Samuelson's Consumption-Loan Model With Country-Specific Fiat Monies

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ABSTRACT

In this paper, we examine various exchange rate regimes, paying particular attention to what difference the monetary-fiscal policy choices of governments make. The exchange rate may be market-determined or fixed, and if fixed, either cooperatively or by one government alone. Further, capital controls may or may not apply. Our most important result, quite general, we believe, is that absent capital controls the equilibrium exchange rate of the floating rate regime is indeterminate. It makes no sense to advocate floating rates and unfettered international borrowing and lending.

I. INTRODUCTION

In this paper we present what seems to us an interesting discrete-time representation or model of the world economy, and then use it to determine and compare the equilibria that obtain under alternative international economic policy (or for short, exchange rate) regimes.

Our model is very much like Samuelson's (1953) overlapping-generations model. Indeed, they differ in only one consequential respect. His is of a one-country world economy with, in its monetary version, one fiat money. In contrast, ours is of a world economy made up of two (or any greater number of) countries, into which has been introduced the social contrivance of not one but two (or several) fiat monies. Each of the countries of our world economy has a government. And as we assume, each of the governments has its own fiat money, the supply of which it controls. So in our world economy there does exist a certain money price, which following accepted practice we refer to as an exchange rate.

As presented herein, our model is of a one-good pure-exchange world economy. It can, however, be thought of as a Heckscher-Ohlin generalization of the Samuelson model, as being of a multi-country world economy in which many different goods are produced and consumed. For on certain assumptions, set out in Appendix A, all goods are produced and consumed in proportions that are independent of what policy regime obtains and do not change with the passage of time.

How we get individuals to want positive amounts of money may be obvious from what we have already said. But consider some member of any generation. That individual begins his (or her) life with an endowment of the one good that is available. He gets no more, though, in the second or final period of his life. And the good is not storable. Consequently, he can consume in the second period of his life only by purchasing money in the first period and then in the second selling it to members of the next generation.

But how much money to buy? And what kind? That of the government of his country? Or that of the government of the other? Or some of both? A single-minded utility maximizer, our individual decides for himself. Of course, he may be influenced by the policy choices of the two governments.

We indicated above that we consider several exchange rate regimes. There is a floating rate regime. And there are two fixed rate regimes: the cooperative, under which the exchange rate is maintained by the two governments acting together; and the non-cooperative, under which the rate is maintained (insofar as possible) by one of the governments acting on its own. And of each regime there are two versions: a laissez-faire version, under which every individual is free to purchase either or both of the two fiat monies; and a portfolio autarky version, under which every member of every generation from the first on is restricted to purchasing only the money of his own government.

As we suppose, though, each of the two governments conducts its own monetary-fiscal policy. In every period each takes some amount of its money from or gives some amount to each of the age-two residents of the country over which it presides. Moreover, each government decides for itself what the tax or transfer (the same for all members of any given generation, but not necessarily for members of different generations) will be. The policies that we consider are simple, being all of a one-parameter family. Nevertheless, we are able to show how the equilibrium (if it exists) of any particular exchange rate regime depends on the monetary-fiscal policy choices of governments. And as will be seen, those choices are of considerable importance. Thus, an equilibrium is Pareto-optimal for some parameter values and suboptimal for others.

But neither government has the slightest bit of what is often referred to as discretion, except maybe at the beginning of the first period, the starting point of our analysis. Governmental policies are policies in the sense of dynamic programming. The taxes and/or transfers of all periods from the first on are known at the beginning of the first. They must be, for the world economy of this paper is inhabited by individuals with perfect foresight. To decide how much of each of the two fiat monies to buy, an age-one individual must have expectations about the prices of the two monies that will prevail in the next period. And as we require, those expectations are correct.

Some if not all of the results reported in this paper will be less than startling to those familiar with the balance-of-payments literature. That does not bother us especially, however, for it is not entirely a waste to have made even familiar results more plausible than they were. And traditional general equilibrium analysis does, we believe, give results that are more plausible than those obtained using an ad hoc macroeconomic model and perhaps some arbitrary government utility function(s). To put the point another way, we think of ourselves as writing in part at least for those who will one day be doing textbooks in international economics. With this paper in hand, they should be a little better able than their predecessors were to treat all of the issues of international economics, balance-of-payments issues as well as those of, say, commercial policy, in the same way, using traditional general equilibrium (or welfare) economics.

In the next section of this paper we introduce many of our assumptions and describe two choice problems, that of the members of generation zero and that of the members of all other generations. Then in section III we describe the family of monetary-fiscal policies that we admit of and work out certain implications for the evolution of the world money supply over time. In sections IV - VI we determine the various exchange rate regime equilibria. And in section VII we give necessary and sufficient conditions for Pareto-optimality of any equilibrium consumption allocation. Finally, in section VIII we make a case for our model.

II. SOME ASSUMPTIONS

It suffices for our purposes that there be only two countries, indexed by the variable k. Each has a population that is constant over time. At the beginning of period t, N_k individuals, all of whom will live for two periods, are born in country k. And since no one ever moves from one country to the other, the period-t population of country k is made up of N_k age-one individuals, the country-k members of generation $t > 1$, and N_k age-two individuals, the country-k members of generation t-1.[1/]

The variable h is the index of the members of any generation t. And $N_k(t)$ is the set of all country-k members of generation t.

There is only one good, produced if at all only in heaven, and in our notation $c_j^h(t) \geq 0$ is the consumption of that good at age j (j = 1,2) by member h of generation t. We assume that all individuals have the same tastes. For all h and t, life-time utility is $u[c^h(t)]$, where $c^h(t) = (c_1^h(t), c_2^h(t))$ is the life-time consumption vector of member h of generation t. The function u is increasing in each of its arguments and thrice-differentiable. It is also homothetic. That is,

$$u_1(x_1,x_2)/u_2(x_1,x_2) = f(x)$$

where (for the moment) the x_j are the arguments of u, $u_j = \partial u/\partial x_j$ and $x_1/x_2 = x$. And f, the marginal-rate-of-substitution function, satisfies the following conditions: $f(x) > 0$, $f'(x) < 0$ and $f''(x) > 0$ for all admissible values of x; $f(x) \to \infty$ as $x \to 0$; $f(x) \to 0$ as $x \to \infty$; and lastly, the gross substitution condition, $xf'(x)/f(x) \geq -1$. Thus, f has an inverse, which we denote by F.

Member h of generation $t \geq 1$ comes into the world in period t with an amount $y^h(t)$ of the one nonstorable good. In that period he consumes a portion of his endowment and trades away what remains, getting money in exchange from one or more members of generation t-1. His period-t budget constraint is therefore

$$(1) \qquad c_1^h(t) + \sum_k P_k(t) m_k^h(t) \leq y^h(t),$$

where $P_k(t)$ is the period-t price of the money of the government of country k and $m_k^h(t)$ is his purchase of that money.

For member h of generation $t \geq 1$, $y^h(t)$ is his life-time income or endowment. That is, he gets no more of the good in period t+1. He does, though, get a transfer of country-k money from the government of country k. That transfer, denoted by $x_k^h(t) \gtreqless 0$, is known in period t and is independent of $m_k^h(t)$. And in period t+1 he trades away both his period-t money purchases and his transfers, getting consumption in return from one or more members of generation t+1. Thus, his period-t+1 budget constraint is

$$(2) \quad c_2^h(t) \leq \sum_k P_k(t+1)[m_k^h(t) + x_k^h(t)].$$

The problem for member h of generation $t > 1$ is to maximize $u[c^h(t)]$ by the choice of a nonnegative vector $c^h(t)$ and a vector $m^h(t) = (m_1^h(t), m_2^h(t))$, subject to (1) and (2) and with $y^h(t)$, the $P_k(t)$, $P_k(t+1)$ and $x_k^h(t)$ known in period t.[2] Under any portfolio autarky regime there is an additional constraint: namely, $m_{k'}^h(t) = 0$ for all h in $N_k(t)$.

The members of generation zero are in the second or final period of their lives in the first period, at the beginning, that is, of our time. So the problem of member h of generation zero is to maximize $u[c^h(0)]$ by the choice of $c_2^h(0)$, subject to the $t = 0$ version of (2) and with $c_1^h(0)$ and the RHS of (2) given.[3]

III. MONETARY-FISCAL POLICIES

As was noted above, member h of generation $t \geq 1$ receives a transfer of country-k money, $x_k^h(t)$, from the government of country k in period t+1. And we assume that

$$(3)\quad x_k^h(t) = \begin{cases} (\alpha_k - 1)\alpha_k^{t-1} M_k/N_k & \text{for all } h \text{ in } N_k(t) \\ 0 & \text{for all } h \text{ in } N_{k'}(t) \end{cases}$$

where α_k is the policy parameter of the government of country k and M_k is some positive but otherwise arbitrary amount of the fiat money of the government of country k. (The values of the M_k, like those of the N_k, are initial conditions.) Thus, the country-k members of generation $t \geq 1$ receive nothing from or pay no taxes to the government of country k′. But all receive identical transfers from or for $\alpha_k < 1$ pay identical taxes to the government of country k. That is, $x_k^h(t) = x_k(t)$ for all h in $N_k(t)$. Since we also assume that $y^h(t) = y_k$, a constant, for all h in $N_k(t)$ and $t \geq 1$, all the country-k members of generation $t \geq 1$ are alike. That is, they all solve precisely the same maximization problem. In effect, then there are only two types of members of generation $t \geq 1$, the residents of the first country and the residents of the second.

The government of country k chooses a value of α_k in the first period and, as all individuals are aware, stays with that value through all subsequent periods. In its choice, though, it is subject to the restriction

$$(4)\quad \alpha_k > 1 - N_k y_k / y \qquad \text{for } k = 1,2$$

where $y = N_1 y_1 + N_2 y_2$.4/

In our notation $M_k(t)$ is the period-t supply of the money of the government of country k, the amount of country-k money with which the members of generation t-1 begin period $t \geq 1$ and in that period sell to the members of generation t. And we assume that under any floating exchange rate regime

$$(5) \quad M_k(1) = M_k \qquad \text{for } k = 1,2.$$

For there is no governmental intervention in the so-called foreign exchange market, and therefore any exchange of first and second-country monies is of necessity between individuals. But under any fixed rate regime an exchange may be between a government and an individual. And so we assume that under any such regime the first-period values of the $M_k(t)$ satisfy not (5), but the weaker restriction

$$(6) \quad M_1(1) + E(1)M_2(1) = M_1 + E(1)M_2,$$

where $E(t) = P_2(t)/P_1(t)$ is the period-t exchange rate, the price of country-two money in units of country-one money.

Under any floating exchange rate regime the supply of country-k money (the amount held by individuals) evolves according to

$$M_k(t) = M_k(t-1) + N_k x_k(t-1).$$

And consequently, since (3) and (5) hold,

$$(7) \quad M_k(t) = \alpha_k^{t-1} M_k$$

under that regime.

If the exchange rate is constant, though, then independent of regime the world money supply,

$$M(t) \equiv M_1(t) + E(t)M_2(t),$$

evolves according to

$$(8) \quad M(t) = \alpha_1^{t-1}M_1 + \alpha_2^{t-1}EM_2.$$

The proof, if such be needed, is by induction. For t = 1, (8) reduces to (6). And with E(t) = E for all $t \geq 1$,

$$M(t) = M(t-1) + N_1x_1(t-1) + N_2Ex_2(t-1).$$

But then it follows from (3) that if M(t-1) is given by (8), M(t) is too.

It is immediate from (8) that

$$(9) \quad \sigma(t) \equiv M(t+1)/M(t) = \alpha_1\lambda(t) + \alpha_2[1-\lambda(t)]$$

where

$$(10) \quad \lambda(t) \equiv (M_1/M_2)(\alpha_1/\alpha_2)^{t-1}/[(M_1/M_2)(\alpha_1/\alpha_2)^{t-1} + E].$$

So $\sigma(t)$ is a weighted average. And for E > 0, the weight of the larger of the α_k increases through time and has unity as its limiting values. Thus, $\sigma(t) \geq \alpha_m$, where $\alpha_m = \min(\alpha_1,\alpha_2)$, is monotonic increasing and has $\alpha_M = \max(\alpha_1,\alpha_2)$ as its limiting value.

Finally, with E constant the money transfers can be expressed as proportions of the world money supply. That is,

$$(11) \quad \begin{aligned} x_1(t) &= (\alpha_1-1)\lambda(t)M(t)/N_1 \\ Ex_2(t) &= (\alpha_2-1)[1-\lambda(t)]M(t)/N_2. \end{aligned}$$

IV. FLOATING RATE REGIMES

Sufficient preliminaries out of the way, we turn now to consideration of our several exchange rate regimes. And in this section we are occupied almost exclusively with floating rate regimes, which we hereafter refer to as F regimes. We begin, though, by indicating what our sense of the word "equilibrium" is, and then establishing a proposition that says something about all of the exchange rate regimes.

For us, an equilibrium is a complete description of the evolution of the world economy from some arbitrary starting date (t = 1) and for arbitrary initial conditions (arbitrary initial stocks of the two fiat monies, distributed in some fashion or other over the members of generation zero). The description takes the form of a set of sequences in t, one for each of the endogenous variables. Those sequences are consistent with consumption and portfolio choices that are optimal, with period-by-period clearing of all markets and with perfect foresight.[5/]

And the first of our propositions is as follows:

Proposition I. For any exchange rate regime, (a) there is a nonmonetary equilibrium; that is, under any regime the money-price sequences satisfying the condition $P_k(t) = 0$ for all k and $t \geq 1$ are equilibrium sequences. And (b) any sequence with $P_k(t) = 0$ for some but not all $t \geq 1$ is not an equilibrium sequence.

The solution to the above-stated maximization problem of member h of generation $t \geq 1$ satisfies (1) and (2) with equality. So we may write

$$(12) \quad u[c^h(t)] = u\{y^h - \sum_k P_k(t)m_k^h(t), \sum_k P_k(t+1)[m_k^h(t)+x_k^h(t)]\},$$

from which part (a) of Proposition I is immediate. For if $P_k(t) = P_k(t+1) = 0$ for all k, then utility is independent of the $m_k^h(t)$. No individual cares what his money holdings are, and therefore any parcelling out of the available stocks of the two monies produces an equilibrium.

Part (b) of the proposition is also easily proved. Suppose that $P_k(t) = 0$ and $P_k(t+1) > 0$ for some value of k. Then, since u is increasing in both of its arguments, there is no optimal quantity of country-k money. For any given quantity, having more would be better. And suppose that $P_k(t) > 0$ and $P_k(t+1) = 0$ for some value of k. Again, there is no optimal quantity. For any given quantity, having less would be better. Thus, at neither pair of prices could the period-t supply-demand condition or conditions be satisfied.

In our further investigation of the F regimes, in our search, that is, for the monetary equilibria of those regimes, we may then limit ourselves to pairs of money-price sequences satisfying the condition $P_k(t) > 0$ for all $t \geq 1$ and at least one value of k.

As we indicated in the introductory section, there are two floating-rate regimes: a laissez-faire regime, to which we refer hereafter as the LF/F regime; and a portfolio autarky regime, the PA/F regime. We consider them in turn, beginning with the former.

Laissez-Faire

Under the LF/F regime there are in a sense too many monies. The existence of more than one gives rise to a multiplicity of equilibria, or to an equilibrium exchange rate that although constant is indeterminate.

Later on it will be convenient to have the following detailed statement of our result.

Proposition II. Under the LF/F regime there are a multiplicity of equilibria, all with $E(t) = E$ for all $t \geq 1$. Thus, (a) for $E = 0$ ($1/E = 0$) there is an equilibrium which is constant and such that the money of the first (second) country has value and the money of the second (first) has none.[6/] And (b) for every choice of $E > 0$ there is a unique equilibrium such that the monies of both countries have value. Further, if $\alpha_1 = \alpha_2$, then the equilibrium, for given E, is constant. And if $\alpha_k > \alpha_{k'}$, then the equilibrium is not constant, but converges, if only in its real part, to the equilibrium with only the money of country k having value.

Under the LF/F regime, portfolio choice is unrestricted. Member h of generation $t \geq 1$ may purchase either or both of the two monies. It follows that if $P_k(t) \geq 0$ for all k and $t \geq 1$, then in equilibrium

$$(13) \quad \beta_k(t) \equiv P_k(t+1)/P_k(t) = \beta(t) \quad \text{for all } k \text{ and } t \geq 1,$$

where $\beta_k(t) - 1$ is the period-t rate of return on country-k money. To verify that, one has only to compute the differential du from (12) and consider how it behaves for variations in the $m_k^h(t)$ that satisfy the condition $\sum_k P_k(t)dm_k^h(t) = 0$.

But then

$$E(t) \equiv P_2(t)/P_1(t) = P_2(t+1)/P_1(t+1) \equiv E(t+1)$$

So if there is an LF/F regime equilibrium with $P_k(t) > 0$ for all k and $t \geq 1$, then the equilibrium exchange rate sequence $\{\hat{E}(t)\}$ is constant.

And it is immediate from the definition of E(t) that if there is an equilibrium with $P_1(t) > 0$ and $P_2(t) = 0$ for all $t \geq 1$, then $\hat{E}(t) = 0$ for all $t \geq 1$. (We use "^" to denote LF/F regime equilibrium values.) Moreover, if there is an equilibrium with $P_1(t) = 0$ and $P_2(t) > 0$ for all t, then $1/\hat{E}(t) = 0$ for all $t \geq 1$. Thus, admitting only of money-price sequences on the restricted set, we may use (8) and all that it implies.

Now, the desired real balance of member h of generation $t \geq 1$ is

$$q^h(t) \equiv \sum_k P_k(t) m_k^h(t).$$

But since all of the country-k members of generation $t \geq 1$ are alike, $q^h(t) = q_k(t)$ for all h in $N_k(t)$. There are then only two optimal quantities, the $\hat{q}_k(t)$. And each satisfies the first-order condition implied by (12), which by (13) and our assumptions about the function u we may write[7/]

$$(14) \qquad y_k - q_k(t) = F[\beta(t)]\beta(t)[q_k(t) + P_k(t)x_k(t)].$$

The k = 1 and k = 2 versions of (14) are both equilibrium restrictions and together imply

$$(15) \qquad y - \sum_k N_k q_k(t) = F[\beta(t)]\beta(t)[\sum_k N_k q_k(t) + \sum_k N_k P_k(t) x_k(t)].$$

which holds for all $t \geq 1$. [To get (15), one multiplies the $k = 1$ version of (14) by N_1 and the $k = 2$ version by N_2 and then adds.]

There is, however, another equilibrium restriction,

$$(16) \quad Q(t) = P_1(t)M(t) \qquad \text{for all } t \geq 1,$$

where $Q(t) \equiv \sum_k N_k q_k(t)$ is the total of the period-t desired real balances of all the members of generation $t \geq 1$. And it follows from (16) and (11) that

$$(17) \quad \begin{aligned} N_1P_1(t)x_1(t) &= (\alpha_1-1)Q(t)\lambda(t) \\ N_2P_2(t)x_2(t) &= (\alpha_2-1)Q(t)[1-\lambda(t)] \end{aligned} \qquad \text{for all } t \geq 1,$$

and from (15) and (17) that

$$(18) \quad Q(t)\{1+\sigma(t)\beta(t)F[\beta(t)]\} - y = 0 \qquad \text{for all } t \geq 1.$$

But we also have as an implication of (16) and (9) that

$$(19) \quad \beta(t) = Q(t+1)/Q(t)\sigma(t) \qquad \text{for all } t \geq 1.$$

So (18) can be rewritten

$$(20) \quad Q(t) + F[Q(t+1)/Q(t)\sigma(t)]Q(t+1) - y = 0$$

In Appendix B we show that for sequences $\{Q(t)\}$ that are positive and bounded away from zero, (20) has a unique solution

$$(21) \quad \hat{Q}(t) = y\phi_A[\sigma(t)]$$

where the function ϕ_A, which is continuous and bounded, depends (although only when the α_k are different) on the vector $A = (\alpha_1,\alpha_2)$ but not on E. Thus, for every choice of $E \geq 0$ there is a LF/F regime equilibrium, given by (21), (19) and $P_2(t) = EP_1(t)$, with $\{\hat{Q}(t)\}$ positive and bounded away from zero. For $\sigma(t)$ is in general determined partly by E. Moreover, even when it is not, $\{\hat{P}_2(t)\}$ is, and hence the equilibrium distribution of utilities is too.

In Appendix B we also show that the monetary equilibrium of the LF/F regime has the following properties: (i) $\{\hat{Q}(t)\}$ is monotonic decreasing; and (ii) $1/\sigma(t) \geq \hat{\beta}(t) \geq \lim \hat{\beta}(t) = 1/\alpha_M$.

To prove part (a) of our Proposition II, we first note that if $E = 0$, then $\lambda(t) = 1$ and $\sigma(t) = \alpha_1$ for all $t \geq 1$. It follows that $\hat{Q}(t) = y\phi_A(\alpha_1)$ and $\hat{\beta}(t) = 1/\alpha_1$ for all $t \geq 1$. The equilibrium is constant and such that only the money of the first country has value. And if $1/E = 0$, then $\lambda(t) = 0$ and $\sigma(t) = \alpha_2$ for all $t \geq 1$. So again the equilibrium is constant, but such that only the money of the second country has value.

As regards part (b), if $E > 0$, then the equilibrium is such that the monies of both countries have value. Moreover, if $\alpha_k = \alpha$ for $k = 1,2$, then $\sigma(t) = \alpha$ for all $t \geq 1$ and the equilibrium associated with any $E > 0$ is constant. But if $\alpha_1 \neq \alpha_2$, then $\lambda(t)$ is not constant and neither is the equilibrium. It is, however, an implication of (10) that with $\alpha_1 > \alpha_2$ the equilibrium in its real part converges to that with the money of the first country having value and that of the second having none. That is, if $\alpha_1 > \alpha_2$, then $\hat{Q}(t) \to y\phi_A(\alpha_1)$ and $\hat{\beta}(t) \to 1/\alpha_1$ as $t \to \infty$. And with $\alpha_1 < \alpha_2$ the equilibrium in its real part converges to that with only the money of the second country having value.

To complete our proof of Proposition II, we have only to show that with $\hat{P}_k(t) > 0$ for all $t \geq 1$ and at least one value of k the feasible consumption set is consistent with $f[c^h(t)]$ having an internal maximum. Since $\hat{P}_k(t) = 0$ implies

$$(22) \quad N_k[y_k+\hat{P}_k(t)x_k(t)] > 0,$$

it suffices to establish that (22) holds for any $\hat{P}_k(t) > 0$. It follows from (3) and (7), though, that (22) is satisfied if and only if

$$\alpha_k > 1 - N_k y_k/P_k(t)M_k(t).$$

But that inequality is implied by (4) and the inequality $\hat{Q}(t) = \sum_k P_k(t)M_k(t) \leq y$, which follows from (1).

And now to conclude this subsection we establish two other facts about the LF/F regime monetary equilibria. The first is that if the α_k are different and $E > 0$, then a version of Gresham's Law obtains. That is, the relatively abundant money drives out the relatively scarce, or comes in the end to make up the world money supply. It follows from (7) that $M_1(t)/M(t) = \lambda(t)$, and from (16) that

$$\hat{P}_1(t)M_1(t)/\hat{Q}(t) = \lambda(t).$$

Further,

$$\sum_k \hat{P}_k(t)M_k(t)/\hat{Q}(t) = 1.$$

And hence, since $\lim \lambda(t) = 1$ if $\alpha_1 > \alpha_2$ and $\lim \lambda(t) = 0$ if $\alpha_1 < \alpha_2$ we have that $\alpha_k > \alpha_{k'}$ implies $\lim \hat{P}_k(t)M_k(t)/\hat{Q}(t) = 1$.

Our second fact is that trade balance is not in general a characteristic of the monetary equilibria of the LF/F regime. By definition, the equilibrium period-t trade surplus of country k, denoted by $\hat{B}_k(t)$, is the difference between $N_k y_k$ and the equilibrium consumption of the period-t residents of country k, the country-k members of generations t-1 and t. Thus, $\hat{B}_k(1)$ depends on the arbitrary distribution of money holdings over the first and second-country members of generation zero. But for $t \geq 2$ we can be quite definite about $\hat{B}_k(t)$.

The equilibrium period-t consumption of, say, the first-country members of generation $t \geq 2$ is $N_1[y_1-\hat{q}_1(t)]$, and that of the first-country members of generation t-1 is $N_1[\hat{\beta}(t-1)\hat{q}_1(t-1)+\hat{P}_1(t)x_1(t-1)]$. And therefore, since (11) and (16) hold,

$$(23) \quad \hat{B}_1(t) = N_1\hat{q}_1(t) - [N_1q_1(t-1)/\hat{Q}(t-1)+(\alpha_1-1)\lambda(t-1)]\hat{Q}(t)/\sigma(t-1).$$

It follows that if $\alpha_k = \alpha$ for k = 1,2, then

$$(24) \quad \hat{B}_1(t) = [N_1\hat{q}_1(t)-\lambda(t)\hat{Q}(t)](\alpha-1)/\alpha \qquad \text{for all } t \geq 2,$$

where $\hat{q}_1(t)$, $\hat{Q}(t)$ and $\lambda(t)$ are all constants. Thus, if $\alpha = 1$, then $\hat{B}_k(t) = 0$ for all $t \geq 2$. But if $\alpha \neq 1$, then the $B_k(t)$ are not necessarily zero. Indeed, if $\alpha_k = \alpha \neq 1$, then $\hat{B}_k(t) = 0$ for all $t \geq 2$ if and only if the period-t equilibrium desired real balances of all the first-country members of generation t, $N_1\hat{q}_1(t)$, is the same as the period-t equilibrium real supply of first-country money, $\hat{P}_1(t)M_1(t) = \lambda(t)\hat{Q}(t)$. There is a unique value of E at which equality obtains.[8/]

It also follows from (23) that if $\alpha_1 \neq \alpha_2$, then for all $t \geq 2$

$$(25) \quad \lim \hat{B}_1(t) = \begin{cases} -[N_2(\alpha_1-1)/\alpha_1] \lim \hat{q}_2(t) & \text{if } \alpha_1 > \alpha_2 \\ [N_1(\alpha_2-1)/\alpha_2] \lim \hat{q}_1(t) & \text{if } \alpha_1 < \alpha_2 \end{cases},$$

where each of the limits on the RHS is positive, a consequence of the fact that if $\alpha_k > \alpha_{k'}$ and country-k money has value, then any country-k′ transfer has a limiting real value of zero. Thus, if $\alpha_1 \neq \alpha_2$, then in the limit $\hat{B}_k(t) = 0$ for all $t \geq 2$ if and only if $\alpha_M = 1$. Note, though, that with $\alpha_M \neq 1$, it makes all the difference whether $\alpha_M \gtrless 1$.

Under the LF/F regime equality between equilibrium exports and imports is then a very special outcome. For almost all choices of the α_k, country k will either have a trade surplus, a surplus just sufficient to offset the import of country-k′ money, or a trade deficit that is just sufficient to offset the export of country-k money.

Portfolio Autarky

What distinguishes the PA/F regime is that under it a country-k member of generation $t \geq 1$ can achieve his desired intertemporal consumption allocation only by buying or going short country-k money. He cannot deal in country-k′ money. Nor can he lend period-t consumption or borrow such from a resident of country k′, getting or giving period-t+1 consumption in exchange. So under the PA/F regime the rates of return on the two monies need not be the same.

Proposition III. Under the PA/F regime (a) there exists a unique and constant equilibrium for every admissible choice of the α_k. Further (b) every equilibrium exhibits independence or autonomy; the country-k equilibrium is independent of $\alpha_{k'}$. And (c) $\tilde{B}_k(t) = 0$ for all k and $t \geq 2$ and every admissible choice of the α_k, where "~" denotes the equilibrium value under the PA/F regime.

Even under the PA/F regime, all country-k members of generation $t \geq 1$ are alike. So $q^h(t) = q_k(t)$ for all h in $N_k(t)$ and $t \geq 1$. And member h of generation $t \geq 1$, a resident of country k, can therefore be thought of as maximizing

$$u\{y_k-q_k(t),\beta_k(t)[q_k(t)+P_k(t)x_k(t)]\}$$

by his choice of $q_k(t)$, subject to the restriction $m^h_{k'}(t) = 0$ or alternatively $q_k(t) = P_k(t)m^h_k(t)$. It follows from (3) and (7), though, that

$$\text{(26)} \quad x_k(t) = (\alpha_k-1)M_k(t)/N_k.$$

And under the PA/F regime it is required for equilibrium that

$$\text{(27)} \quad P_k(t)M_k(t) = N_k q_k(t) \qquad \text{for all k and } t \geq 1.$$

We thus have that in equilibrium

$$\beta_k(t) = P_k(t+1)M_k(t+1)/P_k(t)M_k(t+1) = q_k(t+1)/\alpha_k q_k(t)$$

and hence that the optimal $q_k(t)$ satisfies the first-order condition

$$\text{(28)} \quad q_k(t) + F[q_k(t+1)/q_k(t)\alpha_k]q_k(t+1) - y_k = 0.$$

But (28) is a special case of (20), with α_k substituted for $\sigma(t)$. Consequently,

$$(29)\quad \tilde{q}_k(t) = y_k\phi_A(\alpha_k) = y_k/[1+F(1/\alpha_k)]$$

and

$$(30)\quad \tilde{\beta}_k(t) = 1/\alpha_k > 0.$$

And since $E(t)/E(t-1) = \beta_2(t)/\beta_1(t)$,

$$(31)\quad \tilde{E}(t) = \alpha_1\tilde{E}(t-1)/\alpha_2 = (\alpha_1/\alpha_2)^{t-1}\tilde{E}(1) = (\alpha_1/\alpha_2{}^{t-1})M_1\tilde{q}_2/M_2\tilde{q}_1,$$

an exponential function of time.

With (29) and (30) holding for all k and $t \geq 1$, we thus have part (a) of our Proposition III. And part (b) is immediate.[9/] Part (c) follows from the equality versions of (1) and (2), which give equilibrium consumption, and (26), (27) and (29).

V. COOPERATIVE FIXED RATE REGIMES

We go on now to consider in turn various fixed exchange rate regimes: in this section, our so-called cooperative fixed rate regimes, the C regimes; and in the next, our noncooperative fixed rate or N regimes.

Under the LF/C and the PA/C regimes the two governments carry out their own monetary-fiscal policies. Thus, the government of the first country, once having chosen a value for the policy parameter, distributes or collects its own money in the way indicated by (3). And so does the government of the second country. It is together, though, that the two governments conduct exchange rate policy, or maintain the exchange rate at some agreed upon value. Since each government has an unlimited supply of its own money, the policy $E(t) = \bar{E}$ for all $t \geq 1$, where E is the agreed-upon value of E, is feasible.

But then under either C regime the evolution of the world money supply is given by (8). As we showed in section III, for (8) to hold it is only required that E(t) be the same in all periods. Furthermore, $\beta_k(t) = \beta(t)$ for all k and $t \geq 1$. So under either C regime the equilibrium Q(t) sequence is given by (21) and the additional restriction $E(t) = \bar{E}$ for all $t \geq 1$. Thus, the following proposition.

Proposition IV. Let $\bar{E}$ be the C-regime value of E that is maintained by the two governments. Then under either version of that regime the equilibrium is the $\bar{E}$ equilibrium of the LF/F regime.[10/]

What the two governments do then by agreeing on a value for E, a value to be imposed on the foreign exchange market, is choose one from among the many equilibria of the LF/F regime. It does not quite follow, though, that the LF/C and PA/C regimes, in contrast with the LF/F regime, are therefore practical real-world alternatives. The LF/F regime would not seem to be. How could the real world get along without there being ruling exchange rates? But reaching agreement on exchange rate values would perhaps be extremely difficult, except for certain monetary-fiscal policy choices.

In general, the equilibrium utilities of the first and second-country members of all generations depend on E when the α_k are different.[11/]

And as is easily verified, those utilities depend on E when the α_k have a common value α, different from unity. More particularly, if $\alpha > 1$, then the greater is E the worse off are the first-country members of generation $t \geq 1$ and the better off are the second-country members. For with $\alpha > 1$ the residents of both countries receive transfers from their respective governments. And it follows from (10) and (11) that the greater is E the less valuable are the transfers received by the first-country members of generation t and the more valuable are those received by the second-country members. But with $\alpha < 1$ the residents of both countries pay taxes. Moreover, the greater is E the less burdensome are the taxes paid by first-country members and the more burdensome are those paid by second-country members. Thus, if $\alpha < 1$, then the greater is E the better off are the first-country members and the worse off are the second-country members.

There is another way of making our point, by indicating how the equilibrium trade balances of the two countries depend on E. If the α_k are different, then the limiting values of the $\hat{B}_k(t)$ are independent of E. And except when $\alpha_M = 1$, those limiting values are different from zero.[12/] Furthermore, if the α_k have a common value α, then the $\hat{B}_k(t)$ are independent of E only when $\alpha = 1$. If $\alpha > 1$, then $\hat{B}_1(t)$ is greater the greater is E. And if $\alpha < 1$, then $\hat{B}_2(t)$ is greater the greater is E. [Recall (24) and footnote 8.]

As was pointed out above, if the α_k are different and $\alpha_M \neq 1$, then there is no E such that the lim $B_k(t) = 0$. But if $\alpha_k = \alpha$, $k = 1,2$, then there exists an E such that the $B_k(t) = 0$ for all $t \geq 2$. Indeed, if $\alpha = 1$, then for any E the $B_k(t)$ are identically zero. And if $\alpha \neq 1$, then there is a unique E at which that is so, the PA/F regime equilibrium value $\tilde{E}$.

There would thus seem to be a case for like (coordinated) monetary-fiscal policies. If the α_k are different, then under either of the C regimes the residents of one of the countries permanently subsidize those of the other. And that being so, the two governments, protectors of the interests of their respective citizenries, could find it extremely difficult if not impossible to reach agreement on a value of E to impose on the foreign-exchange market. But if the α_k are the same, then there is a value of E, namely $\tilde{E}$, at which with trade being balanced the residents of neither country subsidize those of the other. So agreement on pegging E at $\tilde{E}$ can be imagined.

And why is the LF/C regime better than the PA/F and PA/C regimes, both of which also give a determinate equilibrium exchange rate? Presumably, the answer is (although from our model it would not appear so) that any PA regime is relatively costly to maintain.

VI. NONCOOPERATIVE FIXED-RATE REGIMES

There are various noncooperative policies or strategies that might be considered. We limit ourselves almost exclusively, though, to one. It is much the same as that which many years ago, before the final crises of the Bretton Woods era, Friedman (1969) urged the United States government to impose on the world, and might be referred to, from the perspective, that is, of the inactive or passive government (for Friedman, the United States), as the policy of benign neglect. Under it, the government of, say, the first country obeys (3) and does nothing else. The government of the second country also obeys (3), but in addition maintains the exchange rate at the previously announced value E until it finds itself with insufficient reserves to carry on, at which time it (i) imposes the portfolio autarky regime, (ii) sells such reserves (money of the government of the first country) as it still holds, and (iii) foreswears any subsequent intervention in the market for foreign exchange.

In principle, the just-described policy can be followed whether or not portfolio autarky obtains, and in this section we consider both possibilities. Before considering the feasibility of the LF/N and PA/N regimes, though, with the intervention scheme as specified, we elaborate on what happens when the active government (for us, the government of the second country) has to give up, or when, as we say, there is a breakdown.

Obviously, for there to be a breakdown in period t, the period-t-1 money holdings of the members of generation t-1 must have been consistent with there having been no breakdown in period t-1. The government of the second country must have held some of the money of its first-country counterpart. And a breakdown occurs in period t if the money demands of the members of generation t, evaluated at $\bar{E}$, cannot be satisfied by the members of generation t-1 and the government of the second country. If they cannot, then the second-country government gives up as residual supplier-demander of the money of the government of the first country. It leaves the exchange market, never to return. It also imposes (or reimposes) portfolio autarky, telling (or reminding) the residents of the second country that they cannot lend to or borrow from first-country residents, and that they cannot own any first-country money or sell any second-country money to residents of the first country. And lastly the government of the second country sells off whatever first-country money it has left, thereby taking an equivalent value of its own money from individuals. Thus, in period t the demands for and supplies of the two monies are what they would be if all along the PA/F regime had obtained. The distribution of "income" over the members of generation t-1 may not be the same, but with everyone having the same tastes that is of no consequence. The equilibrium for period t and beyond is that of the PA/F regime.

Proposition V. If breakdown occurs in any period, say $\bar{t}$, then $E(t) = \tilde{E}(t)$ and $q_k(t) = \tilde{q}_k(t)$ for all k and $t \geq \bar{t}$, where as above $\tilde{E}(t)$ and $\tilde{q}_k(t)$ are the PA/F regime equilibrium values of E(t) and $q_k(t)$ respectively.

It is perhaps apparent why we assume that portfolio autarky is imposed when there is a breakdown. On that assumption, there are known terminal conditions. And with individuals possessed of perfect foresight, such conditions are very much needed. But the assumption is not just convenient. For as will be recalled, under the LF/F regime the equilibrium exchange rate is indeterminate.

Still to be determined, though, is whether there exists an equilibrium for the time interval prior to the breakdown. Is our intervention scheme feasible? Under the LF/N regime? And under the PA/N regime? We first consider the scheme on the assumption that portfolio choice is unrestricted.

Laissez-Faire

Suppose that through period t-1 there has been no breakdown, and consider the members of generation t, having to decide in period t which of the monies to buy. Assuming no breakdown in period t, there are two possibilities. Either there is no breakdown in period t+1, in which case $\beta_1(t) = \beta_2(t)$. Or there is a breakdown then. And in that event, as is implied by Proposition V, which is the more attractive money depends on how $\overline{E}$ and $\tilde{E}(t+1)$ compare. More particularly, if $\overline{E} < \tilde{E}(t+1)$, then $\beta_1(t) < \beta_2(t)$; and if $\overline{E} > \tilde{E}(t+1)$, then $\beta_1(t) > \beta_2(t)$.

Now, a breakdown is caused by an excessive demand for first-country money, by a demand that exceeds the amount that is outstanding under the PA/F regime. And as we assume, the demand is excessive when that money dominates second-country money, or when $\overline{E} > \tilde{E}(t+1)$. There is no breakdown, though, when second-country money dominates first-country money, or when $\overline{E} < \tilde{E}(t+1)$. So the following proposition, which describes how the pre-breakdown equilibrium depends on $\overline{E}$ and the α_k, is almost immediate. Only a few brief remarks are required to establish it.

Proposition VI. Under the LF/N regime, and with the exchange rate policy of benign neglect, a unique equilibrium exists. Further: (a) if $\alpha_1 \geq \alpha_2$ and $\overline{E} \leq \tilde{E}(1)$, then the equilibrium is the $\overline{E}$ equilibrium of the C regime; (b) if $\alpha_1 < \alpha_2$ and $\overline{E} < \tilde{E}(1)$, then the equilibrium exchange rate is

$$E(t) = \begin{cases} \overline{E} \text{ for all } t \leq \overline{t} \\ \\ \tilde{E}(t) \text{ for all } t \geq \overline{t} \end{cases},$$

where $\overline{t}$ is the value of t, assumed to exist, such that $\tilde{E}(t) = \overline{E}$;[13/] and (c) if $\overline{E} > \tilde{E}(1)$, then breakdown occurs in the first period, and therefore, for all $t \geq 1$, the equilibrium is that of the PA/F regime.

To prove part (a), we note that if either of the hypotheses holds with strict inequality, then second-country money dominates in every period and there is never a breakdown. On its own, the government of the second country does what is done cooperatively under the C regime. And if both hypotheses hold with equality, then a breakdown may occur. . . (Money demands are indeterminate.) But for all $t \geq 1$ the $\beta_k(t)$ are independent of whether there is a breakdown.

To prove part (b), we first establish that breakdown occurs in period $\bar{t}$. Since $\alpha_1 < \alpha_2$, $\tilde{E}(t)$ is exponential decreasing. [Recall (31).] And therefore, by the definition of $\bar{t}$, second-country money dominates in all periods $t \leq \bar{t} - 2$, and first-country money dominates in all periods $t \geq \bar{t}$. In period $\bar{t} - 1$, there is no domination. That, however, is only because time is discrete. If it were not, then second-country money would dominate for all $t < \bar{t}$. So ignoring the trouble that discrete time makes, we conclude that breakdown occurs in period $\bar{t}$.

The question then is about periods 1, 2, ..., $\bar{t}$-1. But the equilibrium for those periods is readily enough calculated. There is a single asset $M(t)$, which for all $t < \bar{t}$ is given by (8) with $E = \bar{E}$. It has a positive period-$\bar{t}$ value, from which a unique positive period-$\bar{t}$-1 value can be derived. And proceeding backward in time, the equilibrium values for the earlier periods can be too.[14/]

Part (c) is also easily established. If $\bar{E} > \tilde{E}(1)$, then $\bar{E} \geq \tilde{E}(2)$. (Recall footnote 13.) And if $\bar{E} > \tilde{E}(2)$, then first-country money dominates even in the first period. Of course, if $\bar{E} = \tilde{E}(2)$, then there is no domination in that period. But again, since the possibility exists only because time is discrete, we ignore it.

Our Proposition VI, which has now been proved, says for one thing that absent portfolio restrictions the government of the second country (or, more generally, the active government, the one that on its own is managing the exchange rate) cannot even temporarily overvalue its currency. That is to say, it cannot impose a value of the exchange rate that is greater than any that would obtain under the PA/F regime.

Portfolio Autarky

The question is what it was: under the noncooperative intervention scheme described at the outset of this section, is there an equilibrium? Now, however, the assumption is that portfolio autarky obtains even before a breakdown occurs.

In this subsection we proceed somewhat indirectly, and to begin provide more definitions. The cooperative abandonment strategy $(\bar{E},t^*)$ is the strategy under which the exchange rate $\bar{E}$ is cooperatively maintained for all $t < t^*$, and then, in period t^*, any government reserves are sold, the PA/F regime is imposed, and retained forever after. If the strategy $(\bar{E},t^*)$ satisfies the condition

$$N_1 q_1(t) \leq P_1(t) M_1 \alpha_1^{t-1}$$

for all $t < t^*$, or is consistent with the government of the second country having nonnegative reserves, then it is a feasible noncooperative abandonment strategy, or since $\bar{E}$ is the maintained exchange rate, an $\bar{E}$ feasible noncooperative abandonment strategy. And in our notation, the set of all such strategies, obtained by varying t^*, is $S(\bar{E})$. Finally, the PA/N regime $\bar{E}$ equilibrium strategy (that is, the equilibrium strategy with $E = \bar{E}$) is that element of $S(\bar{E})$ with the maximum t^*.

Obviously, there is an equilibrium for any cooperative abandonment strategy, and it is in principle straightforward to determine whether it is in $S(\bar{E})$. The period-t^* values of the two monies are what they would be under the PA/F regime, namely,

$$\tilde{P}_k(t^*) = (N_k y_k/M_k\alpha_k{}^{t^*-1})/[1 + F(1/\alpha_k)] \qquad \text{for } k = 1,2.$$

And using the $\tilde{P}_k(t^*)$ as terminal conditions, one solves for the equilibrium of period t*-1, then for the equilibrium of period t*-2, and so on back to the equilibrium of the first period. Let $\overline{P}_k(t)$ and $\overline{m}_k(t)$, $t < t^*$, be the equilibrium values of $P_k(t)$ and $m_k(t)$ respectively. Then; with t, $\overline{E}$ and the $P_k(t)$ as parameters, the $\overline{P}_k(t-1)$ and $\overline{m}_k(t-1)$ are given by the following equations:

$$(32)\ [y_k - P_k(t-1)m_k(t-1)]/P_k(t)[m_k(t-1) + (1-\alpha_k)\alpha_k^{t-2}M_k/N_k]$$

$$= F[P_k(t)/P_k(t-1)] \qquad \text{for } k = 1,2$$

$$(33)\ N_1 m_1(t-1) + \overline{E}N_2 m_2(t-1) = \alpha_1^{t-2}M_1 + E\alpha_2^{t-2}M_2$$

$$(34)\ P_2(t-1)/P_1(t-1) = \overline{E}$$

Equation (32) is simply a version of (20), (33) is an equilibrium condition (aggregate money demand equals supply), and (34) is the exchange rate constraint. The given strategy is in $S(\overline{E})$ if $\overline{m}_1(t) \leq \alpha_1^{t-2}M_1/N_1$ for all $t < t^*$.

But our goal is a characterization of the PA/N regime equilibrium like that of the LF/N regime equilibrium given above and in pursuit of that goal we find it convenient to consider the three following possibilities separately: $\alpha_1 = \alpha_2$; $\alpha_1 < \alpha_2$; and $\alpha_1 > \alpha_2$.

The first of our PA/N regime propositions says in effect that with $\alpha_1 = \alpha_2$, there is no difference between the LF/N and PA/N regimes.

Proposition VII. Suppose that $\alpha_1 = \alpha_2$. Then (a) if $\overline{E} > \tilde{E}$, $S(\overline{E})$ is empty. And (b) if $\overline{E} \leq \tilde{E}$, then $(\overline{E},\infty)$ is an element of $S(\overline{E})$.

To establish part (b), we have only to recall that $P_1(t)M_1\alpha_1^{t-1} = Q(t)\lambda(t)$ and, as was shown above (footnote 8), that under the C regime the difference $N_1q_1(t) - Q(t)\lambda(t)$ is strictly increasing in E and zero at $E = \tilde{E}$. It also follows that for $\overline{E} > \tilde{E}$, $(\overline{E},\infty)$ is not in $S(\overline{E})$. So it only remains to prove that for $\overline{E} > \tilde{E}$ and any $t^* \geq 2$, $(\overline{E},t^*)$ is not in $S(\overline{E})$. In doing that, it is convenient to rewrite (32)-(34) in the $P_k(t-1)$ and the new variables.

$$Z_k(t-1) \equiv N_km_k(t-1) - M_k\alpha^{t-2}, \quad k = 1,2.$$

It is then easily shown that the solution $\overline{Z}_1(t^*-1)$ is strictly increasing in $\overline{E}$, and further that $\overline{Z}_1(t^*-1) = 0$ at $\overline{E} = \tilde{E}$.

And if $\alpha_1 < \alpha_2$? Then no positive exchange rate can be maintained indefinitely by the government of the second country. It can, however, maintain any rate at least as long when portfolio autarky obtains as when laissez-faire does.

<u>Proposition VIII.</u> Suppose that $\alpha_1 < \alpha_2$. Then (a) for any $\overline{E} > 0$, $(\overline{E},\infty)$ is not in $S(\overline{E})$. But (b) that set does contain $(\overline{E},\overline{t})$ where $\overline{t}$ is defined as in the preceding subsection.

To prove part (a), we suppose to the contrary that

$$(35) \quad N_1\overline{q}_1(t) \leq \overline{P}_1(t)M_1\alpha_1^{t-1} \qquad \text{for all } t \geq 1,$$

where $\overline{q}_k(t)$ and $\overline{P}_k(t)$ are the PA/C regime equilibrium values of $q_k(t)$ and $P_k(t)$ respectively with $E = \overline{E}$. And since $\lim \lambda(t) = 0$ when $\alpha_1 < \alpha_2$ (14) and (17) imply that $\lim \overline{q}_1(t) > 0$, and hence that the limit of the LHS of (35) is positive. But our version of Gresham's Law tells us that the limit of the RHS is zero. So we have the sought-after contradiction, and hence part (a) of our Proposition VIII.

In proving part (b) we proceed in steps, the first being to show that

$$(36) \quad \overline{Q}(t-1) > \overline{Q}(t) > \tilde{Q} \qquad \text{for all } t < \overline{t},$$

where $\{\overline{Q}(t)\}$, the PA/N regime equilibrium sequence, satisfies (20) for $t < \overline{t}$ and with $\overline{Q}(\overline{t}) = \tilde{Q}$. As noted in Appendix B, (20) is equivalent to

$$\ln Q(t) = \gamma[\ln Q(t+1), \ln\sigma(t)],$$

where, for given $\sigma(t)$, γ is increasing and bounded from above by ln y, and has a unique fixed point. And if lnx is the solution to

$$\ln x = \gamma[\ln x, \ln \sigma(\overline{t}-1)],$$

then, since $\sigma(t)$ is increasing and $\gamma_2 < 0$, $x \geq \tilde{Q}$ implies (36).

Now,

$$(x-\tilde{Q})/y = 1/\{1 + F[1/\sigma(\overline{t}-1)]\} - b/v_1 - (1-b)/v_2 = h(b),$$

where $b = N_1 y_1/y$ and $v_k = [1 + F(\tilde{\beta}_k)]$. Then, using (9) and (10) to write $\sigma(t-1)$ as a function of $\lambda(t)$, we have $\sigma(\overline{t}-1) = \alpha_1/[\delta + (1-\delta)\lambda(\overline{t})]$, where $\delta = \alpha_1/\alpha_2$ and where, by the definition of $\overline{t}$, $\lambda(\overline{t}) = 1/[1 + v_1(1-b)/v_2 b]$. It follows that

$$\sigma(\overline{t}-1)^{-1} = \sigma(b)^{-1} = \{\delta[b + v_1(1-b)/v_2] + b(1-\delta)\}/\alpha_1[b + v_1(1-b)/v_2],$$

and hence that $\sigma(0)^{-1} = \tilde{\beta}_2$ and $\sigma(1)^{-1} = \tilde{\beta}_1$. So h(b) is a twice differentiable function of b with h(0) and h(1) = 0. Thus, $h < 0$ implies that h is a minimum for some b in (0.1), at which value $h < 0$, $h' = 0$ and $h'' > 0$. These, however, are mutually inconsistent conditions, as can be verified by computing the derivatives of h.

Now, it follows from (36) that

$$\overline{P}_1(t)[M_1(t) + \overline{E}M_2(t)] > \tilde{P}_1(t)[M_1(t) + \tilde{E}(t)M_2(t)] \qquad \text{for all } t < \overline{t}.$$

But since $\overline{E} < \tilde{E}(t)$ for all $t < \overline{t}$,

$$(37) \quad \overline{P}_1(t) > \tilde{P}_1(t) \qquad \text{for all } t < \overline{t}.$$

Further, $\overline{Q}(t-1) > \overline{Q}(t)$ for all $t < \overline{t}$ implies $1/\sigma(t) > \overline{\beta}(t)$ for all $t < \overline{t}$. And $\alpha_1 < \alpha_2$ implies $\tilde{\beta}_1 = 1/\alpha_1 > 1/\sigma(t)$ for all $t < \overline{t}$. So

$$(38) \quad \tilde{\beta}_1 > \overline{\beta}(t) \qquad \text{for all } t < \overline{t}.$$

We want to show that $\overline{M}_1(t) \leq \tilde{M}_{1(t)}$ for all $t < \overline{t}$, or equivalently that

$$(39) \quad \overline{q}_1(t)/\overline{P}_1(t) \leq \tilde{q}_1/\tilde{P}_1(t) \qquad \text{for all } t < \overline{t}.$$

It follows from (37) that if $\alpha_1 \geq 1$, then $\overline{P}_1(t)x_1(t) \geq \tilde{P}_1(t)x_1(t)$ for all $t < \overline{t}$. But that ordering of the transfers and (38) imply that $\overline{q}_1(t) \leq \tilde{q}$, for all $t < \overline{t}$. [Recall footnote 7.] And therefore, since (37) holds, we have (39). Alternatively, if $\alpha_1 < 1$, then (39) is equivalent to

$$(40) \quad \overline{q}_1(t)/\overline{P}_1(t)x_1(t) \geq \tilde{q}_1/\tilde{P}_1(t)x_1(t) \qquad \text{for all } t < \overline{t}.$$

But both sides of (40) satisfy $q/z = g(z,\beta)/z \equiv g^*(z,\beta)$ where $g_1^* \leq 0$ and $g_2^* \leq 0$. [Again, recall footnote 7.] So (40) follows from (38) and the inequality $\overline{P}_1(t)x_1(t) < \tilde{P}_1(t)x_1(t)$. And with that observation our proof is complete.

And when $\alpha_1 < \alpha_2$ does $S(\overline{E})$ necessarily contain strategies with $t^* > \overline{t}$? The answer is "no." For some specifications, $S(\overline{E})$ does, we believe, contain such strategies. With the proper choice of u, it will emerge that the government of the second country can overvalue its money, although only temporarily. For awhile, it can maintain a value of the exchange rate greater than the PA/F regime equilibrium value. But if u is Cobb-Douglas, then $S(\overline{E})$ contains no strategies with $t^* > \overline{t}$. (That is to say, the proof makes nontrivial use of the gross substitution condition that is satisfied by, among others, the Cobb-Douglas utility function.) With u Cobb-Douglas, the government of the second country acquires reserves in the first period. But the country has trade deficits, in periods $t = 2,3,\ldots,\overline{t}-1$, the sum of which equals the first-period acquisition of reserves. (But then a period-t trade deficit does not imply, in our sense of the phrase, an overvalued currency, for the second country has a trade deficit in period t, $2 \leq t < \overline{t} - 1$, and the period-t exchange rate is not greater than it would be under the PA/F regime.

And now to consider the third and last possibility, $\alpha_1 > \alpha_2$. With that ordering of the α_k, the threat is not of eventual breakdown. For any $\overline{E}$, there exists a value of t, say t', such that (35) holds for all $t \geq t'$. [The argument is essentially that used to establish Proposition VIII (a).] But if $(\overline{E},\infty)$ is to be in $S(\overline{E})$, then (35) must hold as well for $t < t'$.

Under the C regime, $\overline{E}$ affects the equilibrium only by way of $\sigma(t)$. And further, with $\alpha_1 > \alpha_2$, $\sigma(t)$ gets closer to its limiting value α_1 as $\overline{E}$ gets smaller. So there must be values of $\overline{E}$ that can be maintained indefinitely, and without benefit of cooperation, by the government of the second country. As the following Proposition says, though, not any value of $\overline{E}$ can be so maintained.

Proposition IX. Suppose that $\alpha_1 > \alpha_2$. That even if $\overline{E} \leq \tilde{E}(1)$, $(\overline{E},\infty)$ may not be an element of $S(\overline{E})$.

Let u be such that $g_2 > 0$. [Recall footnote 7.] Also, let $\alpha_2 = 0$ and $\overline{E} = \tilde{E}(1)$. And suppose that $(\overline{E},\infty)$ is in $S(\overline{E})$. Since the residents of the second country get no transfers, and since $1/\alpha_2 > 1/\sigma(t) > \overline{\beta}(t)$, we have (i) $\overline{q}_2(t) < \tilde{q}_2$. But since $(\overline{E},\infty)$ is by assumption in $S(\overline{E})$, we also have (ii) $\overline{M}_2(1) \geq \tilde{M}_2(1)$. And (i), (ii) and $\overline{E} = \tilde{E}(1)$ imply (iii) $\overline{P}_1(1) < \tilde{P}_1(1)$. But the implied ordering of the transfers received under the different regimes by the age-one residents of the first country in the first period, and the inequality $\overline{\beta}_1(t) > 1/\alpha_2 = \tilde{\beta}_1$, which is established in Appendix B, imply that $\overline{q}_1(1) > \tilde{q}_1$. And hence, by (iii), $\overline{M}_1(1) > \tilde{M}_1(1)$, a contradiction.

Other Schemes

We have devoted considerable space to one noncooperative policy or strategy, that which might best be called the abandonment strategy. There are many others, though, among them, the following "feedback" strategies: that under which one government, by itself, manages the exchange rate in accordance with what has been happening to its reserves; and that under which the active government varies its monetary-fiscal policy (for the country-k government, α_k) according to how its reserves have been changing. But the latter strategy, we must confess, does not interest us greatly, since in our view the α_k, being in a fundamental sense fiscal policy variables, cannot be freely varied. Perhaps more to the point, though, having analyzed our several regimes for arbitrary α_k, we have already implicitly said a good bit about what a policy rule relating monetary-fiscal policy and reserves would do.

As regards the first of the alternative schemes, maybe the most important observation is that it is not feasible if portfolio choice is unrestricted. With free choice, there cannot be anticipated changes in the exchange rate. But if portfolio choice is restricted, then some such schemes likely are feasible. When portfolio autarky obtains, individuals cannot avoid entirely the capital losses implied even by anticipated exchange rate changes. And in our model there exists a monetary equilibrium for any finite rate of return on money, however negative that rate may be. So we too conclude (and recall our Proposition IX) that there is more scope for exchange rate management when portfolio autarky obtains than when it does not.

VII. CONDITIONS FOR OPTIMALITY

It remains only for us to provide the necessary and sufficient conditions for the optimality of exchange rate regime equilibria. Before doing so, however, we give a final few definitions.

For us, an allocation is a sequence $\{c\} = c_2(0), c(1), \ldots, c(t), \ldots$, where $c_2(0)$ is the N element vector of the second-period consumption allocation of the members ($N = N_1 + N_2$) of generation zero, and $c(t) = (c^1(t), c^2(t), \ldots, c^N(t))$ is the vector of the life-time consumption allocations of the members of generation $t \geq 1$. And as we say, the allocation $\{c\}$ is convergent if the $c(t)$ go to a limit as $t \to \infty$.

Further, an allocation $\{\bar{c}\}$ is Pareto-superior to another, say $\{\hat{c}\}$, if $\bar{c}_2^h(0) \geq \hat{c}_2^h(0)$ for all h and $u[\bar{c}^h(t)] \geq u[\hat{c}^h(t)]$ for all h and $t \geq 1$, and either the first inequality is strict for at least one value of h or the second is (or both are). And finally, an allocation $\{\hat{c}\}$ is Pareto-optimal if there exists no feasible allocation $\{\bar{c}\}$ that is Pareto-superior to it.

The necessary and sufficient conditions are given in the following proposition, proved in Appendix C.

<u>Proposition X.</u> An equilibrium allocation $\{c\}$ is Pareto-optimal if and only if

(a) $\sum_h c_2^h(t-1) + \sum_h c_1^h(t) = y$ for all $t \geq 1$,

(b) $f[c_1^h(t)/c_2^h(t)] = \eta(t)$ for all h and $t \geq 1$, and

(c) $\lim \eta(t) \geq 1$.

Condition (a) requires for all $t \geq 1$ that the period-t allocations of the members of generations t-1 and t add up to the world endowment, and of course is necessarily satisfied by all of the equilibria of all of our regimes, whatever the α_k may be. Condition (b), also a static condition, requires that there be a common intertemporal marginal rate of substitution for all the members of each generation $t \geq 1$. It does not require that the common rates of different generations be equal. Even so, not all of the equilibria satisfy it. All of the LF regime equilibria do. But unless $\alpha_1 = \alpha_2$ no PA equilibrium does, except that characterized by $P_k(t) = 0$ for all k and $t \geq 1$. And condition (c), truly a dynamic condition, requires that the limiting value of those common marginal rates of substitution (a value which exists, for all equilibrium allocations converge) be not less than unity. Thus, even the LF regime equilibria are not necessarily optimal. Any such equilibrium is optimal if $\alpha_M \leq 1$. And if $\alpha_M > 1$, then no such equilibrium is. Finally, if $\alpha_k > 1$ and $\alpha_{k'} \leq 1$, then the only optimal LF regime equilibrium is that with the money of country k′ having value and that of country k having none.

On some assumptions, it may make all the difference whether exchange rates are fixed or flexible. On ours, however, as the foregoing discussion makes clear, that is in a sense anyway of no consequence. Even when portfolio choice is unrestricted, it is not sufficient (or necessary) for optimality that exchange rates be flexible. But on our assumptions optimality can be guaranteed by the appropriate choice of monetary-fiscal policies.[15/]

Proposition XI. It is sufficient for optimality of the (monetary) equilibrium of any of our regimes that $\alpha_k = \alpha \leq 0$ for k = 1,2. Moreover, the policy $\alpha_k = 0$, k = 1,2, is special, even among those that ensure optimality. Under it, neither government cares what regime obtains, or what the equilibrium value of E is, except insofar as E affects the distribution of income over the members of generation zero.

VIII. CONCLUSION

Why, though, do our governments have those monetary-fiscal policies that period after period they so faithfully follow? We have provided no answer to that question, which we suspect is why we have our Proposition XI. What if we had started off in this paper by creating a world in which taxation serves some worthy purpose, and in which, whatever the type of tax, it costs something to collect revenues? We likely would not then have ended up by establishing that the policies $\alpha_k = \alpha \leq 0$, $k = 1,2$, are all optimal, or that $\alpha_k = 0$, $k = 1,2$ is the "best" of those policies. In such a world, taxing partly by inflation may well be justified. And different countries having different inflation rates may be too. In such a world, there may well be something to be said for the α_k being different, and for one or both being positive. That is still to be verified. But believing as we do, we certainly would not insist that the results given in Proposition XI are quite general.

About one of our results, though, in our judgment, the most basic, we would. We have in mind Proposition II(b), which says in effect that a multiplicity of fiat monies is unnatural, or that in the natural state, in the absence, that is, of government-inspired restrictions on portfolio choice and official exchange-market intervention, equilibrium exchange rates are indeterminate. Of course, if it is right that optimal taxation involves some taxing by inflation, and further that in general optimal tax burdens (and hence inflation rates) differ from country to country, then the existence of several fiat monies is in no sense unnatural. Each government must have its own. But then our Proposition II(b) has only to be interpreted differently, as saying that restrictions on portfolio choice (or, more generally, restrictions that make the several extant fiat monies other than perfect substitutes) are the inevitable consequence of attempts by governments to approximate optimal taxation.

The problem posed by the existence of more than one fiat money can in theory be solved, as our Proposition V suggests, by imposing the LF/C regime. Under that regime there are ruling exchange rates, cooperatively maintained by governments, at which trades of fiat monies can be made. On our reading, however, admittedly rather casual, history denies the practicability of that solution. The fixed-rate regimes of the period between World Wars I and II did not exhibit the cooperation that is the sine qua non of our LF/C regime. Nor did the Bretton Woods regime. The real world seems never to have known a regime closely approximating it. And the explanation could be that under it too much monetary coordination is required.

But portfolio restrictions essentially the same as ours (or to borrow from the lexicon of international economics, capital controls) have often been imposed by real-world governments. And that, it seems to us, argues for the model used in this paper, or for some generalization thereof. Unlike traditional balance-of-payments models, it provides an explanation for why governments have resorted to such restrictions.

Some may, though, be wondering how robust our results are. And more particularly, will our basic result, that even two fiat monies are too many, hold for a world economy in which there are other kinds of assets as well? If some of those other kinds of assets are too good, such that they dominate the fiat monies as stores of value (and whether they do depends in part on what the α_k are), then whatever the regime there will exist no monetary equilibrium. In equilibrium, no fiat money will have value. But if there is no dominance, then the LF/C regime equilibrium will be indeterminate. At any rate, that is our guess. The existence of other kinds of assets would not seem to make fiat monies less than perfect substitutes, each for all the others.

Then, too, there are (or may be) other ways of generating money demands. And if one of those alternative ways is used, will the resulting model tell us, as ours does, that the LF/C regime equilibrium, if it exists, is indeterminate. We believe so. Fiat monies are likely always to be perfect substitutes. Unless of course governments interfere.

The foregoing is by way of saying that the results given in Proposition II and others as well, although obtained on the cheap, using a particularly simple model of a monetary economy, should nevertheless be taken seriously.[16/] There does remain the task, though, of determining which of our results also hold for a risky world economy, for a world economy in which endowments or production functions are random, and monetary-fiscal policies are too. Our guess is that randomness does not by itself make fiat monies imperfect substitutes.

APPENDIX A

We observed in the introductory section of this paper that our model can be interpreted as a Heckscher-Ohlin generalization of Samuelson's overlapping-generations model. What we meant, as we explained there, is that certain assumptions yield a kind of recursiveness. The equilibrium terms of trade and factor rentals (and hence the amounts of the various goods produced and consumed and the factor allocations) are independent of time and, in our sense of the phrase, of the choice of international economic policy regime as well. So it is possible to regard our (within-country) endowments as per capita GNP, or as a composite of many produced goods, say $\sum_i \tau_i y_i$, where y_i is the equilibrium quantity of the i^{th} good, $i = 1,2,\ldots, I$, and τ_i is the price of the i^{th} good in units of the first. And in this appendix we indicate the assumptions that are sufficient to give the appropriately recursive equilibrium solution.[17/]

Utility functions should be identical for all individuals, separable in the consumptions of different periods, and homothetic in the components of within-period consumption. Resource endowments and production functions should be constant over time and such as to yield acceptable production possibility frontiers.[18/] And, finally, there has to be free trade in goods. For if there is a tariff, then in general recursiveness does not obtain.

There is of course the obvious question: Why a recursive equilibrium? Why have the goods terms of trade independent of, say, the choice of exchange rate regime? Why indeed? We do not have a compelling answer, other perhaps than that nothing very definite about the relationship between, for example, the terms of trade and the policy regime can be expected from a model with a generally simultaneous equilibrium solution.

APPENDIX B

In this appendix we first show that for sequences {Q(t)} that are positive and bounded away from zero, (20) has a unique solution, denoted ϕ_A, which is continuous and bounded, and then go on to establish certain properties of the monetary equilibrium given by ϕ_A.

Since Q(t) and $\sigma(t)$ are positive for all $t \geq 1$, we can rewrite (20) as follows:

$$G[\ln Q(t), \ln Q(t+1), \ln \sigma(t)] = 0.$$

Further, the partial derivatives of G, obtained from (20), are

$$G_1 = Q(t)\ \{1 - \sigma(t)[\beta(t)]^2 F\} > 0$$

$$G_2 = Q(t+1)[F+\beta(t)/F'] \leq 0$$

$$G_3 = -Q(t)[\beta(t)]^2\sigma(t)F > 0$$

[We have supressed the argument of F, which recall is $\beta(t)$.] The sign restrictions on G_1 and G_3 are implied by $F' = 1/f'$, and that on G_2 is implied by the gross substitution restriction on u, namely, $xf'(x)/f(x) \geq -1$. Thus, the sequence {Q(t)} satisfies (20) if and only if it satisfies

$$\text{(B1)}\quad \ln Q(t) = \gamma[\ln Q(t+1), \ln \sigma(t)],$$

where $\gamma_1 = -G_2/G_1$ is in [0,1) and $\gamma_2 = -G_3/G_1$ is in (-1,0).

Equation (B1) is a first-order difference equation in ln Q(t), with $\sigma(t)$, exogenously determined by $A = (\alpha_1,\alpha_2)$, as the driving variable. So for us {Q(t)} is an equilibrium sequence if for given A it satisfies

$$\text{(B2)} \quad \ln Q(t) = \phi_A[\sigma(t)] \qquad \text{for all } t \geq 1,$$

where ϕ_A is a continuous (and real-valued) function defined on the interval $I(A) = [\alpha_m, \alpha_M]$ and such that

$$\text{(B3)} \quad \phi_A[\sigma(t)] = \gamma\{\phi_A[\sigma(t+1)], \ln \sigma(t)\}.$$

In our notation, S is the set of all continuous (and real-valued) functions that are defined in the interval I(A), bounded from above by $\ln y$ and bounded from below by $\ln L$, where L is any element of $(0,Q^*]$, and Q^* is the unique number satisfying the conditions $0 < Q^* < y$ and

$$G[\ln Q^*, \ln Q^*, \ln \alpha_M] = 0.$$

[That Q^* is as described can be verified using (20).] And as we now prove:

<u>Proposition BI.</u> For any admissible A, there exists one and only one point (function) is S that satisfies (B3).

To begin our proof, we note that by the definition of $\lambda(t)$

$$\lambda(t+1) = (\alpha_1/\alpha_2)\lambda(t)[1 - \lambda(t) + \alpha_1\lambda(t)/\alpha_2].$$

It follows that

$$\text{(B4)} \quad \sigma(t+1) = \alpha_1 + \alpha_2 - \alpha_1\alpha_2/\sigma(t) \equiv \psi[\sigma(t)].$$

And for $Z \equiv \sigma(t)$, (B3) is equivalent to

$$\text{(B5)} \quad \phi_A(Z) = \gamma\{\phi_A[\psi(Z)], \ln Z).$$

Now, let ϕ be any element of S and let the mapping Γ be defined as follows:

$$\text{(B6)} \quad \Gamma(\phi) = \gamma\{\phi[\psi(Z)], \ln Z\}.$$

Evidently, any element ϕ_A of S is a fixed point under Γ. And vice versa.

As is well known, the metric space S = (S,ρ), where for ϕ_1 and ϕ_2, any two elements of S, $\rho(\phi_1,\phi_2) = \max |\phi_1 - \phi_2|$, is complete. Therefore, all we have to show is (i) that Γ maps S into itself and (ii) that Γ is a contraction. [See Kolmogorov and Fomin (1957), pp. 34, 43.]

The proof of (i) is as follows. Let Z be any element of I(A). Then by (B4) the argument of ϕ on the RHS of (B6) is in I(A). And therefore, since γ is continuous, Γ maps continuous functions into continuous functions. So what remains is to show that $\Gamma(\phi)$ is appropriately bounded.

For any Z in I(A), there is a unique number, say $\bar{x}$, such that $0 < \bar{x} < \ln y$ and $\bar{x} = \gamma(\bar{x}, \ln Z)$. [See (20).] Moreover, since $\gamma_2 < 0$ and $Z \leq \alpha_M$, we have $\bar{x} \geq \ln Q^*$. And since $0 \leq \gamma_1 < 1$, we also have:

(a) if $x \leq \bar{x}$, then $x < \gamma(x, \ln Z) \leq \bar{x}$; and

(b) if $x > \bar{x}$, then $\bar{x} < \gamma(x, \ln Z) < x$.

Now, $\phi[\psi(Z)]$ is either in $[\ln L, \bar{x}]$ or in $(\bar{x}, \ln y]$. And if it is in the first interval, then by (a)

$$\ln L \leq \phi[\psi(Z)] < \gamma\{\phi[\psi(Z)], \ln Z\} \leq \bar{x} \leq \ln y.$$

But if it is in the second interval, then by (b)

$$\ln L \leq \ln Q^* \leq \bar{x} < \gamma\{\phi[\psi(Z)], \ln Z\} < \phi[\psi(Z)] \leq \ln y.$$

Thus, $\Gamma(g)$ is appropriately bounded.

Having established that Γ maps S into itself, we turn now to showing that Γ is a contraction. Suppose that for any Z

$$x_1 = \phi_1[\psi(Z)]$$

$$x_2 = \phi_2[\psi(Z)]$$

where again ϕ_1 and ϕ_2 are arbitrary elements of S. Then by the Mean Value Theorem,

$$\gamma(x_2, \ln Z) - \gamma(x_1, \ln Z) = (x_2 - x_1)\gamma_1,$$

where γ_1 is evaluated at (x, ln Z) for some x in $[x_1, x_2]$. And hence, since $\gamma_1 \geq 0$,

$$\max_Z |\gamma(x_2, \ln Z) - \gamma(x_1, \ln Z| \leq (\max_Z \gamma_1)(\max_Z |x_2 - x_1|)$$

$$= (\max_Z \gamma_1)(\max_Z |\phi_2 - \phi_1|).$$

So we have only to show that $\max_Z \gamma_1 < 1$.

But $\gamma_1 < \mu/(1 + \mu)$, where $\mu = -\sigma(t)[\beta(t)]^2 F' > 0$. Thus, if μ is bounded from above, Γ is a contraction. The elements ϕ_1 and ϕ_2 are, however, bounded from above and below, which as we have seen implies that γ has upper and lower bounds, and it follows that γ_1 (and hence μ) is evaluated at some point which is bounded from above and below. Consequently, μ is bounded.

We have now established our Proposition BI, which gives uniqueness on the space S. It is natural to ask, though, whether there are functions satisfying (B3) that are not in S. Since any equilibrium sequence $\{Q(t)\}$ must be bounded from above by ln y, no equilibria are ruled out by the requirement that the elements of S are bounded from above. But we have not so far succeeded in showing that any sequence satisfying (20) must be bounded from below by some positive number. So there may be equilibrium functions satisfying (B3) that generate sequences which converge to zero. If, however, there are equilibrium sequences with $\lim Q(t) = 0$, then in the limit they are the same as the equilibrium with $Q(t) = 0$ for all $t \geq 1$.

We now establish a second proposition:

<u>Proposition B II.</u> The unique equilibrium function ϕ_A is such that

$$\text{(a)} \quad \phi_A[\sigma(t)] \geq \phi_A[\sigma(t+1)] \qquad \text{for all } t \geq 1$$

and

$$\text{(b)} \quad 1/\sigma(t) \geq \beta(t) \geq \lim \beta(t) \equiv \beta^* \qquad \text{for all } t \geq 1.$$

To prove part (a), we first note that with Γ being a contraction, repeated application of Γ to any element ϕ of S yields a sequence, $\Gamma(\phi)$, $\Gamma(\Gamma(\phi)) = \Gamma^{(2)}(\phi),\ldots,\Gamma^{(m)}(\phi),\ldots$, that converges to ϕ_A. If $\phi^{(0)} = \ln Q^*$ is taken as the starting point, then the resulting sequence is (with the argument of σ suppressed)

$$\phi^{(1)}(\sigma) = \gamma[\ln Q^*, \ln \sigma]$$

$$\phi^{(2)}(\sigma) = \gamma[\phi^{(1)}(\sigma), \ln \sigma]$$

$$\vdots$$

$$\phi^{(n)}(\sigma) = \gamma[\phi^{(n-1)}(\sigma), \ln \sigma]$$

$$\vdots$$

and $\lim \phi^{(n)}(\sigma) = \phi_A(\sigma)$.

Now, since $\gamma_2 < 0$, we have $\phi^{(1)}(\sigma_0) > \phi^{(1)}(\sigma_1)$, where σ_0 and σ_1 are any two values of σ satisfying the restriction $\sigma_0 < \sigma_1$. And since $\gamma_1 \geq 0$, $\phi^{(n)}(\sigma_0) \geq \phi^{(n)}(\sigma_1)$ for all $n \geq 1$. Thus, $\phi_A(\sigma_0) \geq \phi_A(\sigma_1)$. And part (a) follows from the observation that $\{\sigma(t)\}$ is nondecreasing for any admissible choice of A.

Having shown that $Q(t) \geq Q(t+1)$ for all $t \geq 1$, we also have that $P_1(t)M(t) \geq P_1(t+1)M(t)\sigma(t)$, from which the first of the inequalities of part (b) of Proposition B II is immediate. And by the definition of $\beta(t)$ we have that

$$\begin{aligned} \ln \beta(t) - \ln \beta^* &= \ln Q(t+1) - \ln Q(t) - \ln \sigma(t) + \ln \sigma^* \\ &= \ln Q(t+1) - \gamma[\ln Q(t+1), \ln \sigma(t)] - \ln \sigma(t) + \ln \sigma^*, \end{aligned}$$

where $\sigma^* \equiv \lim \sigma(t)$. Also, by the Mean Value Theorem,

$$\begin{aligned} \gamma[\ln Q(t+1), \ln \sigma(t)] &= \gamma[\ln Q^*, \ln \sigma^*] + \gamma_1[\ln Q(t+1) - \ln Q^*] + \gamma_2[\ln\sigma(t)-\ln\sigma^*] \\ &= \gamma_1 \ln Q(t+1) + (1-\gamma_1)\ln Q^* + \gamma_2[\ln \sigma(t) - \ln \sigma^*], \end{aligned}$$

where $Q^* = \lim Q(t)$. Therefore,

$$\ln \beta(t) - \ln \beta^* = (1-\gamma_1)[\ln Q(t+1) - \ln Q^*] - (1+\gamma_2)[\ln \sigma(t) - \ln \sigma^*].$$

But then with the γ_i bounded in the ways indicated above, it must be that $\ln \beta(t) - \ln \beta^* \geq 0$ for all $t \geq 1$. For the sequences $\{Q(t)\}$ and $\{\sigma(t)\}$ are respectively monotonic decreasing and increasing.

APPENDIX C

In this appendix we give a proof of our proposition X, first establishing that the conditions

(a) $\sum_h c_2^h(t-1) + \sum_h c_1^h(t) = y$ for all $t \geq 1$

(b) $f[c_1^h(t)/c_2^h(t)] = \eta(t)$ for all h and $t \geq 1$

(c) $\lim \eta(t) \geq 1$

are sufficient, and then that they are necessary.

Sufficiency

The proof is by contradiction. So to begin we suppose that there is an equilibrium allocation $\{\hat{c}\}$ that satisfies (a)-(c) but is not Pareto-optimal. The Pareto-superior allocation is $\{\bar{c}\}$, which as we assume satisfies (a) and (b).[19]

By hypothesis, the substitution of $\{\bar{c}\}$ for $\{\hat{c}\}$ makes some individuals better off. One better-off individual, perhaps the only one, is as we suppose a member of generation $\bar{t}$. And it suffices to consider only the following two possibilities: the first (I), that $\bar{t} \geq 1$ and $\bar{\eta}(\bar{t}) \leq \hat{\eta}(\bar{t})$; and the second (II), that $\bar{t} \geq 0$ and $\bar{\eta}(\bar{t}) > \hat{\eta}(\bar{t})$.

I. By hypothesis, $u[\bar{c}^h(\bar{t})] \geq u[\hat{c}^h(\bar{t})]$ for all h, with strict inequality obtaining for at least one value of h. It follows that $\bar{c}_1^h(\bar{t}) \geq \hat{c}_1^h(\bar{t})$ for all h, and again with strict inequality obtaining for some values of h.[20] But then, as is implied by (a), $\bar{c}_2^h(\bar{t}-1) < \hat{c}_2^h(\bar{t}-1)$ for some values of h. Since $u[\bar{c}^h(\bar{t}-1)] \geq u[\hat{c}^h(\bar{t}-1)]$ for all h, though, we have that $\bar{c}_1^h(\bar{t}-1) > \hat{c}_1^h(\bar{t}-1)$ for some values of h, and further that $\bar{c}_2^h(\bar{t}-2) < \hat{c}_2^h(\bar{t}-2)$

for some values of h. And so on. Clearly, sufficient repetition of the foregoing argument yields the sought-after contradiction, that $\bar{c}_2^h(0) < \hat{c}_2^h(0)$ for at least one value of h.

II. Letting $C_j(t) = \sum c_j^h(t)$, j = 1,2, and $\gamma(t) = \hat{C}_2(t) - \bar{C}_2(t)$, we show that the sequence $\gamma(t)$ is unbounded, or in other words that contrary to assumption $\{\bar{c}\}$ is not a feasible allocation.[21/]

Using Figure I, we first give a diagrammatic "proof." What is crucial, the life-time consumption bundle or point $\hat{c}^h(t)$ lies on the indifference curve $u[\hat{c}^h(t)]$ where the slope of that curve is in absolute value less than unity. Moreover, the consumption point $\bar{c}^h(t)$ lies, as indicated in Figure I, on a ray that is below

[Insert Figure I here.]

the one labeled $c_1^h(t) = F[\hat{n}(t)]c_2^h(t)$. It may lie, as in Figure I, on the same indifference curve as $\hat{c}^h(t)$ does, or on another of greater value. But in either event the indifference curve on which it lies has a slope at $\bar{c}^h(t)$ that is less than the slope of $u[\hat{c}^h(t)]$ at $\hat{c}^h(t)$.

It follows that any given $\gamma(t-1)$, shown in Figure I as the difference between $\hat{c}_1^h(t)$ and $\bar{c}_1^h(t)$, is less than $\gamma(t)$, which (although only for a one-person economy) is by definition the difference between $\bar{c}_2^h(t)$ and $\hat{c}_2^h(t)$. To see that, one has only to project the latter difference, $\bar{c}_2^h(t) - \hat{c}_2^h(t)$, into the y-axis. That is done of course using a negatively sloped 45° line that passes through the point $\bar{c}^h(t)$. And that line, having a slope in absolute value of unity, is steeper than the line tangent to $u[\hat{c}^h(t)]$ either at $\hat{c}^h(t)$ or at $\bar{c}^h(t)$.

Now, though, for those who regard the foregoing argument as rather less than satisfactory, we present one that is a bit more rigorous. To repeat, what wants showing is that the sequence $\{\gamma(t)\}$ is unbounded.

By hypothesis, $u[\bar{c}^h(\bar{t})] \geq u[\hat{c}^h(\bar{t})$ for all h, with strict inequality obtaining for at least one value of h. We therefore have (the argument is analogous to that given in footnote 20) that $\bar{c}_2^h(\bar{t}) \geq \hat{c}_2^h(\bar{t})$, with strict inequality obtaining for some values of h. So $\gamma(\bar{t}) \neq 0$, from which it follows that $\gamma(t-1) > 0$ for some $t \geq \bar{t}$.

Let $\tilde{c}(t)$ be the solution to the following optimization problem: minimize $C_2(t)$ by the choice of $c(t)$, subject to the constraints

$$\text{(C.1)} \qquad \hat{C}_1(t) - C_1(t) \geq \gamma(t-1)$$

$$\text{(C.2)} \qquad u[c^h(t)] \geq u[\hat{c}^h(t)] \qquad \text{for all } h.$$

It follows that $\tilde{c}(t)$ is unique, satisfies (C.1) and (C.2) in their equality versions, and satisfies (b) with $\eta(t) = \tilde{\eta}(t)$, where $\tilde{\eta}(t)$ is the optimal value of the multiplier associated with (C.2).

But since $\{\bar{c}\}$ is Pareto-superior to $\{\hat{c}\}$, $\bar{c}(t)$ also satisfies (C.2). Further, with $\{\bar{c}\}$ and $\{\hat{c}\}$ both satisfying (a), $\bar{c}(t)$ satisfies (C.1), and so is feasible for the above-stated minimization problem. Hence,

$$\text{(C.3)} \qquad \bar{C}_2(t) - \tilde{C}_2(t) \geq 0.$$

As we observed above, $\tilde{c}(t)$ satisfies (C.2) in its equality version. Therefore, $\tilde{c}(t)$ and $\hat{c}(t)$ both satisfy $c_2^h(t) = \xi[c_1^h(t)]$, the equation of the indifference curve with value $u[\hat{c}^h(t)]$. And appealing to the Mean Value Theorem, we may then write

$$\text{(C.4)} \qquad \tilde{c}_2^h(t) - \hat{c}_2^h(t) = [\hat{c}_1^h(t) - \tilde{c}_1^h(t)]\{\hat{\eta}(t) + \Phi_t[\hat{c}_1^h(t) - \tilde{c}_1^h(t)]\}.$$

The function Φ_t, which has the difference $\hat{c}_1^h(t) - \tilde{c}_1^h(t)$ as its argument, is such that $\Phi_t(0) = 0$. And with ξ being a strictly convex function, $\Phi_t' > 0$.

We may, however, also write

$$\hat{c}_1^h(t) - \tilde{c}_1^h(t) = \zeta_t^h[\gamma(t-1)].$$

And since $\tilde{c}(t)$ satisfies (b) and the equality versions of (C.1) and (C.2) the function ζ_t^h is also such that $\zeta_t^h(0) = 0$ and $(\zeta_t^h)' > 0$. So we have

$$\Phi_t[\hat{c}_1^h(t) - \tilde{c}_1^h(t)] = \Phi_t\{\zeta_t^h[\gamma(t-1)]\} = \Psi_t^h[\gamma(t-1)]$$

where $\Psi_t^h(0) = 0$ and $(\Psi_t^h)' > 0$.

Letting Ψ_t be the minimum over h of the Ψ_t^h, we also have that Ψ_t is continuous, strictly increasing and such that $\Psi_t(0) = 0$. Moreover, as follows from (C.4) and the definition of Ψ_t,

$$\text{(C.5)} \qquad \tilde{c}_2^h(t) - \hat{c}_2^h(t) \geq [\hat{c}_1^h(t) - \tilde{c}_1^h(t)]\{\hat{n}(t) + \Psi_t[\gamma(t-1)]\}.$$

But then, summing (C.5) over h and making use of (C.3), (C.2) and the definition of $\gamma(t)$, we get

$$\text{(C.6)} \qquad \tilde{C}_2(t) - \hat{C}_2(t) = \gamma(t) \geq \gamma(t-1)\{\hat{n}(t) + \Psi_t[\gamma(t-1)]\}.$$

Thus, by the above-stated properties of Ψ_t, $\gamma(t-1) > 0$ implies $\gamma(t) > 0$. So $\gamma(t) > 0$ for all $t \geq \bar{t} + 1$.

We know, though, that there exists a value of t, say t*, such that $\hat{n}(t) \geq 1$ for $t \geq t^*$. For by assumption $\{\hat{c}\}$ satisfies (c). And as was shown in Appendix B, $n(t)$ approaches its limit from above. It follows then from (C.6) that

$$\text{(C.7)} \qquad \gamma(t)/\gamma(t-1) \geq 1 + \Psi_t[\gamma(t-1)] \qquad \text{for } t \geq t^*.$$

Thus, on the restricted domain $t > \max(\bar{t}, t^*)$, $\gamma(t) > 0$ and the sequence $\{\gamma(t)\}$ is strictly increasing. So if $\{\gamma(t)\}$ is bounded, then it has a limit, say $\bar{\gamma} > 0$. And as is implied by (C.7), $\bar{\gamma}$ satisfies the condition

$$\text{(C.8)} \qquad 1 + \lim \Psi_t(\bar{\gamma}) \leq 1.$$

But Ψ_t approaches a limit Ψ which is strictly increasing and such that $\Psi(0) = 0$. Consequently, $\bar{\gamma}$ cannot satisfy (C.8). And as was alleged, $\{\gamma(t)\}$ is unbounded.

Necessity

That (a) and (b) are necessary conditions for optimality is, as we remarked earlier on, quite evident. Accordingly, we show here only that (c) is too. And we proceed in the obvious way, by considering an equilibrium allocation $\{\bar{c}\}$ that satisfies (a) and (b) but not (c), and then producing an allocation that is Pareto-superior.

The allocation $\{\bar{c}\}$ may not be stationary. That is, the lifetime consumption bundles $\bar{c}^h(t)$ and $\bar{c}^h(t+1)$ are not necessarily the same. But the $\{\bar{c}\}$ bundles or allocations of members h of the first and successive generations, $\bar{c}^h(1)$, $\bar{c}^h(2)$,..., converge to some limiting bundle $\bar{c}^h$, which gives utility $u(\bar{c}^h)$. And since $\bar{\eta}$, the limiting value of the $\bar{\eta}(t)$, is by hypothesis less than unity, the indifference curve $u(\bar{c}^h)$ has a slope at $\bar{c}^h$ that is in absolute value greater than unity. As shown in Figure II, it is steeper at $\bar{c}^h$ than the negatively sloped 45° line passing through $\bar{c}^h$. Moreover, indifference curves being strictly convex, it follows that $\hat{c}^h$, the consumption bundle or point that also gives utility $u(\bar{c}^h)$ and lies on

[Insert Figure II here.]

that negatively sloped 45° line passing through $\bar{c}^h$, is to the south-east of $\bar{c}^h$. And what is crucial, so are the bundles or points lying on that line that give utility greater than $u(\bar{c}^h)$. One such point is $\tilde{c}^h$, which as shown in Figure II has a first-period component that is less than $\bar{c}_1^h$ by the amount $\Delta^h/2$ and a second-period component that is greater than $\bar{c}_2^h$ by that same amount $\Delta^h/2$.

Now, with $\bar{c}^h$ being the limiting consumption bundle, there is some generation, say $\bar{t}$, such that the bundles of members h of the generations $\bar{t}$ and following, $\bar{c}^h(\bar{t})$, $\bar{c}^h(\bar{t}+1),\ldots$, are all very much like $\bar{c}^h$. Those bundles, interpreted as points, are all very close to the point $\bar{c}^h$. All lie within the circle of radius δ^h and origin $\bar{c}^h$ that is shown in Figure II. One such point, also shown in Figure II, is $\bar{c}^h(\bar{t}+j)$, $j = 1,2,\ldots$, which gives utility $u[\bar{c}^h(\bar{t}+j)]$. And as we show below, the point $\tilde{c}^h(\bar{t}+j)$, which differs from $\bar{c}^h(\bar{t}+j)$ exactly as $\tilde{c}^h$ differs from $\bar{c}^h$, gives utility greater than $u[\bar{c}^h(\bar{t}+j)]$. Nor is it accidental that $\tilde{c}^h(\bar{t}+j)$ is shown in Figure II as lying within the circle of radius δ^h and origin $\tilde{c}^h$. It must. And that is why we know that it gives greater utility than does $\bar{c}^h(\bar{t}+j)$.

Our proposed allocation $\{\tilde{c}\}$ is defined as follows:

$$\tilde{c}_2^h(0) = \bar{c}_2^h(0) \qquad \text{for all } h$$

$$\tilde{c}^h(t) = \bar{c}^h(t) \qquad \text{for all } h \text{ and } t \leq \bar{t} - 2$$

$$\tilde{c}^h(t) = (\bar{c}_1^h(t),\ \bar{c}_2^h(t)+\Delta^h/2) \quad \text{for all } h \text{ and } t = \bar{t} - 1$$

$$\tilde{c}(t) = (\bar{c}_1^h(t)-\Delta^h/2,\ \bar{c}_2^h(t)+\Delta^h/2) \qquad \text{for all } h \text{ and } t \geq \bar{t}$$

where $\Delta^h > 0$ is such that

$$\text{(C.9)} \qquad u(\tilde{c}^h) - u(\overline{c}^h) = \varepsilon > 0,$$

and where $\overline{t}$ is as specified below.

Now, since $\{\overline{c}\}$ is feasible, $\{\tilde{c}\}$ is too. So all that remains is to show that $u[\tilde{c}^h(t)] \geq u[\overline{c}^h(t)]$ for all $t \geq \overline{t}$. With u being uniformly continuous on a domain that has $\overline{c}^h$ and $\tilde{c}^h$ as interior points, it must be that for any $\varepsilon/4 > 0$, there exists a number $\delta^h > 0$ such that if two points in the domain of u satisfy $|x_1 - x_2| \leq \delta^h$, then $|u(x_1) - u(x_2)| \leq \varepsilon/4$. So let $\overline{t}$ be such that $|\overline{c}^h(t) - \overline{c}^h| \leq \delta^h$. Then by the definition of $\{\tilde{c}\}$, $|\tilde{c}^h(t) - \tilde{c}^h| \leq \delta^h$. (See Figure II.) And hence, as required,

$$u[\tilde{c}^h(t)] - u[\overline{c}^h(t)] \geq u[\tilde{c}^h(t)] - u(\overline{c}^h) - \varepsilon/4$$

$$\geq u(\tilde{c}^h) - \varepsilon/4 - u(\overline{c}^h) - \varepsilon/4 = \varepsilon/2.$$

So we have necessity.

FOOTNOTES

1. The variable t denotes both the period of time and the generation. As the index of generations, though, it ranges over the set {0,1,2,...}. For in the first period (t = 1), the starting point of our analysis, the individuals born in the prior period, all members of generation zero, are still around and of decided relevance. When we write "...generation t" and do not restrict t, then the reference is to all generations 0,1,2,....

2. Note the $m^h(t)$ is not restricted to being non-negative. A country-k member of generation $t \geq 1$ may be short country-k money, having promised delivery in period t+1 in exchange for consumption in period t. Under the laissez-faire version of any exchange rate regime, he may also be short country-k′ money.

3. The initial or first-period stocks of the two monies are distributed arbitrarily over the first and second-country members of generation zero. That is, we require no particular distribution. Thus, in our analysis of alternative exchange rate regimes we do not have to worry about which regime was in force prior to the first period.

4. Together with the conditions imposed on u, (4) guarantees that the equilibrium values of first-period consumption and second-period consumption are positive when the equilibrium $P_k(t)$ are positive.

5. Think of member h of generation $t \geq 1$ knowing the values of all parameters. Possessed of that knowledge, he can solve the appropriate programming problem, thereby correctly calculating the $P_k(t+1)$.

6. If all of the equilibrium sequences of the real variables are constant, then the equilibrium is, as we say, constant. More particularly, the money-price sequences of a constant equilibrium are not necessarily constant.

7. More generally, in a monetary equilibrium under any regime the optimal real balance q^h of any age-one individual is the solution to the following problem: maximize $u(c^h)$ by the choice of q^h (and of course c^h), subject to the constraints $c_1^h = y^h - q^h$ and $c_2^h = \beta(q^h+Z^h)$ for some $\beta > 0$. [For residents of country k, $Z^h = P_k(t)x_k(t)$.] The problem has a unique solution, $q^h = g(Z^h,\beta)$, where $-1 < g_1 < 0$ and $g_2 \geq 0$.

8. There must exist such a value of E. For at $E = 0$, $\lambda = 1$ and $\hat{q}_2 > 0$; and at $E = \infty$, $\lambda = 0$ and $\hat{q}_1 > 0$. And uniqueness follows from

$$\partial(N_1\hat{q}_1-\lambda\hat{Q})/\partial E = \hat{Q}[(\alpha-1)g_1-1]\partial\lambda/\partial E > 0,$$

where the inequality is implied by the fact that $\hat{q}_1 = g[(\alpha_1-1)\lambda\hat{Q}/N_1,\beta]$ and the further fact that with $\alpha_k = \alpha$, $k = 1,2$ $\hat{Q}$ and $\hat{\beta}$ are independent of E.

9. Awhile back Johnson suggested (1972, p. 199) that the "fundamental argument for flexible exchange rates is that they would allow countries autonomy with respect to their use of monetary, fiscal and other policy instruments." It has since become widely acknowledged, though, that autonomy obtains under the portfolio autarky version of the floating rate regime, but not under the

laissez-faire version. See for example Whitman (1975, pp. 518-19). But our conclusion is not quite that. Indeed, we wonder a little how others can be sure that a unique equilibrium exists under the LF/F regime.

10. Under the LF/F regime, as under the PA/C regime, the (equilibrium) values of the $M_k(t)$ are known. (Under the PA/C regime, though, they are demand-determined.) And under the LF/C regime those values are unknown. Only the value of $\bar{M}(t)$ is known. But the indeterminacy of the LF/C regime equilibrium, unlike that of the LF/F regime, is of no real significance. What matters are the consumption choices, and under the LF/C regime they are independent of the composition of $M(t)$.

11. Using the equality versions of (1) and (2) and (15) it is possible to write $u[c^h(t)]$ as depending only on $q_k(t)$ and $\lambda(t)$. And our results are obtained by differentiating wrt E.

12. It follows from (10), (14) and (17) that if $\alpha_k > \alpha_{k'}$, then the limiting value of $\hat{q}_{k'}(t)$ is independent of E. And our results follow then from (25). Note, though, that it matters whether $\alpha_M \gtreqless 1$. Thus, if $\alpha_1 > \alpha_2$, then lim $\hat{B}_1(t) \gtreqless 0$ as $\alpha_1 \gtreqless 1$. And if $\alpha_1 < \alpha_2$, then lim $\hat{B}_1(t) \gtreqless 0$ as $\alpha_2 \gtreqless 1$. The explanation is easy enough. If for example $\alpha_1 > \alpha_2$, then in the limit [recall (17)] country-two members of generation t get no transfers or pay no taxes. And if $\alpha_1 = 1$, then neither do country-one members. But if $\alpha_1 > 1$ (< 1), then country-one members get transfers (pay taxes) and in consequence get a relatively large (small) share of the world endowment.

13. Certain anomalies result from the assumption that time is discrete, and in order not to have to deal with them, we assume, when necessary, that $\bar{E}$ is an element of $\{\tilde{E}(t)\}$.

14. Even for $t \leq \bar{t}$, however, the equilibrium is in general not the same as that of the C regime with $E = \bar{E}$, or equivalently, the $\bar{E}$ equilibrium of the LF/F regime. For the equilibria to be the same, it is necessary, and sufficient too, that $\tilde{Q} = \hat{Q}(\bar{t})$, where as above "^" indicates an equilibrium value of the LF/F regime. And that equality does not always obtain. Consider the utility function

$$f(x_1, x_2) = (x_1^{\frac{1}{2}}+x_2^{\frac{1}{2}})^2.$$

With that choice, (20) becomes

$$Q(t+1) = [Q(t)\sigma(t)]^2/[y - Q(t)].$$

And $\lim Q(t) = y/(1+\alpha_M^2)$. Now, if $\tilde{Q} = \hat{Q}(t)$, then as an initial condition $\tilde{Q}$ generates a sequence converging to the given limit. But for $A = (0,1)$ and $N_k y_k = 1/2$, it does not. As anyone actually doing the calculations will discover, the terms very quickly come to exceed the limit. And monotonicity guarantees that they must continue to do so.

15. But not too much should be made of that. On assumptions more general than ours, portfolio autarky may rule out desirable trades even when $\alpha_1 = \alpha_2$.

16. See Wallace (1977).

17. Those who are more than casually interested may want to consult Kareken and Wallace (1977), wherein the recursiveness is actually established. (See pp. 22-5, 42-3.) In that paper we alleged (p. 32, footnote 10) that Gale's (1974) "one good can be interpreted as a composite good only if additional restrictions are imposed on the utility functions." But that, it would seem, is quite wrong. So an apology is due Professor Gale.

18. If the model is to be of a monetary economy, though, of an economy, that is, with valuable money or monies, and if there is a second factor of production, another, that is, besides labor, then some slight ingenuity is required. It must somehow be guaranteed that the second factor of production, be it land or a durable good referred to as capital, does not as an asset dominate all monies. If land, then it can be assumed that age-one individuals are endowed between them with the labor and land, and that the land is passed from one generation to the next in accordance with some exogenous inheritance scheme.

19. That is without loss of generality. As is evident, (a) and (b) are necessary conditions for optimality. So if $\{\bar{c}\}$ does not satisfy either or both of those conditions, then there is another allocation, say $\{c^*\}$, that does satisfy both and is Pareto-superior to $\{\bar{c}\}$. And by the transitivity property, $\{c^*\}$ is Pareto-superior to $\{\hat{c}\}$.

20. Suppose to the contrary that $\bar{c}_1^h(\bar{t}) < \hat{c}_1^h(\bar{t})$ for at least one value of h. Since $\{\bar{c}\}$ and $\{\hat{c}\}$ both satisfy (b), $F[\bar{n}(\bar{t})]\bar{c}_2^h(t) < F[\hat{n}(\bar{t})]\hat{c}_2^h(t)$. But $\bar{n}(t) \leq \hat{n}(t)$ implies $F[\bar{n}(t)] \geq F[\hat{n}(t)]$, from which it follows that $\bar{c}_2^h(\bar{t}) < \hat{c}_2^h(\bar{t})$ for

some values of h, and hence that $u[\bar{c}^h(\bar{t})] < u[\hat{c}^h(\bar{t})]$ for some values of h. [For a diagrammatic version of the argument, see Kareken and Wallace (1977), pp. 37-8.]

21. With $\{\gamma(t)\}$ unbounded, there must come a period, say t', in which it takes more than the world endowment to make the members of generation $t' - 1$ as well off with their $\{\bar{c}\}$ allocations as with their $\{\hat{c}\}$ allocations.

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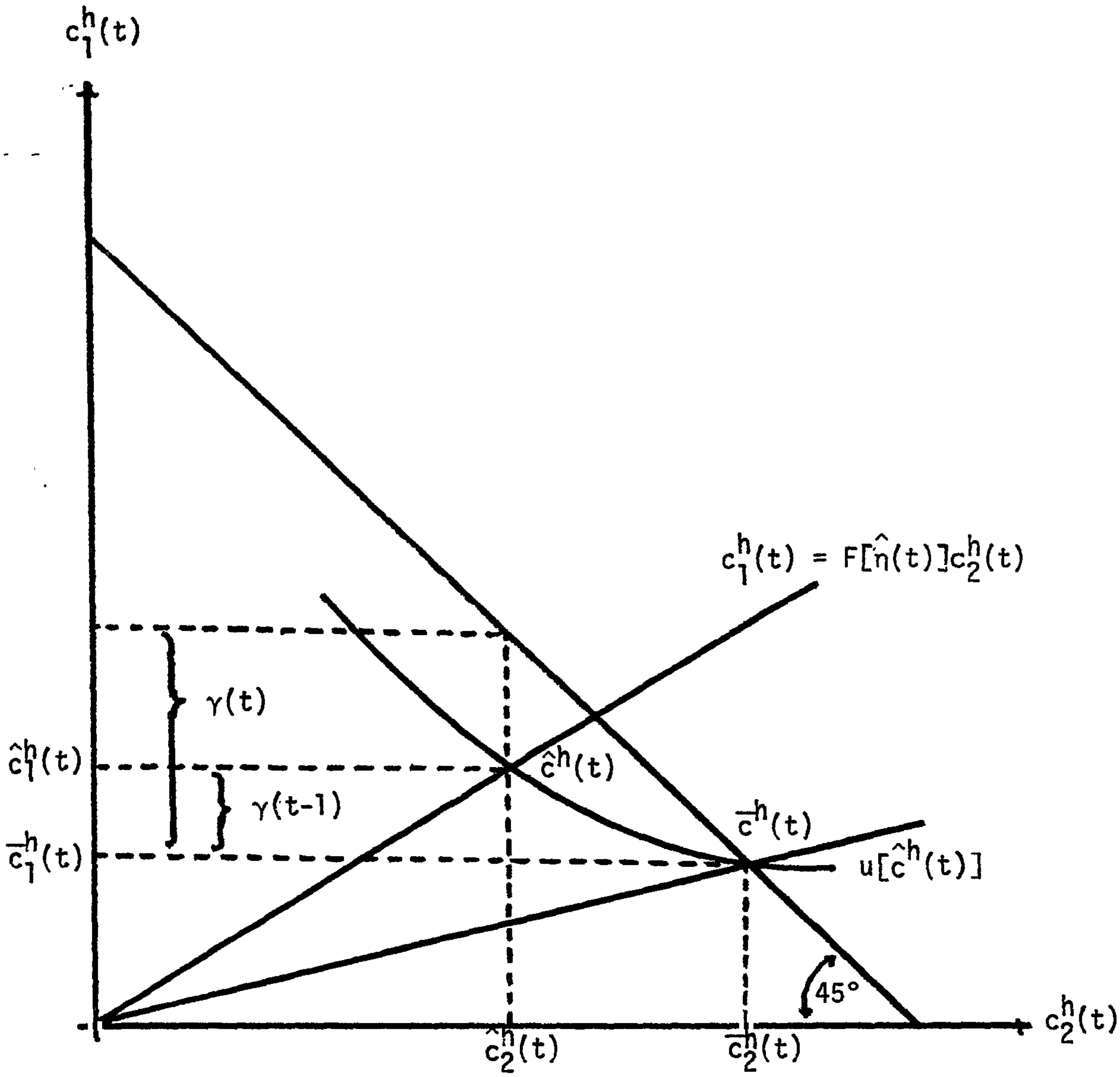
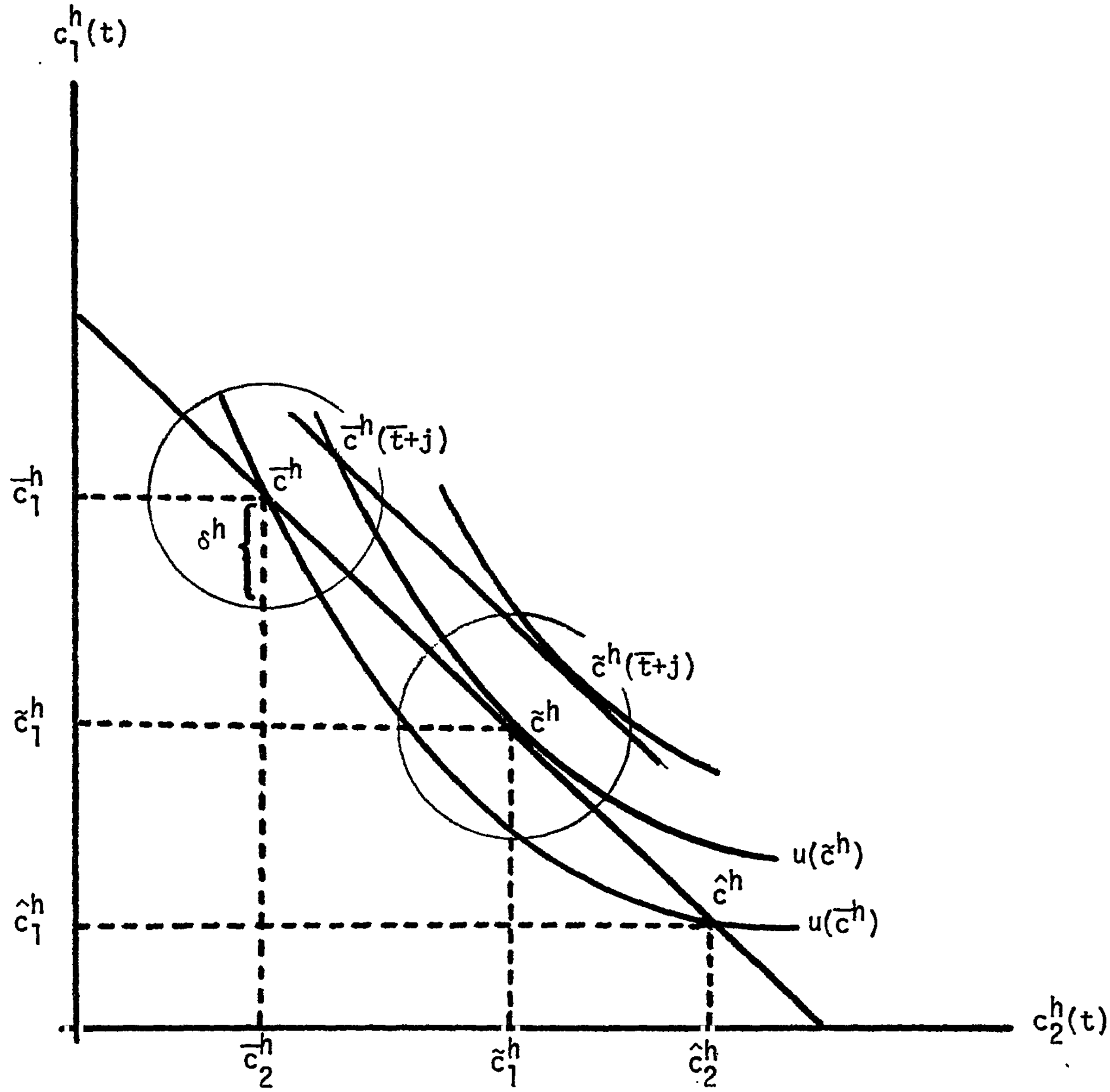


$c_1^h(t)$
$c_1^h(t) = F[\hat{n}(t)]c_2^h(t)$
$\gamma(t)$
$\hat{c}_1^h(t)$
$\hat{c}^h(t)$
$\gamma(t-1)$
$\bar{c}^h(t)$
$\bar{c}_1^h(t)$
$u[\hat{c}^h(t)]$
45°
$c_2^h(t)$
$\hat{c}_2^h(t)$
$\bar{c}_2^h(t)$

FIGURE I



$c_1^h(t)$
$\bar{c}^h(\bar{t}+j)$
$\bar{c}^h$
$\bar{c}_1^h$
δ^h
$\tilde{c}^h(\bar{t}+j)$
$\tilde{c}_1^h$
$\tilde{c}^h$
$u(\tilde{c}^h)$
$\hat{c}^h$
$\hat{c}_1^h$
$u(\bar{c}^h)$
$c_2^h(t)$
$\bar{c}_2^h$
$\tilde{c}_1^h$
$\hat{c}_2^h$

FIGURE II