P, \geqslant)$ be an ordered space and $F\colon P \to P$ an increasing function. Assume there exists a point $b \in P$ such that $b \leqslant F(b)$ and every chain in $i(\{b\})$ has a supremum. Then the set of fixed points of F is not empty.*

To apply this theorem, we just need to establish that every chain in a compact subset of $\mathscr{M}(S)$ has a supremum. A slightly stronger result, also used in Section 5, is now proved.

PROPOSITION 1: *Any chain C in a compact subset Λ of $\mathscr{M}(S)$ has a supremum. Furthermore, the chain converges to the supremum.*

PROOF: Since Λ is compact and C is a net in Λ directed by itself, there exists a subnet C' of C that converges to some element μ^* in Λ. We will now show that μ^* is the supremum of C. To show that μ^* is an upper bound for C, for any $\nu \in C$ let $C'' = \{\mu \in C'\colon \mu \succcurlyeq \nu\}$. C'' is a subnet of C' so it also converges to μ^*. Since $\succcurlyeq$ is a closed order we can conclude that $\mu^* \succcurlyeq \nu$. We will now show that μ^* is the least upper bound. Suppose to the contrary that ν^* is another upper bound and that there exists some $f \in M$ with $\int f\nu^*(ds) < \int f\mu^*(ds)$. Since C' converges to μ^*, there exists some measure $\mu \in C'$ with $\int f\nu^*(ds) < \int f\mu(ds)$. This contradicts ν^* being an upper bound for C. Finally, note that this argument implies that any subnet of C converges to μ^*, so μ^* is the limit of C. *Q.E.D.*

We now present the main result of this section.

THEOREM 1: *Let Λ be a compact subset of $\mathscr{M}(S)$ and $T\colon \Lambda \to \Lambda$ an increasing map. Then T has a fixed point if and only if there exists a measure μ_a in Λ such that $T\mu_a \succcurlyeq \mu_a$.*

PROOF: Since $\succcurlyeq$ is reflexive, necessity is immediate. Sufficiency follows from Proposition 1 and by applying the Theorem of Knaster-Tarski. *Q.E.D.*

EXAMPLE: The following example illustrates that no assumption is redundant.

Fix $\lambda \in (0,1)$. Let g: $[0,1] \to [0,1]$ be given by

$$g(s) = \begin{cases} \lambda x + (1-\lambda) & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

For a measure μ on $[0,1]$ define the mapping T by

$$T\mu(A) = \mu(g^{-1}(A)).$$

It is simple to check that T has no invariant distribution. What assumptions of the theorem are violated?

(a) If we consider the natural order on $[0,1]$, all assumptions are satisfied except monotonicity of T.

(b) If we consider the reflexive order in $[0,1]$, i.e. $x \geqslant y$ iff $x = y$, then all assumptions are satisfied except that no measure is increased.

(c) If we consider the natural order with the only exception that $\{1\}$ is only related to itself, then the only hypothesis that fails is continuity of the order.

(d) If we consider the natural order and let $0 \geqslant 1$, then either the order is not asymmetric or it is not transitive.

A natural question that arises is what happens if T^n (the composition of T n times with itself) rather than T, satisfies the hypotheses of Theorem 1. This occurs, for instance, if T is decreasing and thus T^2 increasing. By Theorem 1, T^n has a fixed point. If in addition T is linear, it will also have a fixed point.

Corollary 1: *If T: $\Lambda \to \Lambda$ is a linear mapping, T^n is increasing, and $T^n\mu_a \succcurlyeq \mu_a$ for some measure μ_a in Λ, then T has a fixed point.*

Proof: Since T^n satisfies the assumptions of Theorem 1, it has a fixed point, say μ_0. Let $\mu_k = T^k\mu_0$; then $T\mu_k = \mu_{k+1}$ for $k = 0, 1, \dots n-1$. Let $\mu = (1/n)\Sigma_{k=0}^{n-1}\mu_k$. Then $T\mu = (1/n)\Sigma_{k=0}^{n-1}T\mu_k = (1/n)\Sigma_{k=1}^{n}\mu_k = (1/n)\Sigma_{k=0}^{n-1}\mu_k$ since $\mu_n = \mu_0$. Thus $T\mu = \mu$ and the proof is complete. *Q.E.D.*

Let $\mathscr{P}(S)$ be the set of probability measures on S. This will be the space of measures considered in the applications to Markov processes considered in the next section. The following Corollary will prove very useful.

Corollary 2: *If T: $\mathscr{P}(S) \to \mathscr{P}(S)$ is increasing in S and has a minimum element (i.e. there exists an $a \in S$ such that $s \geqslant a$ for all $s \in S$), then T has a fixed point.*

Proof: Let δ_a be the measure that assigns probability one to the point set $\{a\}$. Then for all $\mu \in \mathscr{P}(S)$, $\mu \succcurlyeq \delta_a$. Hence for any increasing mapping T: $\mathscr{P}(S) \to \mathscr{P}(S)$ it is the case that $T\delta_a \succcurlyeq \delta_a$. Thus in this case any increasing map has a fixed point. *Q.E.D.*

It is of interest in many economic applications to analyze how changes in some underlying parameters of the economy result in changes in the set of invariant distributions corresponding to these economies. For this purpose, Corollary 3 provides a useful result which, loosely speaking, establishes that if the mappings T and T' on Λ are ordered, the set of invariant measures corresponding to them will also be ordered. To make this statement precise, we will say that T' *dominates* T if for all $\mu \in \Lambda$, $T'\mu \succcurlyeq T\mu$.

COROLLARY 3: *If T' and T are two mappings on Λ that satisfy all the assumptions of Theorem* 1 *and T' dominates T, then for every fixed point μ of T (μ' of T') there exists a fixed point μ' of T' (μ of T) such that $\mu' \succcurlyeq \mu$.*

PROOF: Let μ be a fixed point for T. Let $\Lambda' = \{\mu' \in \Lambda\colon \mu' \succcurlyeq \mu\}$. This is a closed subset of Λ and hence it is compact. For any $\mu' \in \Lambda'$, $T'\mu' \succcurlyeq T'\mu \succcurlyeq \mu$ and hence $T'\colon \Lambda' \to \Lambda'$. By Theorem 1, T' has a fixed point in Λ'. For any fixed point μ' of T' the existence of a fixed point μ of T with $\mu \preccurlyeq \mu'$ can be established in the same way. *Q.E.D.*

This result is particularly useful when the fixed points of T and T' are unique.

3. APPLICATIONS TO MARKOV PROCESSES

In many economic applications, it is of interest to know if the variables that describe the state of the economy at each point in time (state vector) have an invariant distribution, when the state vector follows a stationary Markov process.

In the next corollary we present conditions on the transition function for the Markov process that guarantee the existence of a stationary distribution. Before that we need to define the mapping $T\colon \mathscr{P}(S) \to \mathscr{P}(S)$ induced by the Markov process.

Let $P\colon S \times \mathscr{S} \to [0,1]$ be a transition function describing the Markov process. We will say that P is *increasing* if P is increasing in its first argument in the stochastic order sense, i.e. for s and s' in S, $s \geqslant s'$ implies $P(s,\cdot) \succcurlyeq P(s',\cdot)$. The transition function P induces a mapping $T\colon \mathscr{P}(S) \to \mathscr{P}(S)$ defined by

$$T\mu(A) = \int P(s,A)\mu(ds).$$

COROLLARY 4: *If S is a compact metric space with a minimum element and $P\colon S \times \mathscr{S} \to [0,1]$ is an increasing transition function, then the Markov process corresponding to P has a stationary distribution; i.e., there exists a fixed point for the mapping T induced by the process.*

PROOF: By Corollary 2, it suffices to show that T is increasing. For this purpose, let μ and μ' be in $\mathscr{P}(S)$ and suppose $\mu' \succcurlyeq \mu$. Let f be any increasing, nonnegative, bounded, and measurable function. Since increasing indicator functions are dense in the increasing functions of $L^1(T\mu)$ and $L^1(T\mu')$, we may assume without loss of generality that f is the indicator function of an increasing set A. For $s' \geqslant s$, since P is increasing $P(s', A) \geqslant P(s, A)$. Hence $P(\cdot, A)$ is an increasing, nonnegative, bounded, and measurable function too. So if $\mu' \succcurlyeq \mu$,

$$\int f T\mu'(ds) = T\mu'(A) = \int P(s, A)\mu'(ds)$$

$$\geqslant \int P(s, A)\mu(ds) = T\mu(A) = \int f T\mu(ds).$$

Hence $T\mu' \succcurlyeq T\mu$ so T is increasing and the proof is complete. *Q.E.D.*

In many economic applications the Markov process that the state vector follows is generated in the following manner:

Given a state space X and a random variable z defined on a measure space $(Z, \mathcal{Z})$ with transition function Q the evolution of the state is described by a mapping g: $X \times Z \to X$ with the following interpretation: If at time t the state vector is x_t and the realization of z is z_t, then $x_{t+1} = g(x_t, z_t)$. Let $S = X \times Z$ and let $\mathcal{S}$ be the product σ-algebra. This structure induces a mapping P: $S \times \mathcal{S} \to [0, 1]$ defined by

$$(1) \qquad P(x, z; A \times B) = \begin{cases} Q(z, B) & \text{if } g(x, z) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

If g is measurable, P will be a well defined transition function for a Markov process (see Stokey, Lucas, and Prescott (1989, Theorem 9.13)). The following Corollary can be easily proved.

COROLLARY 5: *Suppose the function g in* (1) *is jointly measurable and increasing, Q is increasing, and X and Z are compact metric spaces endowed with closed orders and with minimum elements. Then the Markov process defined in* (1) *has a stationary distribution.*

4. MONOTONE POLICY FUNCTIONS

The structure presented at the end of Section 3 often arises from the stationary solution to an optimal stochastic control problem. Following Stokey, Lucas, and Prescott (1989) the control problem can be formulated in the following way:

$(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ are measurable spaces, where X is the set of possible values of the endogenous state variable and Z is the set of possible values for a Markov process with transition function Q. Each period the decision maker chooses the value of the endogenous variable x' for the following period from the set $\Gamma(x, z)$, where Γ: $X \times Z \to X$ is a correspondence representing the set of feasible choices. Let $A \subset X \times X \times Z$ be the graph of Γ and F: $A \to \mathbb{R}$ the one-period return function, which is assumed to be bounded. Returns are discounted at a constant rate $\beta \in (0, 1)$. Under fairly standard conditions the principle of optimality holds and there exists an optimal stationary policy g: $X \times Z \to X$ such that g is measurable on $(S, \mathscr{S}) = (X, \mathscr{X}) \times (Z, \mathscr{Z})$, $g(x,z) \in \Gamma(x,z)$ a.s. and maximizes the expected discounted returns of the control problem. Let v: $S \to \mathbb{R}$ be the unique value function associated to this problem. The function g is a measurable selection from

$$\gamma(x,z) = \left\{ y \in \Gamma(x,z) \colon v(x,z) = F(x,y,z) + \beta \int v(y,z') Q(z,dz') \right\}.$$

Under what conditions will g satisfy the assumptions of Corollary 5? What structure on the control problem will lead to the existence of monotone and measurable selections? The basis for this analysis is Topkis' (1978) theory of supermodularity, which provides sufficient conditions for optimizing behavior to be monotone in decision parameters.[10]

The following definitions—most from Topkis (1978)—will be used. Let S be a lattice and T a partially ordered set. The function f: $S \times T \to \mathbb{R}$ is *(strictly)supermodular* in S if for each $t \in T$, $f(x \vee y, t) - f(x,t) \geqslant (>) f(y,t) - f(x \wedge y, t)$ for every x and y in S. This function is said to have *(strictly) increasing differences* if $f(x,t) - f(y,t)$ is strictly increasing in t whenever $x \geq y$ for all x and y in S. A function f that is supermodular and has increasing differences is said to satisfy the *cardinal complementarity conditions*; for short we will say this function is of class $\underline{cc}$. A correspondence Γ: $X \to 2^Y$ from a partially ordered set S to a lattice Y is said to be *ascending* if for any x_1 and x_2 in X such that $x_2 \geqslant x_1$, $y_2 \in \Gamma(x_2)$ and $y_1 \in \Gamma(x_1)$ implies $y_1 \vee y_2 \in \Gamma(x_2)$ and $y_1 \wedge y_2 \in \Gamma(x_1)$. When the domain of the correspondence is a product space $X \times Z$, we will say that Γ has *strict complementarity* if for any two elements x_1 and x_2 in X such that $x_2 \geq x_1$ and any two elements z_1 and z_2 in Z with $z_2 \geqslant z_1$, $s \in \Gamma(x_1, z_2)$ and $t \in \Gamma(x_2, z_1)$ imply that $s \wedge t \in \Gamma(x_1, z_1)$ and $s \vee t \in \Gamma(x_2, z_2)$. Note that if the graph of a correspondence is a sublattice (in the product order), it will be ascending and have strict complementarity.

The following results from Topkis (1978) will be used:

(a) *If f is supermodular on a lattice S, then the set S^* of points at which f attains its maximum on S is a sublattice of S.* (Theorem 4.1).

(b) *If S is a lattice, T a partially ordered set, Γ: $T \to 2^S$ an ascending correspondence, $f(x,t)$ supermodular in x on S for each $t \in T$, and $f(x,t)$ has increasing differences in (x,t) on $S \times T$, then the correspondence $\gamma(t)$ giving the maximizers of f at t, is ascending* (Theorem 6.1).

For the selection arguments a measurability condition will be required on γ, which is immediately satisfied when γ is upper hemicontinuous. The correspondence γ: $X \to 2^Y$ will be said to be *upper measurable* if for any closed set $F \subset Y$, the set $\{x \in X\colon \gamma(x) \cap F \neq \phi\}$ is measurable. If Y is a separable metric space

[10] Milgrom and Shannon (1991) extend this analysis by developing ordinal conditions which are necessary and sufficient for monotonicity. Lovejoy (1987) develops some comparison theorems for dynamic programming problems.

and γ compact valued (as below), this is equivalent to measurability of γ as a function from X to the set of closed subsets of Y endowed with the Hausdorff topology (Debreu (1967)).

Suppose X is a lattice and Z a partially ordered set. Note that the space $S = X \times X$ is also a lattice with the induced componentwise ordering. The following proposition gives conditions so that there exists a measurable selection g that is monotone increasing.

PROPOSITION 2: *Assume F is supermodular as a function of (x, x') for each $z \in Z$ and has increasing differences, that Q is an increasing transition function, Γ has strict complementarity, and for all $z \in Z$ the graph of $\Gamma(\cdot; z)$ is a sublattice. Then v is supermodular in x and has increasing differences and $\gamma(x, z)$ is a sublattice of X for all $x \in X$ and $z \in Z$. If in addition Γ is ascending, γ will be ascending too. Furthermore, if γ is nonempty, compact valued, and upper measurable, and X is a complete separable metric space with a continuous lattice structure,*[11] *the functions $\bar{g}(x, z) = \sup \gamma(x, z)$ and $\underline{g}(x, z) = \inf \gamma(x, z)$ will be monotone increasing and Borel measurable.*

PROOF: We first establish that the functional equation defined by

$$Tv(x, z) = \max_{x' \in \Gamma(x, z)} F(x, x', z) + \int v(x', z') Q(z, dz')$$

maps the class of cc functions into itself. Since the class of cc functions is closed under pointwise convergence, the solution to the Bellman equation will be in this class. Assume v is cc and let $H(x, x', z) = F(x, x', z) + \int v(x', z') Q(z, dz')$. Since supermodular functions are closed under addition and preserved by integration, it follows immediately that H is supermodular. For x_1' and x_2' in $\Gamma(x, z)$, $x_2' \geq x_1'$ implies that $v(x_2', z') - v(x_1', z')$ is increasing in z' and bounded. Since Q is an increasing transition, it follows immediately that

$$\int v(x_2', z') Q(z, dz') - \int v(x_1', z') Q(z, dz')$$

$$= \int (v(x_2', z') - v(x_1', z')) Q(z, dz')$$

is increasing in z. As a sum of functions with increasing differences H will also have increasing differences and hence will be cc. By Lemma 1 in the Appendix Tv is supermodular in x and has increasing differences.

Since $\Gamma(x, z)$ is a sublattice for all $x \in X$ and $z \in Z$ and H is supermodular in x' for fixed (x, z), Theorem 4.1 in Topkis (1978)—stated above—implies that $\gamma(x, z)$ is a sublattice of X. If Γ is ascending, Theorem 6.1 in Topkis

[11]The space S has a continuous lattice structure if the functions from $S \times S \to S$ taking $(s, t) \to s \vee t$ and $(s, t) \to s \wedge t$ are continuous.

(1978)—also stated above—implies γ is ascending. The last results follows immediately from Lemma 2 in the Appendix. *Q.E.D.*

COROLLARY 6: *Under the assumptions of Proposition 2 and if the sets X and Z are compact, their respective orders closed and have minimum elements, there exists a stationary distribution for the induced Markov process.*

A natural question that arises in this context is how the optimal policy functions are affected by changes in the exogenous process for z. The following corollary provides a useful result.

COROLLARY 7: *Let Q_a and Q_b be two transition functions on Z such that for all $z \in Z, Q_b(z,\cdot) \succcurlyeq Q_a(z,\cdot)$. Let g_a and g_b be optimal policy functions for the control problems with transitions Q_a and Q_b, respectively. Then $\bar{g}_b \geqslant g_a$ and $\underline{g}_a \leqslant g_b$, where the functions $\bar{g}$ and $\underline{g}$ denote the two selections indicated in Proposition 2.*

PROOF: Let $\tilde{Z} = Z \times \{a, b\}$ and define a transition $\tilde{Q}$ on $\tilde{Z}$ by letting $\tilde{Q}(z, a, \cdot)$ be the product measure $Q_a(z,\cdot) \times \delta_a$ and $\tilde{Q}(z, b, \cdot)$ the product $Q_b(z,\cdot) \times \delta_b$, where δ_s is the dirac measure on $\{s\}$. The set $\{a, b\}$ is a lattice with the order $\{b \geqslant a, a \geqslant a, b \geqslant b\}$ and $\tilde{Z}$ a lattice with the product order. It is easy to verify that with these orderings $F, \tilde{Q}, X, \tilde{Z}$ satisfying the assumptions of Proposition 2. Since $\bar{g}_b(x, z) = \sup \Gamma(x, z, b)$ and Γ is ascending, $\bar{g}_b(x, z) \vee g_a(x, z) \in \Gamma(x, z, b)$, proving that $\bar{g}_b(x, z) \geq g_a(x, z)$. The second part of the Corollary follows from a similar argument. *Q.E.D.*

REMARK: If there is a unique measurable selection, then letting T_a and T_b be the Markov operators associated to g_a and g_b, respectively, it follows that T_b dominates T_a so by Corollary 3 the invariant distributions for these processes are likewise ordered.

Proposition 2 gives conditions under which there exists a monotone measurable selection. A natural question is whether by strengthening some of the conditions one can establish that *all* measurable selections will be monotone. This is done in the following Proposition.[12]

PROPOSITION 3: *Assume F is strictly supermodular as a function of (x, x') for each $z \in Z$ and has strictly increasing differences, that Q is an increasing transition function, Γ has strict complementarity, and for all $z \in Z$ the graph of $\Gamma(\cdot; z)$ is a sublattice. Then any measurable selection $g: (x, z) \to x'$ is nondecreasing in (x, z).*

[12] The proof follows analogous arguments to that of Proposition 1 and is thus omitted.

5. CONVERGENCE TO THE UNIQUE INVARIANT DISTRIBUTION

The last two sections provided conditions for the existence of invariant distributions for Markov processes. This section considers the question of uniqueness and convergence. We provide a simple easily verified condition under which the invariant distribution for the process is unique and globally stable. An algorithm for successively approximating the invariant distribution is also provided.[13] This result is used in the application presented in Section 6; for the sake of completeness we develop it in this section. Furthermore, our proof is a remarkably simple one and it extends easily to the non-time-homogeneous Markov case and also suggests conditions for the uniqueness and global stability of nonlinear mappings.

THEOREM 2: *Suppose P is increasing, S contains a lower bound (which we will denote by a) and an upper bound (which we will denote by b), and the following condition is satisfied:*

Monotone Mixing Condition (MMC): There exists a point $s^* \in S$ and an integer m such that $P^m(b,[a,s^*]) > 0$ and $P^m(a,[s^*,b]) > 0$.

Then there is a unique stationary distribution λ^* for process P and for any initial measure μ, $T^n\mu = \int P^n(s,\cdot)\mu(ds)$ converges to λ^*.

NOTE: The intuition behind this result is as follows: The MMC condition implies that though the monotonicity of T and its iterates preserve the ordering of two distributions, after finite iterations some of the mass in these distributions reverses ordering. This process taken indefinitely implies a complete reversal of ordering but by monotonicity of T and antisymmetry of $\succcurlyeq$ this can only occur if in the limit both distributions coincide.

PROOF OF THEOREM 2: Choose $\varepsilon > 0$ and m such that $P^m(b,[a,s^*]) > \varepsilon$ and $P^m(a,[s^*,b]) > \varepsilon$. Let δ_s indicate the probability measure that concentrates all the mass on the point set $\{s\}$. We will prove that the following inequality holds:

$$(1-\varepsilon)\delta_a + \varepsilon\delta_{s^*} \preccurlyeq T^m\delta_a \preccurlyeq T^m\delta_b \preccurlyeq (1-\varepsilon)\delta_b + \varepsilon\delta_{s^*}. \tag{2}$$

For this purpose, let f denote an arbitrary element of $\boldsymbol{M}$. Then:

$$\begin{aligned}\int f(s)T^m\delta_a(ds) &\geqslant f(a)\int_{s<s^*} T^m\delta_a(ds) + f(s^*)\int_{s\geqslant s^*} T^m\delta_a(ds)\\ &\geqslant f(a)(1-\varepsilon) + f(s^*)\varepsilon\\ &= \int f(s)\{(1-\varepsilon)\delta_a + \varepsilon\delta_{s^*}\}(ds)\end{aligned}$$

where the second inequality follows from $f(a) \leqslant f(s^*)$ and $\int_{S\geqslant s^*} T^m\delta_a(ds) > \varepsilon$. This establishes the left hand side inequality; the right hand side can be proved in the same way. Since T is increasing, by induction T^k, i.e. the composition of

[13] Battacharya and Lee (1988) prove a similar result for processes in $\mathbb{R}^n$, but do not require compactness. Our work was independently done.

T with itself k times is also increasing so $T^m\delta_a \preccurlyeq T^m\delta_b$, which establishes (2). By monotonicity and linearity of T^k:

$$(3) \qquad (1-\varepsilon)T^k\delta_a + \varepsilon T^k\delta_{s^*} \preccurlyeq T^{k+m}\delta_a \preccurlyeq T^{k+m}\delta_b \preccurlyeq (1-\varepsilon)T^k\delta_b + \varepsilon T^k\delta_{s^*}.$$

By Proposition 1 the monotone sequences $\{T^k\delta_a\}$ and $\{T^k\delta_b\}$ converge. Let these limits be λ_a and λ_b. If necessary along a subsequence, $T^k\delta_{s^*}$ converges to a limit λ^*. By the closed graph property of the stochastic order:

$$(1-\varepsilon)\lambda_a + \varepsilon\lambda^* \preccurlyeq \lambda_a \preccurlyeq \lambda_b \preccurlyeq (1-\varepsilon)\lambda_b + \varepsilon\lambda^*.$$

This inequality implies that $\lambda_a \succcurlyeq \lambda^* \succcurlyeq \lambda_b$, which by asymmetry and transitivity imply that $\lambda_a = \lambda^* = \lambda_b$. It is easy to see that λ^* is a fixed point for T. Using the monotonicity of T and the definition of λ_a and λ_b, $T^k\mu \to \lambda^*$ for any measure μ on S. *Q.E.D.*

REMARKS: (i) Since the Markov structure was used in the proof only to obtain linearity of T, the conclusions of the theorem are also valid for any non-time-homogeneous Markov process $\{P_t\}$ such that P_t is increasing for all t and there exists $s^* \in S$, $\varepsilon > 0$, and m such that $P_t^m(b,[a,s^*]) > \varepsilon$ and $P_t^m(a,[s^*,b]) > \varepsilon$ uniformly in t.[14] (ii) Since linearity of T was only used in proving (3), the results will also hold for any monotone map T that satisfies (2) and for which

$$(1-\varepsilon)T^k\delta_a + \varepsilon T^k\delta_{s^*} \preccurlyeq T^k\big((1-\varepsilon)\delta_a + \varepsilon\delta_{s^*}\big) \quad \text{and}$$
$$T^k\big((1-\varepsilon)\delta_b + \varepsilon\delta_{s^*}\big) \preccurlyeq (1-\varepsilon)T^k\delta_b + \varepsilon T^k\delta_{s^*}.$$

The following corollary is useful in some economic applications.

COROLLARY 8: *Let $\{s_t\}$ be a monotone Markov process on space $(S, \geqslant)$, where $(S, \geqslant)$ satisfies the assumptions of the previous theorem. Let a and b be, respectively, the lower and upper bounds in S and assume they have recurrent neighborhood systems; i.e., for any $\varepsilon > 0$ and $s \in S$, the probability of eventually reaching an ε-neighborhood of a (resp. b) is equal to one. Then s_t has a unique, asymptotically stable distribution.*

There is a sense in which the invariant distribution can be successively approximated. From equation (2) and given that $\delta_b \succcurlyeq \delta_a$, for any nondecreasing, nonnegative, bounded, and measurable function g the following holds:

$$0 \leqslant \int g\lambda^*(ds) - \int gT^{km}\delta_a(ds) \leq (1-\varepsilon)^k[g(b) - g(a)].$$

Thus if ε and m were known, for any given increasing function we could obtain

[14] Here P_t^m denotes the transition from period t to period $t+m$, i.e., $P_t^m = P_{t+m-1}P_{t+m-2}\cdots P_t$. Letting T_j^k be the linear operator associated to P_j^k, equation (3) must be replaced by $(1-\varepsilon)^k\delta_a + A(k)\delta_{s^*} \preccurlyeq T_0^{km}\delta_a \preccurlyeq T_0^{km}\delta_b \preccurlyeq (1-\varepsilon)^k\delta_b + A(k)\delta_{s^*}$, where $A(k)$ is the linear operator $\varepsilon\sum_{j=1}^k(1-\varepsilon)^{k-j}T_{(k-j+1)m}^{(m-1)j}$ and $A(k)\delta_{s^*}$ has mass $1-(1-\varepsilon)^k$. Along a converging subsequence for $A(k)\delta_{s^*}$ the same conclusion follows.

an approximation as close as desired to the expectation of the function with respect to the invariant distribution. Moreover, Battacharya and Lee (1988) show that for any initial distribution μ, $T^k\mu$ converges to λ^* exponentially fast in a metric that defines a topology stronger than the weak* topology.

6. APPLICATIONS

In this section we illustrate the applications of our results to three areas: the theory of investment, stochastic growth, and industry equilibrium.

A. *Investment Theory*

Consider the following optimal capital accumulation problem for a firm faced with stochastic demand:

$$\max_{\{q_t, k_t\}} E \sum_{t=0}^{\infty} \beta^t [R(q_t, z_t) - C(q_t, k_t) - g(k_t, k_{t+1})] \quad \text{subject to}$$

$$q_t \in \Gamma_1(k_t), \qquad k_{t+1} \in \Gamma_2(k_t), \qquad \text{and} \quad (k_0, z_0) \text{ given.}$$

R is a revenue function which depends on the output of the firm q_t and a demand shock z_t that follows a Markov process with transition function Q. Output is constrained by the capital stock k_t of the firm through the correspondence Γ_1 and production cost is given by $C(q_t, k_t)$. Capital accumulation is constrained by the correspondence Γ_2 and the adjustment cost $g(k_t, k_{t+1})$. Net flows are discounted at a constant rate $\beta \in (0, 1)$.

Assume $q \in X$ and $z \in Z$, where X and Z are compact metric spaces endowed with a closed order and Z has minimum element $\underline{z}$. Γ_1: $K \to 2^X$ and K is a compact metric space with continuous lattice structure and with minimum element $\underline{k}$. Assume R is continuous in q, bounded, supermodular in q and with strictly increasing differences in z; C and g are continuous and strictly submodular;[15] Γ_1 and Γ_2 are continuous, compact valued, ascending, and satisfy strict complementarity; Q is increasing.

We now show that Proposition 3 applies to this optimal accumulation problem, and thus there exists a stationary distribution for the capital stock of the firm.

Let $\Pi(k, z) = \max_{q \in \Gamma_1(k)} R(q, z) - C(q, k)$ and let $F(k, z, k') = \Pi(k, z) - g(k, k')$. By Lemma 1 Π is strictly supermodular in k and has strictly increasing first differences. F and Γ_2 define a stochastic control problem which satisfies the assumptions of Proposition 3. It is also easy to check that F is continuous and bounded, and the optimal choice correspondence is upper hemicontinuous (and thus upper measurable). By Proposition 3 all measurable selections from it are nondecreasing. Any such policy function together with transition Q imply a Markov process for (z, k) on the compact set $Z \times K$ with an increasing

[15]A function h is (strictly) submodular if $-h$ is (strictly) supermodular.

transition and minimum point $(\underline{z}, \underline{k})$. By Corollary 5 this process has an invariant distribution.

The assumptions provided are satisfied in many commonly used setups. If $X = \mathbb{R}_+$ and R is C^2, R will automatically be supermodular in q and it will have increasing differences if and only if marginal revenue is increasing in z. For a competitive firm this holds if output price is increasing in z. Likewise, for C twice continuously differentiable, submodularity is equivalent to the statement that marginal cost is decreasing in k. Submodularity of g is satisfied for many of the adjustment cost functions used in the investment literature. The following examples, when K is a subset of the real line, are easily verified: (i) $g(k, k') = h(k'/k)$ where h is increasing and convex; (ii) $g(k, k') = kh(k'/k)$ with h increasing and convex; (iii) $g(k, k')$ a convex function of $\alpha k' - \beta k$, where α and β are positive constants. This includes the standard quadratic case and many types of one sided adjustment costs. Also for the case when K and X are subsets of $\mathbb{R}$ the restrictions needed on Γ_1 and Γ_2 are satisfies if $\Gamma_1(k)$ and $\Gamma_2(k)$ are decreasing sets and ordered by inclusion, i.e. if $d(\Gamma_i(k)) = \Gamma_i(k)$ and $k' \geqslant k$ implies $\Gamma_i(k) \subset \Gamma_i(k')$ for $i = 1, 2$. If Γ_1 is the graph of an increasing production function, it will automatically satisfy these conditions. With free disposal of capital and cumulative investment these hypotheses will also be satisfied for Γ_2. Note that the often imposed restriction of irreversible investment can be accommodated by specifying a one sided adjustment cost function, e.g. $g(k, k') = c \cdot \max(k' - (1 - \delta)k, 0)$, where δ is a depreciation factor. Finally if K is a compact subset of $\mathbb{R}^n$ with the canonical order, it will satisfy the above assumptions.

It is worth indicating that the general assumptions given do not guarantee that there exists a continuous selection and thus standard continuity arguments may not apply. In particular, if k takes discrete values and z is a continuous variable, for fixed k_t the optimal capital k_{t+1} will be discontinuous in z unless it is constant. As an example, consider the following entry/exit problem studied by Dixit (1989).

A firm is faced with an exogenously given stochastic process for the price of a good. There are positive entry and exit costs to the industry. While the firm is in the industry it produces a constant flow of one unit of output q_t. Given the conditional distribution for the price process each period the firm faces the following decision problem: if it is out of the industry it must decide whether to enter or not; if it is in the industry it decides whether to stay or leave.

In this case $k_t \in \{0, 1\}$, where 0 denotes 'out' and 1 'in'. Denoting the entry and exit costs by e and f, respectively, $g(0,1) = e$, $g(1,0) = f$, and $g(0,0) = g(1,1) = 0$. It is easy to see that g is submodular. $R(q, z) = p(z)$ which is trivially supermodular. So assuming the transition function for the demand shock is increasing, the optimal decision rule will be increasing. This means that if the firm is *in* it needs a lower shock to leave the market than to stay and if it is *out* it needs a higher shock to enter than to remain outside. The optimal decision rule is thus the standard two sided *sS* policy involving two trigger prices $p_1 < p_h$ that correspond to the entry and exit barriers, respectively. Assuming p_h or p_1 satisfy the MMC, by Theorem 2 the Markov process for

(p_t, k_t) has a unique stable distribution. This process does not have a continuous transition so standard continuity arguments do not apply.

B. Stochastic Growth Theory

Consider the standard one sector growth model, with objective:

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{subject to}$$

$$k_{t+1} + c_t \leqslant f(z_t, k_t), \qquad c_t \geqslant 0, \qquad k_{t+1} \geqslant (1-\delta)k_t.$$

Throughout this analysis the following will be assumed: (i) there is a maximum capital stock $\bar{k}$ and $0 < \delta \leqslant 1$; (ii) u is continuous, strictly increasing and strictly concave; (iii) f is increasing in both arguments and continuous. With these assumptions, the optimal policy is a selection from the solutions to:

$$\max u(f(z,k) - k') + \beta \int v(z', k') Q(dz' \mid z) \qquad \text{subject to}$$

$$(1-\delta)k \leqslant k' \leqslant f(z,k).$$

(i) *I.I.D. Shocks*[16]

In this case the conditional expectation in the right-hand side is independent of z. So to establish monotonicity of investment in z it suffices to show that $F(z,k,k') = u(f(z,k) - k')$ is strictly supermodular in (z, k') for any $k \in [0, \bar{k}]$. This follows from the assumptions on u and f.[17] Similarly, to establish that k' is increasing in k, it suffices to show that F is supermodular in (k, k') for fixed z, which follows from the assumptions by using similar arguments. The best choice correspondence is thus ascending and, by continuity and u and f, upper hemicontinuous. The assumptions of Propositions 1 and 2 are satisfied, so there exist measurable selections all of which are increasing. The assumptions of Corollary 5 are easily verified, so there exists an invariant distribution for (z, k). Since no restrictions other than continuity and monotonicity were placed on f, the policy function need not be continuous, so standard fixed point arguments are not applicable.

We now specialize this case to Brock and Mirman (1972) to illustrate how the conditions in Theorem 2 can be verified and thus uniqueness and global stability of the invariant distribution obtained. In addition to the assumptions made above, assume that u is strictly concave and continuously differentiable with $u'(0) = \infty$; $f(z,k) = zf(k)$, where f is strictly concave and continuously differentiable; $z_t \in [1, \bar{z}]$ are i.i.d. with probability distribution ψ; there is a maximal sustainable capital stock $\bar{k}$ and $\beta f'(0) > 1$.

For this case the policy function is unique, increasing in both arguments and continuous, the consumption function $c(k, z)$ is increasing in both arguments and continuous, and the value function is differentiable.

[16] This problem is studied in Majumdar, Mitra, and Nyarko (1989), using different techniques.

[17] If u is twice continuously differentiable and f continuously differentiable, this follows from the fact that $F_{12} = -u''f' > 0$.

Let $s^* = (k^*, z^*)$, where $z^* = \int z\psi(dz)$ and k^* is the solution of $1 = \beta z^* f'(k^*)$. We now show that s^* satisfies the MMC of Theorem 2.

Let $\{k_n\}$ be the sequence obtained from the optimal policy rule $g(k_n, 1)$ starting at $k_0 = \bar{k}$, i.e. the sequence of capital stocks starting at the maximal sustainable capital stock and provided thereafter the productivity shock is at its minimum. By monotonicity of the optimal policy rule g, the sequence $\{k_n\}$ is decreasing and since it is bounded $k_n \to b \in [\underline{k}, \bar{k}]$. Further, continuity of g implies $b = g(b, 1)$. For any $\delta > 0$, $\psi([1, 1+\delta)) > 0$, since $\{1\}$ is in the support of ψ. This, together with the continuity of g imply that the probability of eventually being in any neighborhood of b is positive.

The first order condition in the optimization problem $s = (b, 1)$ is

$$u'[c(b,1)] = \beta \int \frac{\partial v}{\partial k}(b, z')\psi(dz') = \beta f'(b) \int u'[c(b,z')] z' \psi(dz')$$

as under the optimal plan $b = g(b, 1)$. But $u'[c(b, z')] \leqslant u'[c(b, 1)]$ with strict inequality if $z' > 1$ given that u is strictly concave and c is strictly increasing in z. Thus

$$u'[c(b,1)] < \beta f'(b) u'[c(b,1)] \int z' \psi(dz')$$

or

$$\frac{1}{\beta} < z^* f'(b).$$

But since f was assumed to be strictly concave and $(1/\beta) = z^* f'(k^*)$, $b < k^*$.

Let $k > 0$ satisfy $g(\underline{k}, 1) = \underline{k}$.[18] Starting from any $0 < k < \underline{k}$, and applying iteratively the decision rule $g(k, \bar{z})$, an argument analogous to the above implies that the increasing sequence generated converges to a point $b > k^*$. Thus, the process generated from any $k > 0$ satisfies the MMC, and possesses a unique globally stable distribution. Since $g(k, 1) \geqslant k$ for all $k < \underline{k}$, this distribution will have support in $[\underline{k}, \bar{k}]$. In consequence, the limiting distribution is identical for all initial $k > 0$.

(ii) *Correlated Shocks*

Assume that Q is an increasing transition. Though under these assumptions the policy function will be increasing in k, it may not be increasing in z: *higher* z will lead to an increase in consumption in the present as well as in the future; but since *higher* z also implies *higher* expected productivity of capital in the

[18]That such $\underline{k} > 0$ exists can be shown in the following way. For the deterministic case with $s = 1$ (the lowest shock), $\beta f'(0) > 1$ implies there exists a unique strictly positive steady state k^*. Furthermore, $k < k^*$ implies $g(k) > k$, where $g(k)$ is the optimal policy function for that deterministic case. For any $k > 0$, the value function for the stochastic problem analyzed $v(k, 1)$ has a higher value than the one that would correspond to the deterministic problem discussed. But in both cases the value at $k = 0$ is zero. Hence locally around 0 the value function for the stochastic problem increases faster than the one for the deterministic problem. In consequence, there exists $0 < k < k^*$ where investment is higher in the stochastic case, so $g(k, 1) > k$. This in turn implies there exists some $\underline{k} > 0$ with $g(\underline{k}, 1) = \underline{k}$.

future, less capital may be necessary to sustain the higher consumption. The analysis that follows will assume that investment is interior and for simplicity that u and f are twice continuously differentiable. So provided the constraints on k' are not binding, the problem can be formulated as:

$$\max_{k'} u(f(z,k)-k')+\beta\int v(z',k')Q(dz' \mid z).$$

To use Theorems 2 and 3 we just need to establish that u is supermodular in (k,k') for fixed $z \in Z$ and has increasing differences in z. The first condition was already established above. Increasing differences can be evaluated separately for (z,k) and (z,k'). The latter, which is equivalent to supermodularity in (z,k'), was also established above. Finally, increasing differences in (z,k) is equivalent to

$$F_{12}(z,k,k')=u''f_2f_1+u'f_{12} \geqslant 0$$

or

$$-\frac{u''}{u'} \leqslant \frac{f_{12}}{f_1 f_2}$$

where all arguments are implicit.[19] The economic interpretation is that the degree of complementarity between capital and shocks must be high relative to the curvature in the utility function, for otherwise the productivity gain may be more than offset by the decrease in marginal utility. If f is a CES function in (z,k), the right side goes to infinity as the elasticity of substitution goes to zero. As expected for the extreme case, namely when $f(z,k)=\min\{\alpha_1 z, \alpha_2 k\}$, F will have increasing differences in (z,k) so the policy function will be increasing in z. Again, if the policy function is monotone assuming z is bounded, Corollary 5 applies and thus there exists a stationary distribution.

C. *Industry Equilibrium*

Lambson (1988) considers a model of entry and exit to an industry with the following characteristics: There is a continuum of firms indexed by the positive real line which, at any time period, can either be 'active' or 'inactive'. In any given period (t) only active firms can produce, incurring a cost $c(q,m_t)$, where m_t is the realization of a stochastic process *common* to all firms in the industry. Note that except for the active/inactive distinction, all firms are identical. Also in any period inactive firms can become active, incurring an entry cost $\mathscr{E}(m_t)$ while active firms may exit and become inactive, obtaining a scrap value $\chi(m_t)$. Let y_t denote the mass of active firms in period t and let q_t denote their output choice. Prices will then be given by $p(y_t q_t, m_t)$, where $p(\cdot, m_t)$ is a nondecreasing inverse demand function with $\lim_{x\to\infty} p(x,m)=0$. An equilibrium is a joint process for (y,q,p,m) adapted to the filtration induced by the process $\{m_t\}$

[19] Donaldson and Mehra (1983) arrived at a similar expression following a different analysis. In contrast to their analysis, the approach followed here makes very few assumptions on the shape of the production function.

such that (i) given this process, output and the implied entry and exit decisions maximize expected discounted profits, and (ii) $p = p(y_t q_t, m_t)$.

Lambson shows that the equilibrium is unique and that it is characterized by a pair of functions $N(m_t)$ and $X(m_t)$, where $N(m_t) \leqslant X(m_t)$ such that when condition m_t occurs, (i) if $y_{t-1} \in [N(m_t), X(m_t)]$ then $y_{t+1} = y_t$; (ii) if $y_{t-1} < N(m_t)$, $y_t = N(m_t)$; (iii) if $y_{t-1} > X(m_t)$, $y_t = X(m_t)$.

Let the state of the industry $s_t = (y_t, m_t)$. Under what conditions does there exist an invariant distribution to which the system converges? For this purpose, Lambson assumes the process for m_t is Markov over a countable state space $M = 1, 2, \ldots$ recurrent and positive with transition $p_{ij} > 0$ for all pairs i, j and that $\sup_i N_i > \inf_i X_i$.

An alternative route, suggested by our results, is to exploit the monotonicity that the equilibrium law of motion for y_t displays. For simplicity, we will consider the case where market conditions only affect the market demand for the product. In particular, suppose market conditions $m \in M$, where M is a compact metric space with a closed partial order with minimal and maximal points $\underline{m}$ and $\overline{m}$, respectively and that $m' \geqslant m$ implies $p(\cdot, m') \geqslant p(\cdot, m)$. Assume $\{m_t\}$ follows a Markov process with an increasing transition P. Under these assumptions $m' \geqslant m$ implies $N(m') \geqslant N(m)$ and $X(m') \geqslant X(m)$. Assume there exist demand shocks m_1 and m_2 such that $N(m_1) \leqslant X(m_1) \leqslant N(m_2) \leqslant X(m_2)$ and $(n, \delta > 0)$ such that $P^n[m \leqslant m_1 | \overline{m}] > \delta$ and $P^n[m \geqslant m_2 | \underline{m}] > \delta$. Then choosing $m^* \in [m_1, m_2]$ and $y^* \in [X(m_1), N(m_2)]$, the point $s^* = (m^*, y^*)$ satisfies MMC so Theorem 2 applies and a unique, asymptotically stable distribution exists.[20] Note that though extra monotonicity assumptions were needed to get this result, the requirements on the degree of communication between states are much weaker and the set of market conditions need not be countable.

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APPENDIX

LEMMA 1: *Let X and Y be lattices and Z a partially ordered set. Suppose f: $S \times Y \times Z \to \mathbb{R}$ is supermodular in $(s, y) \in X \times Y$ for all $z \in Z$, has increasing differences, and is bounded. Let Γ:*

[20] Alternatively suppose $\sup\{N(m) | m \in M\} > \inf\{X(m) | m \in M\}$, and the points $\underline{m}$ and $\overline{m}$ have a recurrent neighborhood system, i.e. for any m and $\varepsilon > 0$ the probability that an ε-neighborhood of $\underline{m}$ (resp. $\overline{m}$) will *eventually* be reached is equal to one. Also assume $p(\cdot)$ is a continuous function. Letting $\underline{y} = X(\underline{m})$ and $\overline{y} = N(\overline{m})$, the points $[\underline{y}, \underline{m}]$ and $[\overline{y}, \overline{m}]$ satisfy the assumptions of the Corollary to Theorem 2, so there exists a unique, asymptotically stable distribution of s_t.

$S \times Z \to 2^Y$ be a correspondence with strict complementarity and for each $z \in Z$, the graph of $\Gamma(\cdot; z)$ is a sublattice. Then

$$g(x,z) = \sup_{y \in \Gamma(x,z)} f(x,y,z) \text{ is supermodular on } x \text{ and has increasing differences.}$$

PROOF: Supermodularity follows from Theorem 4.3 in Topkis (1978). To prove that g has increasing differences, let x_1 and x_2 be two elements in X such that $x_2 \geqslant x_1$ and similarly let z_1 and z_2 be two elements of Z with $z_2 \geqslant z_1$. Let $y_{12} \in \Gamma(x_1, z_2)$ and $y_{21} \in \Gamma(x_2, z_1)$. Then

$$\begin{aligned}
g(x_2,z_2)+g(x_1,z_1) &\geqslant f(x_2,y_{12}\vee y_{21},z_2)+f(x_1,y_{12}\wedge y_{21},z_1)\\
&= f(x_2,y_{12}\vee y_{21},z_2)+f(x_1,y_{12}\wedge y_{21},z_2)+f(x_1,y_{12}\wedge y_{21},z_1)\\
&\quad - f(x_1\, y_{12}\wedge y_{21},z_2)\\
&\geqslant f(x_1,y_{12},z_2)+f(x_2,y_{21},z_2)+f(x_1,y_{12}\wedge y_{21},z_1)\\
&\quad - f(x_1,y_{12}\wedge y_{21},z_2)\\
&= f(x_1,y_{12},z_2)+f(x_2,y_{21},z_1)+f(x_2,y_{21},z_2)\\
&\quad - f(x_1,y_{12}\wedge y_{21},z_2) - (f(x_2,y_{21},z_1) - f(x_1,y_{12}\wedge y_{21},z_1)\\
&\geqslant f(x_1,y_{12},z_2)+f(x_2,y_{21},z_1)
\end{aligned}$$

where the first inequality follows from strict complementarity, the second inequality from the definition of supermodularity, and the third one by using the fact that f has increasing differences. Taking the supremum of the right-hand expression, the result follows. *Q.E.D.*

LEMMA 2: *Let S be a partially ordered set and Y a separable metric space with a continuous lattice structure. Let γ: $S \to 2^Y$ be a nonempty, compact valued, and upper measurable correspondence, and assume that for each $s \in S$, $\gamma(s)$ is a sublattice of Y. Then the functions $\bar{g}(s) = \sup \gamma(s)$ and $\underline{g}(s) = \inf \gamma(s)$ are borel measurable selections of γ. Furthermore, if γ is ascending, these selections are increasing.*

PROOF: Since γ is compact valued in a topology finer than the order topology, $\gamma(s)$ is a complete sublattice for all $s \in S$ (see Birkoff (1967)). Hence $\bar{g}$ and g are well defined selections from γ. We now show that sup is a continuous function on the set $\mathscr{Y}$ of compact subsets of Y with the Hausdorff topology. Let $C \in \mathscr{Y}$ and $s = \sup C$. Let U be an open set containing s and for each $t \in C$ let $\delta_t > 0$ and $\varepsilon_t > 0$ be chosen so that for all $y \in B_{\delta_t}(t)$ and $x \in B_{\varepsilon_t}(s)$, $x \vee y \in U$. Let $\delta = \min_{t \in C}\{\min\{\delta_t, \varepsilon_t\}\}$. Since C is compact, $\delta > 0$. Pick any set $C' \in \mathscr{Y}$ from the open δ neighborhood of C and let $s' = \sup C'$. By choice of C', $s' \in B_{\delta_t}(t)$ for some $t \in C$ and $B_{\varepsilon_t}(s) \cap C' \neq \phi$, so $s' = s' \vee x$ for some $x \in B_{\varepsilon_t}(s)$ and thus $s' \in U$. Since $\bar{g}$ is the composition of a continuous function with a measurable function, it is measurable. Finally, that $\bar{g}$ is nondecreasing follows immediately from the fact that γ is ascending. *Q.E.D.*

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