Dynamic Bargaining Theory

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ABSTRACT

The goal of this paper is to extend the analysis of strategic bargaining to nonstationary environments, where preferences or opportunities may be changing over time. We are mainly interested in equilibria where trade occurs immediately, once the agents start negotiating, but the terms of trade depend on when the negotiations begin. We characterize equilibria in terms of simply dynamical systems, and compare these outcomes with the myopic Nash bargaining solution. We illustrate the practicality of the approach with an application in monetary economics.

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1 Introduction

The approach to the bilateral bargaining problem introduced by Nash (1950) is inherently static, or timeless: it specifies preferences over a set of possible agreements and over failure to reach agreement, and establishes that there exists a unique outcome satisfying a number of axioms. By way of contrast,

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the strategic bargaining model developed by Rubinstein (1982) is dynamic: it specifies a sequence of moves, and preferences over the time of agreement as well as the terms of agreement. The model has a unique subgame-perfect equilibrium, in which there is immediate trade (i.e., the first offer is accepted). Moreover, as shown by Binmore (1987) and Binmore, Rubinstein and Wolinsky (1986), as the time between offers in the game vanishes, the equilibrium outcome converges to an appropriately specified Nash bargaining solution.

Although the Rubinstein model is dynamic in the above sense, it is stationary in the sense that the horizon is typically assumed to be infinite and the specification time-invariant. The goal of this paper is to extend the analysis of strategic bargaining to general nonstationary environments. For example, the payoffs implied by settlement may change over time, or the payoffs implied by a breakdown in the negotiations may change over time, because either preferences or economic conditions change (a special case of this is when the horizon is finite). Additionally, the probability of a breakdown may change over time, or the bargaining power of the agents (in terms of who gets to make the next offer) may change over time, for any number of reasons.[1]

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[1]We do not mean to suggest that no one has thought of these issues before. Some authors, going back to Stahl (1972), do consider finite horizons, which can be handled relatively easily using backward induction. Another source of nonstationarity that has been considered in some of the search and bargaining literature is the fact that when traders leave the market, if they are not replaced, the arrival rates for the remaining agents can change; see the survey in Osborne and Rubinstein (1990). Finally, Merlo and Wilson (1993a, 1993b) analyze models where the payoffs and bargaining power both follow discrete-time stochastic processes; however, they do not pursue the main issue of interest here, which is the limiting case when the time between offers goes to zero. In this case, we

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We would argue that there are many situations of economic interest where these dynamic considerations are relevant. A trivial but illustrative example concerns bargaining with a scalper over the price of a ticket while an event is in progress. One consideration is simple time preference: you may prefer to see the entire event sooner rather than later. But a different and potentially more important consideration is that your utility from going to events changes as events transpire. We are mainly concerned here with situations where trade takes place immediately. Nonetheless, the terms of agreement should depend, in general, on when negotiations begin. Thus, it is not that real time elapses while bargaining with the scalper, but rather that the price of a ticket will generally be a function of the time at which you locate him.

As is standard in strategic bargaining theory, even if there is immediate trade, it is the threat of delay that drives the solution. Since things are changing over time, our agents have to be forward looking when evaluating the effects of delay, and therefore an equilibrium in our model is the solution to a dynamical system. We think the assumption of forward looking agents is superior to the alternative of simply imposing the myopic Nash bargaining solution at each point in time, as has been done in some applications in the past. Although this alternative delivers the same steady states as our forward looking solution, the dynamic paths will generally be different.[2]

are able to characterize equilibria in terms of simple differential equations, and we are also able to provide a general discussion of the realtionship between equilibria and the Nash bargaining solution, things Merlo and Wilson do not attempt.

[2]Examples where the myopic Nash solution is imposed in dynamic models inlcude Pissarides (1987) and Trejos and Wright (1994); see also Mortensen (1989) and Drazen (1988). In some of these models agents' utility functions are assumed to be linear, in which case we will actually show that the myopic Nash solution coincides with the forward looking solution along the entire path, and not just in steady state. However, it is not

At the same time, we show that there does exist a time-varying Nash representation of equilibria in our forward looking model. This Nash representation generally differs from the one suggested by Binmore et al. (1986), unless we restrict attention to steady states. Restricting attention to steady states, we derive Nash representations for a fairly general class of environments and demonstrate how several results in the literature (including those in Binmore et al.) emerge as special cases.

We then illustrate the practicality of the approach with an application in search theory. In particular, we analyze the model of search and bargaining in a monetary economy studied in Trejos and Wright (1994) and Shi (1994). Those papers focus on steady states, or, when dynamics are discussed, impose the myopic Nash bargaining solution. Using our forward looking bargaining solution, we find that in some cases the set of equilibria is qualitatively the same but quantitatively different. In other cases the nature of the equilibrium changes qualitatively. For example, there can exist limit cycles in the price level if we adopt the forward looking solution, but not if we impose the myopic Nash solution.

The rest of the paper is organized as follows. In Section 2 we present a simple version of the model in order to make the key points. In Section 3 we present several extensions and discuss Nash representations. In Section 4 we take up applications. The final section provides some brief concluding remarks. The Appendix contains proofs of some technical results.

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only interesting to consider nonlinear utility, it is sometimes essential. For example, in the monetary model in Trejos and Wright (1994), there does not exist a steady state with valued fiat currency unless we allow nonlinear utility.

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2   The Basic Model

Time proceeds as an infinite sequence of discrete periods of length $\Delta >$ 0; eventually, we will consider the limiting case where $\Delta \to 0$. There are two agents, labeled $i = 1, 2$. Agent 1 has an indivisible good and agent 2 has a divisible good, and they are potentially interested in trade. We could interpret the indivisible good as an *action*; for example, agent 2 may want agent 1 to enter into some sort of relationship, such as employment or marriage, and is willing to pay some amount of his divisible good in order to achieve this end.

If at time t agent 1 trades the indivisible good to agent 2 for q units of the divisible good, their instantaneous utilities are $u_1(q, t)$ and $u_2(q, t)$. Additionally, they discount the future at rate r, so that the payoff from this trade for i, discounted back to the present, is $e^{-rt}u_i(q, t)$. Assume $u_i \in C^\infty$, $\partial u_1/\partial q > 0$ and $\partial u_2/\partial q < 0$, for all t. Assume that u_1 and u_2 are concave in q for all t. Also, agents derive some utility from not trading at all, normalized to 0. For all t, assume that $u_i(q, t) > 0$ for some $q > 0$, so that there exist gains from trade, and that $u_1(q, t) < 0$ for q sufficiently small (potentially negative) and $u_2(q, t) < 0$ for q sufficiently large.[3] We also assume that $u_i(q, t)$ is bounded in t, and that the time derivative $\partial u_i(q, t)/\partial t$ exists, and is bounded, for all (q, t).

The agents meet at some time t, at which point nature chooses one of them at random to propose a value of q, to which the other can respond by either accepting or rejecting the offer. If he accepts, exchange takes place,

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[3]This insures that q is bounded. In principle, we could also impose constraints on the amount of the divisible good that can be traded, say $q \in [0, \hat{q}]$; but we simply assume that such constraints are not binding in most of what follows.

the payoffs are realized, and the game ends. If he rejects, they realize no instantaneous utility that period, and the game moves on to the next period where nature again chooses a proposer at random.[4] This continues until an offer is accepted. The original Rubinstein (1982) model is a special case where $u_1(q,t) = q$ and $u_2(q,t) = 1 - q$; the extensions to nonlinearity and nonstationarity can both be important.

Our goal is to characterize subgame-perfect equilibria in strategies that are history independent, although potentially nonstationary. By definition, history independent strategies do not depend on offers made at previous points in time. However, offers can depend on time because, in general, preferences or opportunities do. We focus here mainly on the case where trade occurs as soon as the agents meet - that is, the first proposal is accepted - an outcome we call an Immediate Trade Equilibrium, or ITE. Although there is considerable interest in the literature in conditions under which there may be a delay in reaching agreement (see, e.g., Merlo and Wilson [1993a, 1993b], or the models surveyed in Osborne and Rubinstein [1990]), our objective is different. We are interested in the case where agreement is immediate, but the nature of the agreement may depend on when the agents meet and negotiations begin.

Generally, immediate trade requires that the present discounted value of the surplus over which the agents are bargaining is decreasing with time. Let

$$\mathcal{A}(t) = \{q : u_i(q,t) > 0, i = 1, 2\}. \tag{1}$$

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[4]Actually, the Rubinstein (1982) model assumed that the identity of the proposer alternated deterministically between periods, rather than being determined randomly each period. We can consider alternating offers as a special case of the generalized version of the model in Section 3.

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By assumption, $\mathcal{A}(t)$ is nonempty. Then suppose that $e^{-rt}u_i(q,t)$ is decreasing in t as long as $q \in \mathcal{A}(t)$ for all t, and strictly decreasing for at least one agent. We call this the "shrinking pie" assumption, and show below how it is used to guarantee immediate trade. First, we describe the features of an ITE assuming that one exists.

In any equilibrium with history independent strategies, if 1 is willing to accept q at t and $q' > q$, then 1 must also be willing to accept q' at t. Similarly, if 2 is willing to pay q at t and $q' < q$ then 2 must also be willing to pay q' at t. Hence, there exist *reservation values*, $q_1(t)$ and $q_2(t)$, such that at time t agent 1 accepts any $q \geq q_1(t)$ and agent 2 accepts any $q \leq q_2(t)$. Moreover, given an agent wants to trade, the best he can do is to propose the reservation value of the other agent. This implies that we can identify an ITE strategy profile with $[q_1(t), q_2(t)]$, where at time t agent 1 proposes $q_2(t)$ if it is his turn to make an offer and accepts any $q \geq q_1(t)$ if it is his turn to respond, while agent 2 proposes $q_1(t)$ if it is his turn to make an offer and accepts any $q \leq q_2(t)$ if it is his turn to respond.

In any ITE, therefore, the reservation values satisfy the following recursive relations:

$$u_1[q_1(t), t] = \frac{1}{1 + r\Delta} \left\{ \tfrac{1}{2} u_1[q_1(t + \Delta), t + \Delta] + \tfrac{1}{2} u_1[q_2(t + \Delta), t + \Delta] \right\} \quad (2)$$

$$u_2[q_2(t), t] = \frac{1}{1 + r\Delta} \left\{ \tfrac{1}{2} u_2[q_1(t + \Delta), t + \Delta] + \tfrac{1}{2} u_2[q_2(t + \Delta), t + \Delta] \right\} \quad (3)$$

For example, (2) says that agent 1 is indifferent between accepting his reservation value at t, or delaying until $t+\Delta$ when a new proposer will be determined at random. These equations are forward looking, in the sense that reservation values today are defined in terms of reservation values next period.

In a model where nothing changes over time, (2) and (3) determine a pair

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of numbers (q_1, q_2). Here, they constitute a system of difference equations which determine paths $[q_1(t), q_2(t)]$. Of course, generally there are many solutions to these difference equations, and so we need to discuss appropriate boundary conditions. We begin with some preliminary technical results.

Lemma 1 *In ITE, $u_i[q(t), t] \geq 0$ and $q_i(t)$ is bounded in t.*

Proof: First, $u_i[q_i(t), t] \geq 0$ because i can always reject all offers. Then q_i must be bounded, since otherwise $u_i[q_i(t), t]$ becomes negative for one i. $\square$

Now consider the average of the two offers,

$$q(t) = \tfrac{1}{2}\left[q_1(t) + q_2(t)\right]. \tag{4}$$

The next result indicates that, as $\Delta \to 0$, $q_1(t)$ and $q_2(t)$ both approach $q(t)$. Hence, as is standard in the sequential bargaining framework, the advantage to being the proposer vanishes with the time between offers.

Lemma 2 *In ITE, for all t, as $\Delta \to 0$, $q_1(t)$ and $q_2(t)$ converge to $q(t)$ at rate Δ.*

Proof: See the Appendix. $\square$

The next step is to describe the behavior of the average offer $q(t)$, which, by Lemma 2, may be regarded as an approximation to $q_1(t)$ and $q_2(t)$.

Theorem 1 *In ITE, in the limit as $\Delta \to 0$, $q(t)$ satisfies the following differential equation:*

$$\dot{q} = \frac{1}{2}\left[\frac{ru_1(q, t) - \partial u_1(q, t)/\partial t}{\partial u_1(q, t)/\partial q}\right] + \frac{1}{2}\left[\frac{ru_2(q, t) - \partial u_2(q, t)/\partial t}{\partial u_2(q, t)/\partial q}\right]. \tag{5}$$

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Proof: Let $\varepsilon(t) = q_1(t) - q(t) = q(t) - q_2(t)$. Then a first order Taylor approximation allows us to write (2) and (3) as

$$u_1[q(t),t] + \varepsilon(t)\frac{\partial u_1[q(t),t]}{\partial q} = \frac{u_1[q(t+\Delta),t+\Delta]}{1+r\Delta} + o(\Delta)$$

$$u_2[q(t),t] - \varepsilon(t)\frac{\partial u_2[q(t),t]}{\partial q} = \frac{u_2[q(t+\Delta),t+\Delta]}{1+r\Delta} + o(\Delta)$$

where $o(\Delta)$ denotes any function with the property that $\frac{o(\Delta)}{\Delta} \to 0$ as $\Delta \to 0$. If we multiply the first of these by $\partial u_2[q(t),t]/\partial q$ and the second by $\partial u_1[q(t),t]/\partial q$, then add the equations and rearrange, we have

$$\left\{ u_1[q(t),t] - \frac{u_1[q(t+\Delta),t+\Delta]}{1+r\Delta} \right\} \frac{\partial u_2[q(t),t]}{\partial q}$$

$$+ \left\{ u_2[q(t),t] - \frac{u_2[q(t+\Delta),t+\Delta]}{1+r\Delta} \right\} \frac{\partial u_1[q(t),t]}{\partial q} = o(\Delta). \tag{6}$$

Now multiply by $1 + r\Delta$, divide by Δ, and take the limit as $\Delta \to 0$ to get

$$\left(ru_1 - \frac{\partial u_1}{\partial q}\dot{q} - \frac{\partial u_1}{\partial t} \right) \frac{\partial u_2}{\partial q} + \left(ru_2 - \frac{\partial u_2}{\partial q}\dot{q} - \frac{\partial u_2}{\partial t} \right) \frac{\partial u_1}{\partial q} = 0,$$

where for ease of notation we drop the arguments of $u_i[q(t),t]$. Solving for $\dot{q}$, we arrive at (5). $\square$

Equilibrium condition (5) may be contrasted with the solution to a static or myopic Nash bargaining problem,[5]

$$q(t) = \arg\max u_1(q,t)u_2(q,t). \tag{7}$$

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[5]This particular Nash problem — with equal bargaining power and zero threat points — is the appropriate one for the setup at hand *if* we restrict attention to steady states. We shall discuss this in detail in Section 3.

The solution to (7) is characterized by

$$u_1(q,t)\frac{\partial u_2(q,t)}{\partial q} + u_1(q,t)\frac{\partial u_2(q,t)}{\partial q} = 0,$$

which generates a time path for q that differs from the solution to (5), except in special cases (see below). However, if the functions u_i settle down over time, then in ITE q converges to a steady state that coincides with the myopic Nash solution.

Theorem 2 *Suppose $u_i(q,t) \to \bar{u}_i(q)$ as $t \to \infty$, $i = 1, 2$. Then, if an ITE exists it is unique, and $q \to \bar{q}$ as $t \to \infty$, where*

$$\bar{q} = \arg\max \bar{u}_1(q)\bar{u}_2(q).$$

Proof: If $u_i(q,t) = \bar{u}_i(q)$ then (5) becomes

$$\dot{q} = \tfrac{1}{2}\left(\frac{r\bar{u}_1}{\bar{u}_1'} + \frac{r\bar{u}_2}{\bar{u}_2'}\right) \equiv \Upsilon(q).$$

The solution to $\Upsilon(q) = 0$ is $\bar{q}$. Moreover, $\Upsilon'(q) > 0$. This implies that if $q(t) > \bar{q}$ in the limit then $q(t)$ increases without bound, and if $q(t) < \bar{q}$ in the limit then $q(t)$ decreases without bound. But Lemma 1 says that q is bounded, and so it must converge to $\bar{q}$. Given $q \to \bar{q}$, there is a unique solution to differential equation (5) and therefore a unique ITE. $\square$

A more general discussion of the relation between steady states of forward looking equilibria and the myopic Nash solution will be provided in Section 3. First, we show that there is a special case in which the equilibrium q agrees with the myopic Nash solution along the entire path, and not just in steady state.

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Theorem 3 *Suppose $u_i(q,t) = \eta_i q + \varphi_i(t)$, where $\eta_1 > 0 > \eta_2$ and $\varphi_i(t)$ is bounded for all t. Then, if an ITE exists it is unique, and q satisfies the myopic Nash condition (7) for all t.*

Proof: With no loss in generality, normalize $\eta_1 = 1$ and $\eta_2 = -1$. Then the equilibrium condition (5) reduces to

$$\dot{q} = \tfrac{1}{2}\left(rq + r\varphi_1 - \dot{\varphi}_1\right) - \tfrac{1}{2}\left(-rq + r\varphi_2 - \dot{\varphi}_2\right),$$

which implies

$$\dot{q} - rq = \tfrac{1}{2}\left[(\dot{\varphi}_2 - \dot{\varphi}_1) - r(\varphi_2 - \varphi_1)\right].$$

Solutions to this equation are of the form

$$q = \tfrac{1}{2}(\varphi_2 - \varphi_1) + \eta_0 e^{rt},$$

where η_0 is a constant. Since q is bounded by Lemma 1, we have $\eta_0 = 0$ and $q = \tfrac{1}{2}(\varphi_2 - \varphi_1)$. In the case under consideration, this is also the myopic Nash solution. $\square$

Up to now, we have been analyzing ITE under the assumption that it exists. The next step is to discuss the conditions that make this valid.

To this end, let $\Pi_i(t) = e^{-rt}u_i[q(t), t]$ be the discounted payoff to i, given a solution q to (5). Then immediate trade at q for all t constitutes an equilibrium if and only if

$$\Pi_i(t) \geq 0 \quad and \quad \Pi_i'(t) \leq 0 \tag{8}$$

for all t and $i = 1, 2$. The first inequality says the agents always want to trade, while the second says they always want to trade sooner rather than

later. In any application, given a path q that solves (5), one must check that (8) is satisfied in order to verify that q constitutes an equilibrium.

Rather than verifying (8) directly in each application, note that $\Pi_i'(t) < 0$ in general if and only if

$$\left(ru_1 - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial q} - \left(ru_2 - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial q} < 0. \qquad (9)$$

The "shrinking pie" assumption guarantees that (9) holds. Furthermore, although a general existence proof is beyond the scope of this paper, under the "shrinking pie" assumption we easily can guarantee that a subgame-perfect equilibrium exists and entails immediate trade, at least as long as $u_i(q,t) \rightarrow \bar{u}_i(q)$ and $\mathcal{A} \equiv \lim \mathcal{A}(t) \neq \emptyset$, where $\mathcal{A}(t)$ was defined in (1). Existence follows because Theorem 2 implies the unique limiting solution satisfies $\bar{q} \in \mathcal{A}$, and continuity plus the "shrinking pie" assumption guarantee that (8) holds along the entire path.

Even if there exists a unique ITE, there could in principle exist other subgame-perfect equilibria, even given the "shrinking pie" assumption.[6] However, uniqueness is straightforward in one special case, which is the case in which there exists $\hat{t}$ such that $u_i(q,t) = \bar{u}_i(q)$ for all $t \geq \hat{t}$, $i = 1, 2$. Then there is a unique subgame-perfect equilibrium for $t \geq \hat{t}$ by standard arguments (see, e.g., Binmore [1987] or Shaked and Sutton [1984]). Backward induction then yields the desired result. In any case, we are more concerned here with characterizing ITE, rather than either the general existence or uniqueness of subgame-perfect equilibria.

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[6]Binmore (1987) constructs a discrete-time example with a "shrinking pie" where there may be multiple equilibria. However, if we consider his example in the limit as $\Delta \rightarrow 0$, it can be shown that multiplicity can only arise if (in our notation) $u_i(q,t)$ has an unbounded time derivative, which we have ruled out by assumption.

We close this section with an example. First, suppose $u_1(q,t) = q^\rho$, with $0 < \rho < 1$, so that agent 1's utility is stationary but nonlinear. Then suppose $u_2(q,t) = e^{-\delta t} - q$. One interpretation is that the indivisible good is depreciating at rate δ; or, if $\delta < 0$, it is appreciating, but as long as $r > 0$, the agents may still want to trade sooner rather than later. Since the u_i functions settle down over time, Theorem 2 implies $q(t) \to \bar{q}$, where in this case $\bar{q} = 0$.

For these functional forms, equilibrium condition (5) can be written

$$\dot{q} = \frac{r(1+\rho)q - \rho(r+\delta)e^{-\delta t}}{2\rho}.$$

The solution to this differential equation subject to the boundary condition $q(t) \to 0$ is easily verified to be

$$q^* = \frac{\rho(r+\delta)e^{-\delta t}}{r(1+\rho) + 2\delta\rho},$$

and the implied payoffs are

$$\Pi_1 = \left[\frac{\rho(r+\delta)}{r(1+\rho)+2\delta\rho} \right]^\rho e^{-(r+\rho\delta)t}$$

$$\Pi_2 = \frac{(r+\delta)}{r(1+\rho)+2\delta\rho} e^{-(r+\delta)t}.$$

If $r + \delta > 0$ then $\Pi_i'(t) < 0$, which guarantees that immediate trade is an equilibrium.

By way of comparison, with these utility functions, the solution to the myopic Nash problem (7) is

$$q^n = \frac{\rho e^{-\delta t}}{1+\rho}.$$

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Observe that

$$q^* - q^n = \frac{\delta\rho(1-\rho)e^{-\delta t}}{(1+\rho)\left[r(1+\rho) + 2\delta\rho\right]}.$$

Hence, $q^* > q^n$ if and only if $\delta > 0$. The intuition is that Π_1 falls more slowly than Π_2 along the equilibrium path when $\delta > 0$, effectively making agent 1 less averse to delay. The myopic Nash solution ignores this, while the forward looking solution takes it into account and therefore gives a bigger payoff to agent 1.[7]

3 Generalization and Discussion

In this section we analyze some extensions to the basic model, and pursue the relation between the equilibria of strategic bargaining games and the Nash solution. We incorporate several generalizations at once, including different rates of time preference, different probabilities of getting to make the next offer, and exogenous breakdowns.

Let r_i be the discount rate of agent i. Let γ_i be a flow utility that i gets while bargaining is in progress. Let λ_i be the Poisson arrival rate with which i believes exogenous breakdowns in the bargaining will occur, and b_i his utility in this event (we do not necessarily impose $\lambda_1 = \lambda_2$). If there is no breakdown, the next offer is made by agent i with probability π_i, where $\pi_1 + \pi_2 = 1$. Note that we allow r_i, γ_i, π_i, λ_i, and b_i to depend on time, although to save space we do not make this dependence explicit in the notation.[8]

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[7]These results hold for $\rho < 1$. If $\rho = 1$, then $q^n = q^*$, as predicted by Theorem 3.

[8]In particular, in discrete time, we can allow $\pi_1(t)$ to equal 1 at one date and 0 at the next date, which is the original Rubinstein (1982) assumption of alternating offers.

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Theorem 4 *In ITE, in the limit as* $\Delta \to 0$, $q(t)$ *satisfies the following generalization of (5):*

$$\dot{q} = \pi_2 \left[\frac{(r_1 + \lambda_1)u_1 - \gamma_1 - \lambda_1 b_1 - \partial u_1/\partial t}{\partial u_1/\partial q} \right]$$

$$+ \pi_1 \left[\frac{(r_2 + \lambda_2)u_2 - \gamma_2 - \lambda_2 b_2 - \partial u_2/\partial t}{\partial u_2/\partial q} \right].$$

(10)

Proof: The generalized versions of (2) and (3) are

$$u_1[q_1(t), t] = \frac{1 - \lambda_1 \Delta}{1 + r_1 \Delta} \left\{ \pi_1 u_1[q_2(t + \Delta), t + \Delta] + \pi_2 u_1[q_1(t + \Delta), t + \Delta] \right\}$$

$$+ \frac{\gamma_1 \Delta + \lambda_1 \Delta b_1}{1 + r_1 \Delta} + o(\Delta)$$

$$u_2[q_2(t), t] = \frac{1 - \lambda_2 \Delta}{1 + r_2 \Delta} \left\{ \pi_1 u_1[q_2(t + \Delta), t + \Delta] + \pi_2 u_1[q_1(t + \Delta), t + \Delta] \right\}$$

$$+ \frac{\gamma_2 \Delta + \lambda_2 \Delta b_2}{1 + r_2 \Delta} + o(\Delta),$$

where $o(\Delta)$ appears because $\lambda_i \Delta + o(\Delta)$ is the probability of a breakdown, by the Poisson assumption. For any t, let $q = \pi_1 q_1 + \pi_2 q_2$ and $\varepsilon = q_1 - q_2$. Notice that $q_1 - q = \pi_1 \varepsilon$ and $q_2 - q = -\pi_2 \varepsilon$. Then, as in the proof of Theorem 1, approximate the above equations around q:

$$u_1[q(t), t] + \pi_1 \varepsilon \frac{\partial u_1[q(t), t]}{\partial q} = \frac{1 - \lambda_1 \Delta}{1 + r_1 \Delta} u_1[q(t + 1), t + 1]$$

$$+ \frac{\gamma_1 \Delta + \lambda_1 \Delta b_1}{1 + r_1 \Delta} + o(\Delta)$$

$$u_2[q(t), t] - \pi_2 \varepsilon \frac{\partial u_2[q(t), t]}{\partial q} = \frac{1 - \lambda_2 \Delta}{1 + r_2 \Delta} u_2[q(t + 1), t + 1]$$

$$+ \frac{\gamma_2 \Delta + \lambda_2 \Delta b_2}{1 + r_2 \Delta} + o(\Delta).$$

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Multiply the first by $\pi_2 \partial u_2[q(t), t]/\partial q$ and the second by $\pi_1 \partial u_1[q(t), t]/\partial q$, then add these equations to get the following generalization of (6):

$$\pi_2\{u_1[q(t), t] - \frac{\gamma_1 \Delta + \lambda_1 \Delta b_1}{1 + r_1 \Delta} - \frac{1 - \lambda_1 \Delta}{1 + r_1 \Delta} u_1[q(t + \Delta), t + \Delta]\} \frac{\partial u_2[q(t), t]}{\partial q}$$

$$+\pi_1\{u_2[q(t), t] - \frac{\gamma_2 \Delta + \lambda_2 \Delta b_2}{1 + r_2 \Delta} - \frac{1 - \lambda_2 \Delta}{1 + r_2 \Delta} u_2[q(t + \Delta), t + \Delta]\} \frac{\partial u_1[q(t), t]}{\partial q}$$

$$= o(\Delta). \tag{11}$$

Finally, multiply (11) by $(1 + r_1 \Delta)(1 + r_2 \Delta)$, divide by Δ, let $\Delta \to 0$, and simplify to arrive at the differential equation (10). □

There are several reasons for considering these extensions to the basic model. One is that, in some applications, things like different probabilities of making the offer or breakdowns can be interesting. Another is that we can provide a general discussion of the relation between the equilibrium of the dynamic bargaining game and the myopic Nash solution.

Suppose we are interested in representing $q(t)$ at each point in time (and not just in steady state) for arbitrary $u_i(q, t)$ (and not just for special functional forms), by the generalized Nash bargaining problem

$$q(t) = \arg\max [u_1(q, t) - T_1]^\theta [u_2(q, t) - T_2]^{1-\theta}, \tag{12}$$

where T_i is the threat point point of of agent i and θ is the bargaining power of agent 1. Note that T_i and θ may depend on time, in general. Then we have the following result.

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Theorem 5 *Suppose we have an ITE, and consider the case where Δ is small. If we want a Nash representation for the equilibrium path for q that applies for arbitrary utility functions, the unique choice for the threat points and bargaining power at each date t is given by:*

$$T_i(t) = \frac{\gamma_i \Delta + \lambda_i \Delta b_i + (1 - \lambda_i \Delta) u_i[q(t+\Delta), t+\Delta]}{1 + r_i \Delta}$$

$$\theta(t) = \pi_1.$$

Proof: A necessary and sufficient condition for q to solve the generalized Nash problem (12) is

$$(1 - \theta)[u_1(q,t) - T_1]\frac{\partial u_2}{\partial q} + \theta[u_2(q,t) - T_2]\frac{\partial u_1}{\partial q} = 0. \tag{13}$$

Comparing (13) and (11), the result is immediate. $\square$

Theorem 5 has the natural interpretation that the bargaining power of an agent is the probability that he gets to make an offer, and the threat point is the continuation payoff he can expect to get by rejecting an offer: the flow $\gamma_i \Delta$, plus the probability of a breakdown times b_i, plus the probability of no breakdown times the equilibrium utility from settling next period, appropriately discounted. Although it is the only interpretation that applies in general, we can come up with an alternative Nash representation if we restrict attention to the linear functional forms used in Theorem 3. Recall that these utility functions imply the equilibrium q agrees with the myopic Nash solution along the entire path in the simple model of the previous section; the next result extends this to the generalized model of this section.

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Theorem 6 *Suppose we have an ITE, and consider the case where Δ is small. Assume $u_i(q,t) = \eta_i q + \varphi_i(t)$, where $\eta_1 > 0 > \eta_2$ and $\varphi_i(t)$ is bounded for all t. Without loss of generality, let $\eta_1 = 1$ and $\eta_2 = -1$. Then, if π_i does not depend on t and $r_i + \lambda_i = r + \lambda$ does not depend on i (but could depend on t) there is an alternative Nash representation for the equilibrium path q, given by*

$$T_i(t) = \int_t^\infty e^{-R(t,\tau)} \left[\gamma_i(\tau) + \lambda_i(\tau)b_i(\tau)\right] d\tau$$

$$\theta(t) = \pi_1,$$

where $R(t,\tau) = \int_t^\tau [r(\sigma) + \lambda(\sigma)]d\sigma$. In particular, if γ_i, λ_i, b_i, and r_i do not depend on t then

$$T_i(t) = \frac{\gamma_i + \lambda_i b_i}{r_i + \lambda_i}.$$

Proof: Normalize $\eta_1 = 1$ and $\eta_2 = -1$. Then for these preferences the solution to the myopic Nash problem (12) is

$$q = \theta(\varphi_2 - T_2) - (1 - \theta)(\varphi_1 - T_1). \tag{14}$$

By virtue of (10), the equilibrium path satisfies

$$\dot{q} - [\pi_2(r_1 + \lambda_1) + \pi_1(r_2 + \lambda_2)]q = \pi_2[(r_1 + \lambda_1)\varphi_1 - \dot\varphi_1]$$

$$\tag{15}$$

$$-\pi_1[(r_2 + \lambda_2)\varphi_2 - \dot\varphi_2] - \pi_2[\gamma_1 + \lambda_1 b_1] + \pi_1[\gamma_2 + \lambda_2 b_2].$$

We want to know when we can find a solution to (15) of the form (14).

Notice (15) is linear and first order, and can be integrated using an integrating factor. In general, such a solution will be a complicated integral of future values of φ and $\dot\varphi$. There is a special case, however, where this does not occur: when π_i does not depend on t, and $\dot{q}$, $\dot\varphi_1$ and $\dot\varphi_2$ have a common

integrating factor. Since $\pi_1 + \pi_2 = 1$, a necessary and sufficient condition for this to occur is $r_i + \lambda_i = r + \lambda$ for both i.

In this case, (15) simplifies to

$$\dot{q} - (r + \lambda)q = \pi_2[(r + \lambda)\varphi_1 - \dot{\varphi}_1] - \pi_1[(r + \lambda)\varphi_2 - \dot{\varphi}_2]$$

$$- \pi_2[\gamma_1 + \lambda_1 b_1] + \pi_1[\gamma_2 + \lambda_2 b_2].$$

Integrating, applying Lemma 1, and using the assumption that π_i does not depend on t yields

$$q = \pi_1\varphi_2 - \pi_2\varphi_1 + \int_t^\infty e^{-R(t,\tau)} \left[\pi_2(\gamma_1 + \lambda_1 b_1) - \pi_1(\gamma_2 + \lambda_2 b_2)\right] d\tau,$$

where $R(t,\tau)$ is defined in the statement of the theorem. Comparing this with (14) yields the desired threat points and bargaining power. The special case where the parameters are time-invariant follows from simplifying T_i. $\square$

In the Nash representation of Theorem 6, the threat points are the pay-offs from disagreeing forever, appropriately discounted. This has an advantage over the representation in Theorem 5, in that T_i depends only on the exogenous parameters and not the endogenous continuation values $u_i[q(t + \Delta), t + \Delta]$. A similar result emerges if we allow general functional forms, but restrict attention to steady states.

Theorem 7 *Suppose we have an ITE, and consider the case where Δ is small. Assume the parameters and utility functions settle down with time. Then, in steady state there is a Nash representation for the equilibrium given*

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by

$$T_i = \frac{\gamma_i + \lambda_i b_i}{r_i + \lambda_i}$$

$$\theta = \frac{\pi_1(r_2 + \lambda_2)}{\pi_1(r_2 + \lambda_2) + \pi_2(r_1 + \lambda_1)}.$$

If $r_i + \lambda_i = r + \lambda$ does not depend on i, as in Theorem 6, then

$$\theta = \pi_1.$$

Proof: In steady state, $u_i(q,t) = \bar{u}(q)$ and q is constant. Inserting this into equation (11) and rearranging yields:

$$\pi_2(r_1 + \lambda_1)\left(\bar{u}_1 - \frac{\gamma_1 + \lambda_1 b_1}{r_1 + \lambda_1}\right)\bar{u}_2' + \pi_1(r_2 + \lambda_2)\left(\bar{u}_2 - \frac{\gamma_2 + \lambda_2 b_2}{r_2 + \lambda_2}\right)\bar{u}_1' = 0 \quad (16)$$

Comparing this with (13), the desired result follows immediately. □

With Theorem 7 in hand, it seems worth mentioning that several results in the literature can be obtained as special cases. For example, set $\lambda_i = \gamma_i = 0$; then $T_i = 0$ and $\theta = \pi_1 r_2/(\pi_1 r_2 + \pi_2 r_1)$, which is the Nash representation derived in Binmore et al. (1986) for what they call their time preference model. In particular, if $\pi_1 = \pi_2$ and $r_1 = r_2$, this is the equal weight Nash solution with zero threat points we used in (7). Now set $r_i = \gamma_i = 0$ but relax the assumption $\lambda_i = 0$; then $T_i = b_i$ and $\theta = \pi_1 \lambda_2/(\pi_1 \lambda_2 + \pi_2 \lambda_1)$, which is the representation in Binmore et al. for their model with exogenous breakdowns.

An important case is the one where after a breakdown agent i may meet a new partner, as in many models of search and bargaining. Let the arrival rate of new partners for i be α_i; then the usual dynamic programming equation from search theory implies that (in steady state)

$$r_i b_i = \alpha_i(\bar{u}_i - b_i). \quad (17)$$

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In this case, it is useful to rewrite (16) as

$$\pi_2\left[(r_1 + \lambda_1)(\bar{u}_1 - b_1) + r_1 b_1\right]\bar{u}_2' + \pi_1\left[(r_2 + \lambda_2)(\bar{u}_2 - b_2) + r_2 b_2\right]\bar{u}_1' = 0, \quad (18)$$

where purely for simplicity we have set $\gamma_i = 0$.

Inserting (17) into(18), we have

$$\pi_2(r_1 + \lambda_1 + \alpha_1)(\bar{u}_1 - b_1)\bar{u}_2' + \pi_1(r_2 + \lambda_2 + \alpha_1)(\bar{u}_2 - b_2)\bar{u}_1' = 0. \quad (19)$$

This implies that q has a Nash representation with $T_i = b_i$ and

$$\theta = \frac{\pi_1(r_2 + \alpha_2 + \lambda_2)}{\pi_1(r_2 + \alpha_2 + \lambda_2) + \pi_2(r_1 + \alpha_1 + \lambda_1)}. \quad (20)$$

Alternatively, we can eliminate b_i entirely from (18) and write

$$\frac{\pi_2(r_1 + \lambda_1 + \alpha_1)r_1}{r_1 + \alpha_1}\bar{u}_1\bar{u}_2' + \frac{\pi_1(r_2 + \lambda_2 + \alpha_1)r_2}{r_2 + \alpha_2}\bar{u}_2\bar{u}_1' = 0. \quad (21)$$

This implies that q has an equivalent Nash representation with $T_i = 0$ and

$$\theta = \frac{\pi_1 r_2(r_1 + \alpha_1)(r_2 + \alpha_2 + \lambda_2)}{\pi_1 r_2(r_1 + \alpha_1)(r_2 + \alpha_2 + \lambda_2) + \pi_2 r_1(r_2 + \alpha_2)(r_1 + \alpha_1 + \lambda_1)}. \quad (22)$$

A special case in much of the search and bargaining literature is the one where the only source of breakdowns is that new agents may arrive during the bargaining, and when a new type i agent arrives he replaces the incumbent (see, e.g., Rubinstein and Wolinsky [1985], Wolinsky [1987], or Binmore and Herarro [1988]). Hence, the breakdown rate for type 1 is the arrival rate for type 2 and vice-versa: $\lambda_1 = \alpha_2$ and $\lambda_2 = \alpha_1$.

To pursue this case, let $r_1 = r_2$ and $\pi_1 = \pi_2$ in order to reduce notation. Then for the representation with $T_i = b_i$ and θ given by (20), it is easy to check that $\theta = \frac{1}{2}$ even if $\alpha_1 \neq \alpha_2$; in this case, different arrival rates show up in terms of different threat points but the same bargaining power.

Alternatively, in the representation with $T_i = 0$ and θ given by (22), we have $\theta \neq \frac{1}{2}$ when $\alpha_1 \neq \alpha_2$; in this case, different arrival rates show up in terms of different bargaining power and the same threat point. This illustrates how features of a model like different arrival rates can be captured either by different bargaining power or by different threat points, at least if we are interested only in steady states.

To close this section, suppose that agent i has the choice of whether to enter into the bargaining game at t, and let Φ_i be his payoff if he declines. This may be interpreted as the utility he derives from autarchy, or, in a search context, the expected utility of waiting for another bargaining opportunity in the future. As with all of the other parameters, Φ can vary with time. Then i is always willing to bargain if and only if the following condition holds for all t:

$$u_i[q(t), t] \geq \Phi_i(t). \tag{23}$$

In applications where there is some utility associated with not bargaining, it is important to check that this constraint holds in order for our solution to constitute an equilibrium.[9]

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[9]It is possible for our bargaining solution to violate (23) for one or both agents. If it violates it for both agents, there are no gains from trade, and they will not bargain. If it violates if for one agent i, then there are gains from trade, but i will not accept the q implied by our bargaining solution. In this case, the best the other agent can do is to offer i the q that solves (23) with equality, in which case i gets none of the gains from trade but is still willing to agree.

4 An Application in Monetary Theory

In this section, we apply our bargaining solution to a version of the monetary model described in Shi (1994) and Trejos and Wright (1994). The model has a large number of infinitely lived agents who meet in an anonymous random matching process, where all exchange is bilateral and quid pro quo. There are k types of agents and k goods, $k > 2$, with the property that type j only consumes good j and only produces good $j + 1$ (modulo k). This rules out direct barter. Also, goods are nonstorable. This rules out commodity money. Hence, if trade occurs at all in this economy, it requires the use of fiat currency.[10]

At $t = 0$, a fraction $M \in [0, 1]$ of the population are each endowed with one unit of fiat currency, and the rest with production opportunities. For simplicity, it is assumed that when agents spend their money they spend all of it, and, except for those initially endowed with production opportunities, no agent can produce until after he consumes. This implies that at *every* point in time there will be M agents with one unit of money (called buyers) and $1 - M$ agents with production opportunities (called sellers).

If a buyer trades his unit of money to a seller for q units of output, the nominal price is $p = 1/q$. Consumption of q units of one's consumption good generates utility $U(q)$, while production of one's production good generates

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[10]Versions of the model that allow direct barter, commodity money, and other complications are contained in Kiyotaki and Wright (1989, 1991, 1993), although those papers only consider steady states, and impose simplistic bargaining rules. A somewhat related paper by Casella and Feinstein (1990) studies bargaining when the aggregate price level is changing. In that model, however, buyers are in the market for a finite number of periods, and so the solution is easily computed using backward induction.

disutility $c(q)$. With no loss in generality, assume that $U(q) = q$. Also assume that $0 < c'(0) < 1$, $c''(q) > 0$ for all $q \geq 0$, and $c(q) < q$ for large q. For now, assume $c(0) = 0$, although we consider a fixed cost below by allowing $c(0) > 0$.

Agents meet randomly according to a Poisson process with arrival rate α, so that the rate at which a seller meets buyers is αM and the rate at which a buyer meets sellers is $\alpha(1 - M)$. When a buyer meets a seller who can produce his consumption good, if they trade then the buyer becomes a seller and the seller becomes a buyer. A random seller produces the consumption good of a buyer with probability $1/k$. We normalize time so that $\alpha k = 1$, with no loss in generality. Then, if we let V_b and V_s denote the value functions for sellers and buyers, we have the usual dynamic programming equations of search theory

$$rV_b = (1 - M)(q + V_s - V_b) + \dot{V_b} \qquad (24)$$

$$rV_s = M[-c(q) + V_b - V_s] + \dot{V_b}, \qquad (25)$$

given that each meeting results in immediate trade at q (see Trejos and Wright [1994], e.g., for a derivation).

The goal is to determine the time path of q as the equilibrium of sequential bargaining when the time between offers is small. For now, we assume there are no breakdowns in the bargaining (but see below). We can then apply (10) directly, with $r_i = r$, $\pi_i = \frac{1}{2}$, $\lambda_i = \gamma_i = 0$, $u_1 = q + V_s$, and $u_2 = -c(q) + V_b$. This yields

$$\dot{q} = \frac{rq + rV_s - \dot{V_s}}{2} - \frac{-rc(q) + rV_b - \dot{V_b}}{2c'(q)}. \qquad (26)$$

We also have to impose the constraint (23) for both buyers and sellers, which in this context means $q + V_s \geq V_b$ and $-c(q) + V_b \geq V_s$.

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We can reduce this model to two dimensions by letting $x = V_b - V_s$. Then the constraints can be written $(q, x) \in \mathcal{B}$, where

$$\mathcal{B} = \{(q, x) : c(q) \leq x \leq q\}.$$

By subtracting (24) and (25), $\dot{x}$ can be expressed as a function of q and x. By inserting (24) and (25) into (26) and simplifying, $\dot{q}$ can also be expressed as a function of q and x. The end result is that the model reduces to the dynamical system

$$\begin{bmatrix} \dot{q} \\ \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dfrac{rq + M[x - c(q)]}{2} + \dfrac{rc(q) + (1 - M)(x - q)}{2c'(q)} \\ \\ (1 + r)x - Mc(q) - (1 - M)q \end{bmatrix}. \qquad (27)$$

An (immediate trade, perfect foresight) *equilibrium* is defined to be solution to (27) that stays in $\mathcal{B}$ and also satisfies the conditions for immediate trade: $\Pi_b = e^{-rt}(q + V_s)$ and $\Pi_s = e^{-rt}[-c(q) + V_b]$ decreasing in t. A special case is a *steady state*, which is an equilibrium with the property that $\dot{q} = \dot{x} = 0$. We also distinguish between monetary and nonmonetary equilibria, where the latter entails $q = 0$ for all t.

Shi (1994) and Trejos and Wright (1994) study steady states by imposing at the myopic Nash solution,

$$q = \arg\max \, [q + V_s][-c(q) + V_b], \qquad (28)$$

consistent with the discussion in Section 3. Additionally, Trejos and Wright (1994) study dynamic equilibria by looking for solutions to (24) and (25) that satisfy (28) at each point in time. While (28) yields the same q as our forward looking solution in steady state, it does not yield the same q

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outside of steady state.[11] To the extent that one thinks of the Nash solution as a reduced form for strategic bargaining, imposing (28) is subject to the objection that agents ought to be forward looking in the bargaining game, as they are in other aspects of the model. Hence, we regard using the forward looking solution as superior to imposing the myopic Nash solution.

First we look for steady states. It is obvious that $(q, x) = (0, 0)$ is a nonmonetary steady state. The next result says that whenever a monetary steady state exists it is unique, and it exists if and only if $c'(0)$ is below some threshold $\tilde{c}$.

Theorem 8 *If a monetary steady state exists, it is unique. It exists if and only if $c'(0) < \tilde{c}$, where $\tilde{c}$ is the smaller root of the quadratic*

$$\tilde{c}^2 - \frac{2(r + M)}{M}\tilde{c} + \frac{(r + M)(1 - M)}{M(r + 1 - M)} = 0.$$

Proof: From (27), $\dot{q} = \dot{x} = 0$ is equivalent to $\Psi(q) = 0$, where

$$\Psi(q) = (r + M)[(1 - M)q - (r + 1 - M)c(q)]$$

$$\tag{29}$$

$$-(r + 1 - M)[(r + M)q - Mc(q)]c'(q).$$

Moreover, a necessary and sufficient for $(q, x) \in \mathcal{B}$ is that $q \le \hat{q}$, where $\hat{q}$ is defined by $(1 - M)\hat{q} = (r + 1 - M)c(\hat{q})$. Note that $\hat{q} > 0$ as long as $c'(0) < \tilde{c}$, where $\tilde{c}$ is defined in the statement of the theorem. One can show $\Psi(0) = 0$, $\Psi(\hat{q}) < 0$, and $\Psi'(q) < 0$ at any $q \in (0, \hat{q}]$ such that $\Psi(q) = 0$. One can also show that $\Psi'(0) > 0$ if and only if $c'(0) < \tilde{c}$. Hence, if $c'(0) < \tilde{c}$ then there

[11]The myopic Nash solution does coincide with the forward looking solution if we assume both the utility of consumption and the disutility of production are linear, by Theorem 3; unfortunately, however, a steady state monetary equilibrium does not exist in this case.

is a unique $q^* \in (0, \hat{q})$ such that $\Psi(q^*) = 0$, and therefore a unique monetary steady state; otherwise, there is no such q and no monetary steady state. $\square$

For the remainder of the analysis we impose $c'(0) < \tilde{c}$, so that the monetary steady state (q^*, x^*) exists. We now proceed to consider dynamic equilibria. By way of comparison, when we impose the myopic Nash solution, we have the following results. The monetary steady state is a source and the nonmonetary steady state is a saddle point. All orbits eventually leave $\mathcal{B}$, and therefore cannot be equilibria, except for the steady states and the saddle path that begins at the monetary steady state and converges to the nonmonetary steady state. Hence, the complete set of equilibria consists of the two steady states, and orbits starting on the saddle path, which converge monotonically to the nonmonetary steady state (see Trejos and Wright 1994).

Consider now the model with forward looking bargaining. The Jacobian of (27) is

$$
J = \begin{bmatrix} r - \frac{Mc'}{2} - \frac{1-M}{2c'} - \frac{[rc + (1-M)(x-q)]c''}{2(c')^2} & \frac{M}{2} + \frac{1-M}{2c'} \\ -Mc' - (1-M) & 1+r \end{bmatrix}.
$$

It is routine but tedious to show that $\det(J) = -\Psi'(q)/2c'(q)$ in steady state, where Ψ is defined in (29). Since $\Psi'(0) > 0$ by the argument in the proof of Theorem 8, $\det(J) < 0$ at the nonmonetary steady state, and it is a saddle point. Since $\Psi'(q^*) < 0$, $\det(J) > 0$ at the monetary steady state, and it is either a sink or a source. After some algebra, it is also possible to show that at any steady state

$$
trace(J) = 1 + r + \frac{(1 - M - Mc')^2 - \Psi'}{2(1+r)c'}.
$$

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Since $\Psi'(q^*) < 0$, $trace(J) > 0$ at the monetary steady state, and it is a source.

One can also show that along the boundary of the set $\mathcal{B}$ the flow is outward; thus, as shown in Figure 1, orbits never enter $\mathcal{B}$ from outside of $\mathcal{B}$. Since $(0,0) \in \mathcal{B}$, the saddle path leading to the nonmonetary steady state lies entirely in $\mathcal{B}$, and so any orbit beginning on the saddle path is an equilibrium. Furthermore, since it cannot come from outside of $\mathcal{B}$, the saddle path must either emanate from (q^*, x^*) or from a cycle surrounding (q^*, x^*).

It is easy to construct examples where the saddle path emanates from (q^*, x^*) and converges monotonically to $(0,0)$, and examples where it emanates from (q^*, x^*) and spirals before converging to $(0,0)$. We have not been able to construct an example of a limit cycle in the model as described, and neither have we been able to rule out limit cycles. However, we can construct an example with cycles by introducing a fixed cost, $c(0) > 0$. As shown in Figure 2, if the fixed cost is not too big, it shifts the $\dot{q} = 0$ and $\dot{x} = 0$ curves so that the intersection at the origin moves to $(q^0, x^0) \in int(\mathcal{B})$. Hence, there are now two monetary steady states, (q^0, x^0) and (q^*, x^*) (as well as a nonmonetary steady state at the origin, not shown in the figure).

With a fixed cost, if we impose the myopic Nash bargaining solution, it is still the case that (q^*, x^*) is a source, (q^0, x^0) is a saddle, and the saddle path converges monotonically from the former to the latter. That is, there are no spirals, let alone cycles. This, if we find cycles in our model, it must be due to the forward looking nature of the bargaining solution.

Our strategy proceeds as follows. First fix M, and set $r = \bar{r}$. Then let $c(q) = a_0 + a_1 q + a_2 q^2$ in the neighborhood of (q^*, x^*), and choose the coefficients a_j so that $(q^*, x^*) \in int(\mathcal{B})$ and $trace(J) = 0$ at (q^*, x^*). Note

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that this is impossible when $a_0 = 0$, as we argued earlier that $trace(J) > 0$ at the monetary steady state under the assumption $c(0) = 0$; but it is possible if $a_0 > 0$. Now for a range of values of r in the neighborhood of $\bar{r}$, one can study the system numerically (we used the program PHASEPLANE).

For $r < \bar{r}$, (q^*, x^*) is a source and (q^0, x^0) a saddle, exactly as with myopic bargaining (or as in the model with no fixed cost). See Figure 2. The important thing about the global dynamics in this case is that the unstable manifold of (q^0, x^0) loops around the branch of the stable manifold connecting (q^*, x^*) and (q^0, x^0). As we increase r, these branches of the stable and unstable manifolds get closer together, until at some $\hat{r} < \bar{r}$, they coalesce to form a homoclinic orbit that starts at (q^0, x^0), loops around (q^*, x^*), and returns to (q^0, x^0).

For $r > \hat{r}$, the branch of the unstable manifold lies inside of a region formed by the two branches of the stable manifold and the vertical axis in Figure 3. Notice that orbits that start in this region cannot escape. Hence, the branch of the unstable manifold in this region must either converge to (q^*, x^*) or to a cycle around (q^*, x^*). But for $r < \bar{r}$, we have $trace(J) > 0$, and (q^*, x^*) is a source. Applying the Poincare — Bendixson Theorem (see, e.g., Guckhenheimer and Holmes 1983), there exists a stable limit cycle around (q^*, x^*) for all $r \in (\hat{r}, \bar{r})$. The size of the cycle is decreasing in r, and for $r > \bar{r}$, it collapses into (q^*, x^*), as shown in Figure 4.

The essential point is that for all $r \in (\hat{r}, \bar{r})$ any orbit that starts in the region formed by the stable manifold and the vertical axis depicted in Figure 3 converges to a limit cycle. To argue that such a path is an equilibrium, we need to verify two more things: that it stays within $\mathcal{B}$, and that it satisfies the immediate trade condition, $\Pi_i'(t) \leq 0$ for all t. Since $(q^*, x^*) \in int(\mathcal{B})$, at

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least for r near $\bar{r}$, the cycles are sufficiently small they must stay in $int(\mathcal{B})$. We then verified numerically in examples that $\Pi_i'(t) \leq 0$ along the cycle. Hence, there exist stable limit cycles that satisfy all of the conditions for monetary equilibria.

Theorem 9 *There exist monetary equilibria that converge to limit cycles in the model with forward looking bargaining, at least in the presence of a fixed cost. The same model with myopic Nash bargaining does not admit cycles.*

Proof: This is clear from the preceding discussion. $\square$

To close this section, we briefly discuss the monetary model with exogenous breakdowns caused by the potential arrival of new agents during the bargaining. With breakdowns, the forward looking solution satisfies (10) with $r_i = r$, $\pi_i = \frac{1}{2}$, $\gamma_i = 0$, $\lambda_1 = M$, $\lambda_2 = 1 - M$, $b_1 = V_b$, and $b_2 = V_s$. This yields

$$\dot{q} = \frac{(r + M)q - Mc(q)}{2} - \frac{(1 - M)q - (r + 1 - M)c(q)}{2c'(q)}. \tag{30}$$

Meanwhile, $\dot{x}$ satisfies the same equation as in the model without breakdowns,

$$\dot{x} = (1 + r)x - Mc(q) - (1 - M)q. \tag{31}$$

As one can see from (30), $\dot{q}$ does not depend on x. In the (q, x) plane, this means that the $\dot{q} = 0$ curve is vertical at q^*. As in the model with no breakdowns, the $\dot{x} = 0$ locus goes through the origin and is upward sloping. The monetary steady state (q^*, x^*) is a source and the nonmonetary steady state $(0, 0)$ a saddle point, with a saddle path that converges monotonically from the monetary to the nonmonetary steady state.

By the results in Section 3, in steady state, a Nash representation for q is provided by

$$q = \arg\max \ [q + V_s - V_b][-c(q) + V_b - V_s]. \tag{32}$$

If we impose (32) as an equilibrium condition instead of the forward looking solution, the set of equilibria is qualitatively the same but quantitatively different. In particular, (32) implies $x = [c(q) + qc'(q)]/[1 + c(q)]$ for all t, which is the actual equation of the saddle path. If we differentiate it with respect to t and equate the resulting expression for $\dot{x}$ with (31), we see that $\dot{q} = g(q)f(q)$, where $f(q)$ is the formula for $\dot{q}$ from the model with forward looking bargaining given in (30) and $g(q) < 1$. Thus, $|\dot{q}|$ is smaller in the model with myopic Nash bargaining than in the forward looking model.

5 Conclusion

This paper has extended the analysis of strategic bargaining to environments where preferences or opportunities vary over time. We concentrated on situations satisfying a "shrinking pie" assumption, so that we can look for equilibria with immediate trade. Although trade is immediate, dynamics are important in that the terms of settlement depend on when the parties meet. As in stationary models with strategic bargaining, as the length of time between a rejection and the next offer shrinks toward zero, the offers of the two agents converge to the same value, q. As a function of time, q can be characterized in terms of a simple dynamical system.

One can always construct a time-varying Nash representation of our forward looking equilibrium. Naive application of the myopic Nash bargaining solution, however, will not generate the right path for q outside of steady

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state, except in special cases. Even in steady state, the analysis sheds light on the appropriate Nash representation in a variety of circumstances, and several results in the literature emerge as special cases.

We applied these ideas in the search and bargaining approach to monetary theory and demonstrated how the myopic Nash solution compares to the forward looking solution. In particular, we constructed an example with stable price cycles in the forward looking model, something that cannot happen if we impose the myopic Nash solution. Search and bargaining theory is applied in a variety in other areas, like macro and labor economics (e.g., Diamond [1982], Mortensen [1982], Pissarides [1987, 1990], Mortensen and Pissarides [1993]), in which many interesting questions are inherently dynamic. Our solution to the forward looking bargaining problem potentially has application in these areas too.

Appendix

We prove Lemma 2, which says that for all t, for small Δ, $q_2(t) - q_1(t) = O(\Delta^a)$ where $a \geq 1$. By way of contradiction, suppose that at some t we have $q_2(t) - q_1(t) = O(\Delta^a)$ with $a < 1$. Notice that ITE requires $q_2(t) > q_1(t)$, while Lemma 1 requires $a \geq 0$. Now let $h = h_0\Delta^b$, where $h_0 > 0$ and $a < b < 1$, and consider the time interval $T_h = [t, t + h]$. By construction, $h \to 0$ as $\Delta \to 0$. Also, if N denotes the number of Δ time periods in T_h then $N \to \infty$ as $\Delta \to 0$.

The following result sets up the required contradiction.

Lemma 3 *Fix $\Delta > 0$ and $k > 0$. Let $n = 1, 2, ...,$ and let M be the number*

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of time periods in an ITE where $t + n\Delta \in T_h$ and

$$\frac{u_1[q_2(t+n\Delta), t+n\Delta]}{(1+r\Delta)^n} > u_1[q_1(t), t] + k\Delta. \tag{33}$$

Then, as $\Delta \to 0$, $\frac{M}{N} \to 0$.

Proof: Let $P_1(t)$ be the expected payoff to player 1 at t if agreement is not reached at t. Player 1 can always use the following strategy in the subgame:

1. Always reject player 2's offer;

2. In period $t + n\Delta$, propose $q > q_2(t + n\Delta)$ if (33) does not hold;

3. In period $t + n\Delta$, propose $q = q_2(t + n\Delta)$ if (33) holds.

Given player 2's strategy in ITE, this strategy implies

$$P_1(t) \geq \{u_1[q_1(t), t] + k\Delta\} \left[1 - \left(\tfrac{1}{2}\right)^M\right]. \tag{34}$$

Settlement occurs in the third contingency in the above list; the probability that this never occurs is $(1/2)^M$, in which case $u_1 \geq 0$. Now ITE requires $P_1(t) \leq u_1[q_1(t), t]$. This and (34) imply

$$\left(\frac{1}{2}\right)^M \geq \frac{k\Delta}{u_1[q_1(t), t] + k\Delta},$$

or, equivalently,

$$M \leq \frac{\log(u_1 + k\Delta) - \log(k\Delta)}{\log(2)}.$$

Now consider the limit as $\Delta \to 0$. If $u_1 = 0$ then $M = 0$. If $u_1 > 0$ (but bounded) then, noting that $N = O(\Delta^{1-b})$, we have

$$\frac{M}{N} \leq O(-\Delta^{1-b} \log \Delta).$$

Hence, $\frac{M}{N} \to 0$. $\square$

By symmetry, the same result holds for player 2. Hence, as $\Delta \to 0$, most time periods $t + n\Delta \in T_h$ are characterized by

$$\frac{u_1[q_2(t + n\Delta), t + n\Delta]}{(1 + r\Delta)^n} \leq u_1[q_1(t), t] + k\Delta \tag{35}$$

$$\frac{u_2[q_1(t + n\Delta), t + n\Delta]}{(1 + r\Delta)^n} \leq u_2[q_2(t), t] + k\Delta \tag{36}$$

By concavity, (35) implies

$$\frac{u_1[q_2(t + n\Delta), t + n\Delta]}{(1 + r\Delta)^n} \leq u_1[q_2(t + n\Delta), t]$$

$$+ [q_1(t) - q_2(t + n\Delta)]\frac{\partial u_1[q_2(t + n\Delta), t]}{\partial q} + k\Delta.$$

This can be rewritten as

$$q_2(t + n\Delta) \leq q_1(t) + R_1(t, t + n\Delta, \Delta),$$

where $R_1(t, t + n\Delta, \Delta)$ is defined to make the statements equivalent.

We know q_1 is bounded, u_1 is continuous with a bounded time derivative, $\partial u_1/\partial q > 0$, and $n \leq N = O(\Delta^{b-1})$. Hence, it can be shown that $|R_1(t, t + n\Delta, \Delta)| \to 0$ as $\Delta \to 0$ for all $t + n\Delta \in T_h$, and the rate of convergence is at least order b. Similarly,

$$q_1(t + n\Delta) \geq q_2(t) + R_2(t, t + n\Delta, \Delta)$$

where $|R_2(t, t + n\Delta, \Delta)| \to 0$ as $\Delta \to 0$, and the rate of convergence is at least order b. Subtracting,

$$q_2(t+n\Delta) - q_1(t+n\Delta) \leq -[q_2(t) - q_1(t)] + R_1(t, t+n\Delta, \Delta) - R_2(t, t+n\Delta, \Delta).$$

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But $q_2(t) - q_1(t) > 0$ and is $O(\Delta^a)$, where $a < b$. Hence, as $\Delta \to 0$, there must exist many time periods $t + n\Delta \in T_h$ where $q_2(t + n\Delta) - q_1(t + n\Delta) < 0$, which is a contradiction. This completes the proof. $\square$

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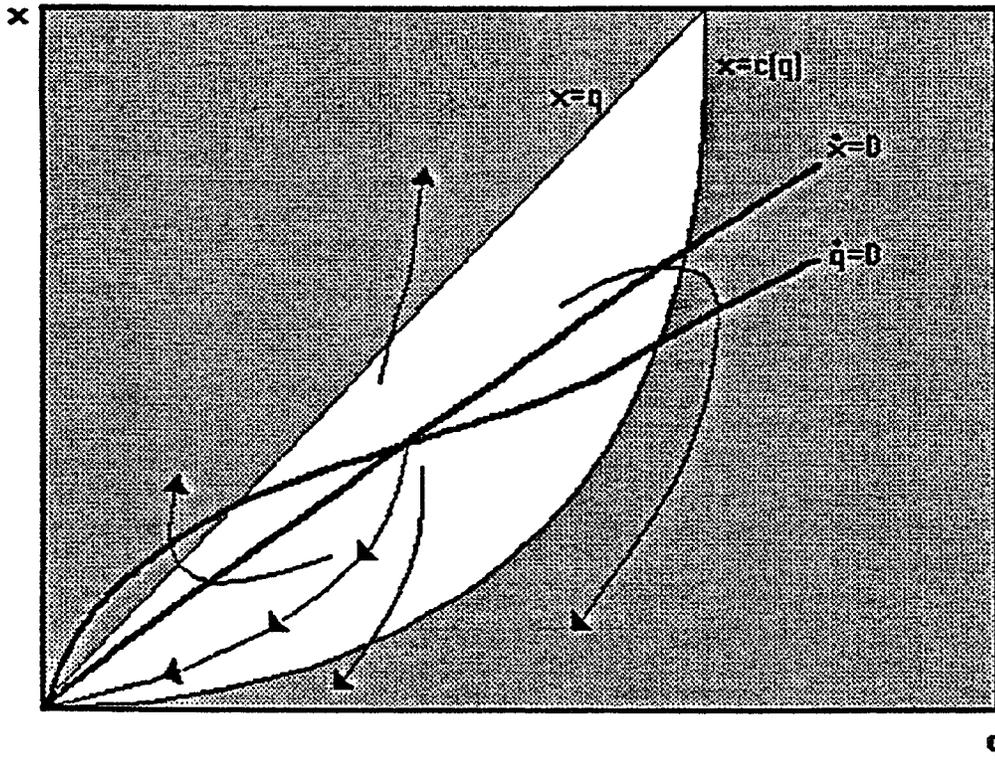A. Wolinsky (1987) "Matching, Search, and Bargaining," *Journal of Economic Theory* 42, 311-33.

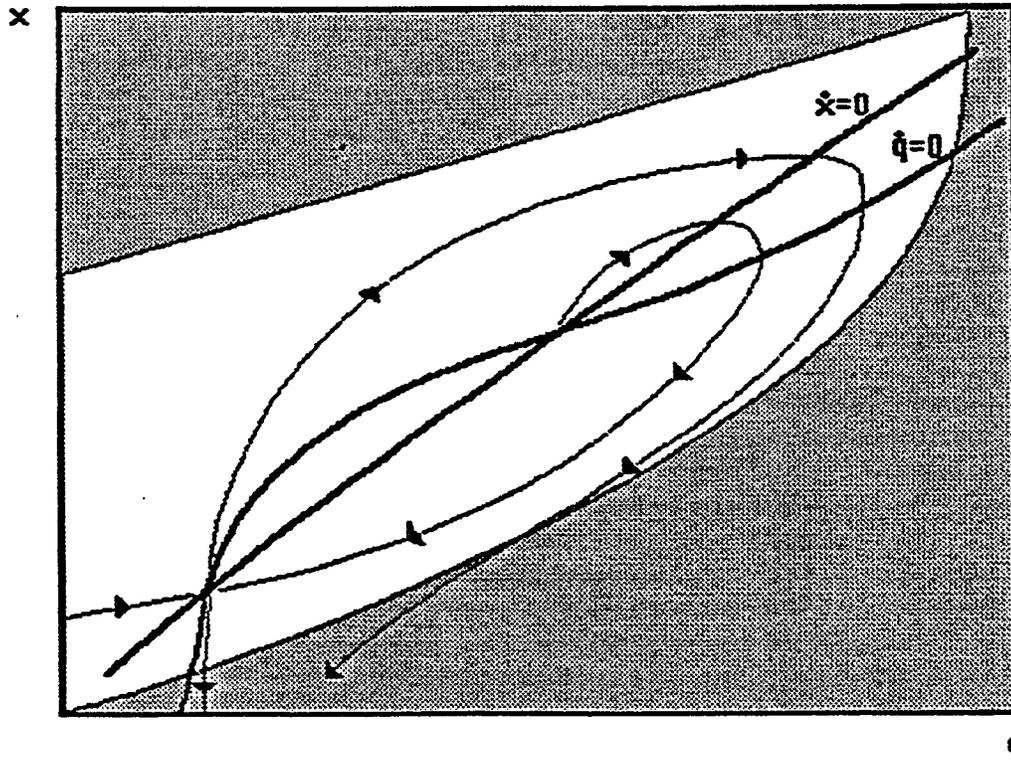Figure 1: Monetary and nonmonetary steady states (source and saddle).

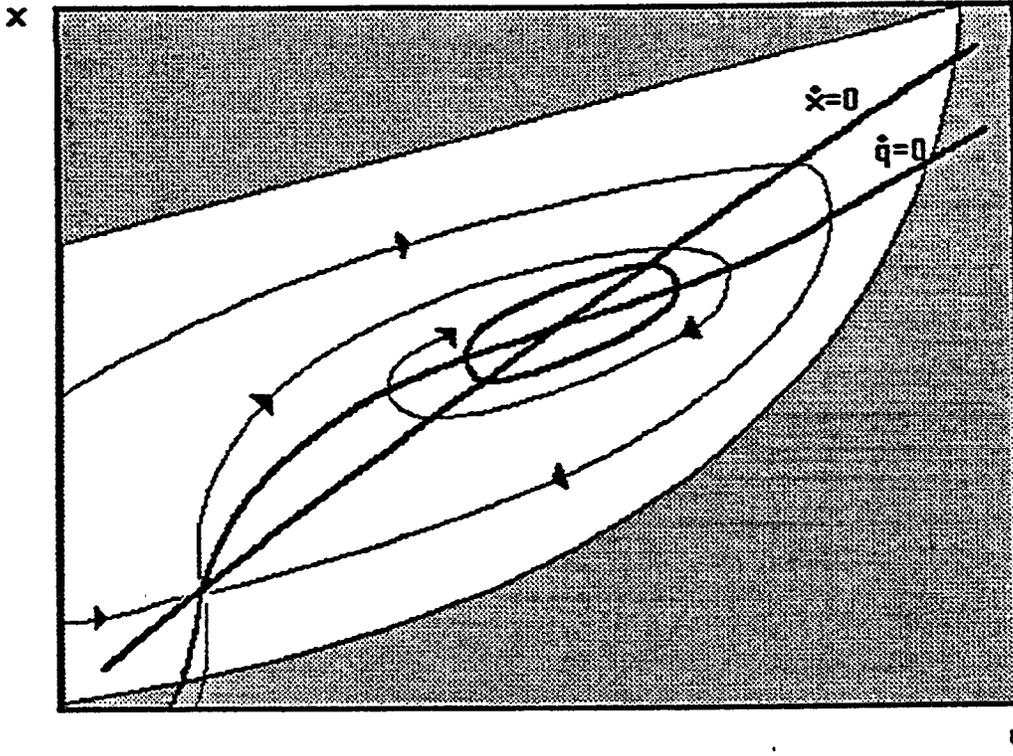Figure 2: Two monetary steady states, one source and one saddle.

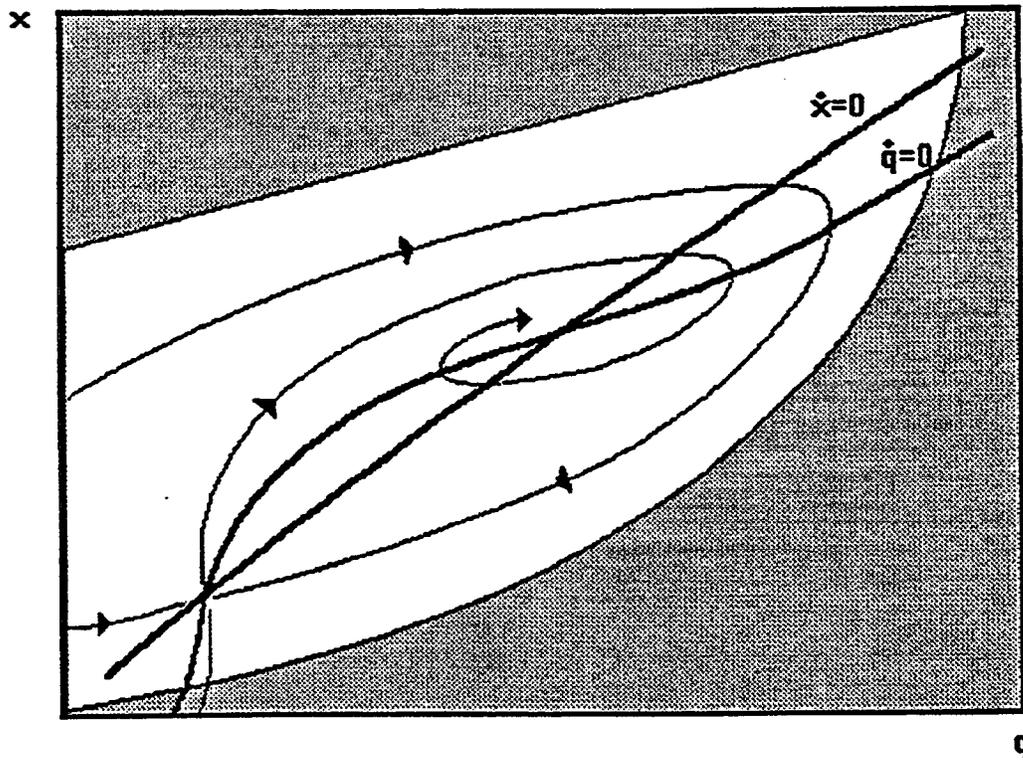Figure 3: Stable limit cycle around the steady state which is a source.

Figure 4: Cycle collapses into steady state, which becomes a sink.