Growth Model: Part 1

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In applied dynamic economic theory, there are two standard ways to analyze the consumption-savings decision. They are

1. The infinitely lived families with parents caring about their children's utility.
2. The long but finite-lived people who leave their children no bequests.

For some issues it matters which abstraction is used. For example if one were to use economic theory to assess the quantitative consequences of the American pay-as-you-go social security system, it matters which abstraction is used. If, on the other hand, one wishes to assess the consequences of technology shocks, public finance shocks or oil shocks for business cycle fluctuations, for all practical purposes it does not matter. Our view is that a better abstraction than either of these is one that has people of both types. Some people leave their children bequests and some people do not. But, given that we are interested in business cycles and the infinitely lived family is the easiest to deal with, we will use the infinitely lived family construct in these notes.

In what follows we will develop a set of necessary conditions for a competitive equilibrium by considering the maximization problems facing families, banks and firms. This set of conditions will be used to determine the unique constant growth competitive

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[1] Fernando Alvarez and Jessica D. Tjornhom were co-authors of earlier versions of these. Any errors or omissions are the sole responsibility of the author.

equilibrium for a parametric set of model economies. We consider the problem facing each sector in turn.

Households

A household has one unit of time every period that can be used to provide labor services *h* or for non-market activities. Non market activities will be referred to as leisure, even though many activities such as cleaning the house is work. The productive time endowment is normalized to 1 and is about 100 hours per week. Sleeping and personal time is not part of the productive time endowment. Leisure is $1 - h$.

Households' preferences are defined over infinite streams of consumptions $\{c_t\}$ and leisure $\{1-h_t\}$. The initially the sizes of families are assumed constant in order to simplify the analysis. The structure will be extended to permit population growth, as this will be necessary when matching the theory to U.S. data. Households' preferences are ordered by

$$u(c_0, 1-h_0) + \frac{u(c_1, 1-h_1)}{(1+\rho)} + \frac{u(c_2, 1-h_2)}{(1+\rho)^2} + \cdots$$

The parameter $\rho > 0$ is the time discount rate. The larger ρ is the more people prefer consumption now relative to consumption in the future.

The prices facing the households at each period t are

$$\begin{aligned} w_t &: \text{ rental price of labor (real wage).} \\ r_t &: \text{ rental price of capital.} \\ i_t &: \text{ interest rate on deposits.} \end{aligned}$$

The households' period budget constraints are

$$d_{t+1} = (1 + i_t)[d_t + w_t h_t - c_t]$$

for $t = 0,1,2,\ldots$. Here the d_t's are beginning of period deposits at the bank.

A Notation Convention and Marginal Utilities

For differentiable functions with multiple arguments, there are partial derivatives. A partial derivative is just the derivative with respect to one of the arguments. Consider differentiable function $f(x,y)$. The partial derivative with respect to the first argument is the derivative with respect to x and will be denoted $f_1(x,y)$. Similarly the derivative with respect to its second argument will be denoted $f_2(x,y)$. In this example there are two arguments of a function, but in general there can be any number.

Marginal utility of consumption is the rate of change of utility with respect to consumption. Thus,

$$MU(c) = u_1(c,1-h)$$

From the definition of a derivative:

$$MU(c) = u_1(c,1-h) = \lim_{\varepsilon \to 0} \frac{u(c+\varepsilon,1-h) - u(c,1-h)}{\varepsilon}$$

In textbooks, which do not use derivatives, the definition of marginal utility of consumption is typically $MU(c) = u(c+1,1-h) - u(c,1-h)$. This is an approximation to the derivative with respect to *c*. Similarly the marginal utility of leisure is

$$MU(1-h) = u_2(c,1-h) .$$

Subsequently in these notes, the marginal utility of consumption is u_1 and the marginal utility of leisure is u_2.

The marginal rate of substitution between two goods is the ratio of the marginal utilities. Thus

$$MRS(c,1-h)=\frac{MU(c)}{MU(1-h)}=\frac{u_1(c,1-h)}{u_2(c,1-h)}$$

Utility maximization requires the marginal rate of substitution between any two goods be equal to their price ratio. This leads to the following two sets of necessary first-order conditions for maximization:

(H1) $$MRS(c_t,1-h_t)=\frac{u_1(c_t,1-h_t)}{u_2(c_t,1-h_t)}=\frac{1}{w_t}$$

(H2) $$MRS(c_t,c_{t+1})=\frac{(1+\rho)u_1(c_t,1-h_t)}{u_1(c_{t+1}1-h_{t+1})}=1+i_t\,.$$

Via the budget constraint, a family can transform consumption c_t into leisure $1-h_t$ at rate $1/w_t$. Given that the ratio of the marginal utilities is the willingness to substitute, (H1) follows. Similarly the household can transform c_t into c_{t+1} at rate $1+i_t$ via saving. This implies (H2).

Firms

The firms face a sequence of static maximization problems, one for each date. The period t problem is

$$\max\left\{F(h_t,k_t,t)-w_th_t-r_tk_t\right\},$$

where h_t is the input of labor services and k_t is the input of capital services. Over time, the production function can change as the result of changes in policy and in the growth in the stock of knowledge useful in production.

The marginal product of a factor is the rate of change in output with respect to that input, holding other inputs constant. Thus,

$$MP(h) = F_1(h,k)$$

$$MP(k) = F_2(h,k)$$

Subsequently, the marginal product of labor is F_1 and the marginal product of capital is F_2.

Two necessary first-order conditions for a maximum are

(F1) $$w_t = MP(l_t) = F_2(k_t, h_t, t)$$

(F2) $$r_t = MP(k_t) = F_1(k_t, h_t, t) .$$

Given that F displays constant returns to scale, payments to factors exhaust the product and there are no dividends.

Banks

In our model world, the banks own the capital and rent it to the firms. Banks also accept deposits from households and pay interest i. Finally fraction δ of the capital stock wears out, that is depreciates, in the period.

The banking industry is competitive and the intermediation technology has constant returns to scale. An implication of these assumptions is that in equilibrium banks neither make a profit nor suffer a loss. They break even in equilibrium. Suppose a bank accepts one unit of deposits at date t and uses it to buy one unit of capital. Suppose further that this bank rents this capital at rate r_{t+1} at date $t+1$ and sells the not depreciated part, $(1-\delta)$, at the end of period $t+1$. At date t its net cash flow is zero. At date $t+1$ its cash flow is

$$r_{t+1} + (1-\delta) - (1+i_t),$$

which must be zero in equilibrium. Note that the bank returns the deposit and pays the promised interest at date $t+1$. The equilibrium requirement that this bank's net cash flows resulting from this set of transactions be zero implies the following necessary condition for equilibrium

(B1) $$r_{t+1} = i_t + \delta .$$

This just says that the rental rate on capital must cover both interest costs and depreciation costs if the bank is to break even.

An additional necessary condition for banks' profit maximization is that they do not throw capital away. Thus

(B2) $$k_{t+1} = (1-\delta)k_t + x_t ,$$

where x is investment good purchases by banks.

A final assumption is that initial deposits are

$$d_0 = k_0(1 + r_0 - \delta).$$

The reason for this assumption is that it results in the net worth of banks being zero and, therefore, zero dividends paid by banks. For all t

(B3) $$d_t = (1 + r_t - \delta) k_t = \left(1 + i_{t-1}\right) k_t ,$$

as equilibrium net worth of banks must be zero at all dates.

Market Clearing:

(M) $$c_t + x_t = F(h_t, k_t, t) \text{ for all } t.$$

The demand for the consumption good by the household plus the demand for the investment good by the bank must equal the supply of output by the firm in every period.

Definition of Competitive Equilibrium:

A competitive equilibrium is (a) a vector for the households $\{c_t, d_t, h_t\}_{t=0,1,2,\dots}$, (b) a vector for the banks $\{d_t, k_t, x_t\}_{t=0,1,2,\dots}$, (c) a vector for the firms $\{c_t, x_t, k_t, h_t\}_{t=0,1,2,\dots}$, and (d) a price system $\{r_t, w_t, i_t\}_{t=0,1,2,\dots}$ such that

(i) Households maximize utility subject to their budget constraints.

(ii) Banks maximize profits subject to their technology constraints.

(iii) Firms maximize profits subject to their technology constraints.

(iv) Markets clear.

Market clearing means the following: (1) deposits of households equal deposits held by banks; (2) labor sold by households equals labor purchased by firms; (3) consumption goods purchases by households equal sales of consumption goods by firms; (4) investment goods purchases by banks equal investment goods sales by firms; and (5) capital services sold by the banks equal capital services bought by the firms at every date.

Definition of Constant Growth Competitive Equilibrium:

A constant growth competitive equilibrium is a competitive equilibrium with the properties that for all t,

$$c_t = c(1+g)^t$$

$$x_t = x(1+g)^t$$

$$y_t = c_t + x_t = y(1+g)^t$$

$$d_t = d(1+g)^t$$

$$h_t = h$$

$$k_t = k(1+g)^t$$

$$w_t = w(1+g)^t$$

$$r_t = r$$

$$i_t = i$$

for some set of numbers $\{c, x, y, d, h, k, w, r, i, g\}$. These variables are all per capita numbers.

A Parametric Set of Economies

Parameters are real numbers that specify the particular economy in a parametric set of economies. All the functions below that describe preferences and technology have one or more parameters:

Utility function: $u(c_t, 1-h_t) = \log c_t + \alpha \log(1-h_t)$.

Production function: $y_t = F(h_t, k_t, t) = A_t^{1-\theta} k_t^{\theta} h_t^{1-\theta}$.

Value of the A_t $\qquad A_t = A(1+\lambda)^t$, where $A > 0$ is given,

Law of motion of capital: $\qquad k_{t+1} = (1\text{-}\delta)\, k_t + x_t$.

The parameters are $\{\rho, \alpha, \theta, A, \lambda, \delta\}$. Numerical values for these parameters along with the assumed functional forms define an economy.

Finding Constant Growth Equilibrium

We have specified a parametric set of model economies. The units in which output is measured is such that A=1. Specifying the parameter vector, that is the values of α, δ, ρ, λ, and θ, defines a model economy. To find the constant growth equilibrium, the expressions for the constant growth equilibrium are substituted into necessary conditions, {(H1)–(H2), (B1)–(B3), (F1)—F(2) and (M)} and the resulting equations simplified. The resulting equations are: Note: in (1) and (3) I have replaced the l, with h.

(1) $$\alpha c = w(1-h) \qquad \text{(H1);}$$

(2) $$i = g + \rho + \rho\, g \qquad \text{(H2);}$$

(3) $$w = \left(1-\theta\right)(k/h)^{\theta} \qquad \text{(F1);}$$

(4) $$r = \theta \ \ (h/k)^{(1-\theta)} \qquad \text{(F2);}$$

(5) $$c + x = k^{\theta}\, h^{(1-\theta)} \qquad \text{(M);}$$

(6) $$r = i + \delta \qquad \text{(B1);}$$

(7) $$x = \left(g+\delta\right) k \qquad \text{(B2);}$$

(8) $$d = \left(1+i\right) k \qquad \text{(B3).}$$

Exercise 1: Show that g for any constant growth equilibria must be

(9) $$g = \lambda.$$

[Hint: Condition (M) must hold for all t.]

Exercise 2: Find the constant growth path for the model economy defined by $A = 1$, $\delta = 0.015$, $\alpha = 1$, $\lambda = .005$, $\theta = 0.30$, and $\rho = .01$. The time period is a quarter of a year. The depreciation is a quarterly rate as is the rate of productivity growth.

Given values for the parameters, the problem is to solve equations (1)–(9) for $(c, x, y, d, h, k, w, r, i \text{ and } g)$. Given the number of variables equals the number of equations, there is hope for a unique solution.

An *algorithm* is a sequence of steps to numerically compute the solution to a problem. A feature of an algorithm is that at any step variables whose values have been determined in earlier steps can be used. Variables whose values have not been determined in earlier steps, however, can not be used.

An algorithm to solve this system of equations, which establishes existence and uniqueness in a constructive way, is as follows:

Step 1: Use (9) to compute g and then (2) to compute *i.*

Step 2: Use (6) and *i* to compute *r*.

Step 3: Use (4) and *r* to compute h/k.

Step 4: Use (3) and the h/k. ratio computed in step 3 to compute w.

Step 5: Use (7) to eliminate *x* in (5) and then use the ratio computed in step 3 to eliminate *k*. The resulting equation is linear in *h* and *c*.

Step 6: Solve equation (1) and the equation determined in step 5 for *h* and *c*. Note this entails solving two linear equations in two unknowns. Note also that w has previously been computed in step 4.

Step 7: Given ratio h/k from step 3 and h from step 6, compute k.

Step 8: Given this k, use (7) to compute x and (8) to compute d.

Calibration

Calibrating to a constant growth path is the inverse operation of finding the constant growth path. Given some constant growth observations for an actual economy, the problem is to find values for the parameters such that the constant growth behavior of that model economy matches the growth observations for that actual economy.

Before the model economy can be calibrated to American data, the model must be extended to permit population growth, as the American population has not been constant as in the model economies that we have dealt with so far. Therefore, we extend the parametric class of model economies to ones where population grows at a constant rate—namely,

$$N_{t+1} = (1+\eta)N_t \qquad (10) \qquad (H3)$$

The preference of the households whose size is growing over time is now

$$\sum_t \frac{1}{(1+\rho)^t} N_t \left(\log c_t + \alpha \log\left(1-h_t\right) \right) ,$$

where c_t is per family member date t consumption and where h_t is the fraction of household members' date t productive time that is allocated to the market sector.

We now interpret all variables, except of course prices, as per capita numbers. With this change, equation (7) must be changed to

$$N_{t+1}k_{t+1} = (1-\delta)N_t k_t + N_t x_t$$

Substituting steady growth expressions for the variables

$$\frac{N_{t+1}}{N_t} k(1+g)^{t+1} = (1-\delta)k(1+g)^t + x(1+g)^t$$

Using H3

(7)′ $$x = (\delta + g + \eta + \eta g)k .$$

Why condition (2) must hold when there is population growth is not obvious. One way to see it to consider the *intertemporal budget constraint*

$$\sum_{t=0}^{\infty} p_t N_t c_t \le d_0 + \sum_{t=0}^{\infty} p_t N_t w_t h_t$$

The p_I's are the price of the consumption good at different dates. Therefore,

$$1 + i_t = p_t / p_{t+1} .$$

This just says that the relative price of the date t and date t+1 consumption good is $(1+i_t)$. Equating the ratio of the marginal utilities of c_t and c_{t+1} to their price ratio gives

$$\frac{(1+\rho)N_t / c_t}{N_{t+1} / c_{t+1}} = \frac{p_t N_t}{p_{t+1} N_{t+1}} = (1+i_t)\frac{N_t}{N_{t+1}} .$$

The population numbers cancel and when constant growth values are substituted, equation (2) is obtained.

Exercise 3: Suppose some economy displays the following growth facts:

(i) $k / y = 12.0$,

(ii) $x / y = 0.24$,

(iii) $N_{t+1} / N_t = 1.005$,

(iv) $y_{t+1} / y_t = 1.005$,

(v) $r\,k / y = 0.30$,

(vi) $h = 0.20$.

Here $y = c + x$ is aggregate output or income. A period is a quarter of a year. Calibrate the model economy to these observations; that is, find the δ, θ, α, η, λ and ρ for which the constant growth path has these values.

An algorithm to solve exercise 3 follows: [(i)–(vi) refers to growth facts, (1)–(6), (7') and (8)–(10) refer to equation numbers.]

Step 1: Compute η using equation (10) and fact (iii).

Step 2: Compute λ using fact (iv).

Step 3: Compute θ using fact (v). [Note from (4): $r\,k = \theta y$.]

Step 4: Compute δ using equation (7)′ and facts (i) and (ii).

Step 5: Compute r using facts (i) and (v).

Step 6: Compute ρ using r, g and δ and equations (2) and (6).

Step 7: Compute α using equations (1), (3), and (5) and facts (ii) and (vi). [Note that $(1-\theta)\ y = w\,h$. Note that this along with (1) implies $c\alpha\ / [y\ (1-\theta)] = (1-h)/h$. Note that $c / y = 1 - x / y$.]

There is one final check. You have found the model economy in the parametric class that matches the specified set of growth facts if there is one. Sometimes no model economy in the parametric set matches all the facts. Therefore, as a final step, for the parameters that you have found in the calibration you should determine the constant growth path and check whether all the facts hold for this path.

Another problem that often arises in calibrations is that there are many model economies in the parametric set of economies that have equilibria that match the observations. If so, the list of facts must be expanded and/or a smaller parametric set of model economies considered.

Exercise 4: Show that if the utility function is replaced by

$$\frac{(\ c^{\sigma}(1-h)^{1-\sigma})^{1-\gamma}\ -\ 1}{1-\gamma},$$

this set of observations does not determine all the parameters $\{\alpha, \delta, \rho, \gamma, \theta, \eta, \lambda\}$. Find parameters $\{\alpha, \delta, \theta, \eta, \lambda\}$ and the *set* of γ and ρ such that the balanced growth path displays facts (i)–(vi).