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Approximately Complete Markets: A Generalization of the States of the World Model

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A Generalization of the States of the World Model*

One goal of the study of finance is to determine the objective function of firms. The usual assumption made is that firms follow the dictates of their stockholders. As long as firm decisions do not alter the portfolio opportunities available the implied objective function of the firm is to maximize market value. The states of the world model allows the study of this provisio in a more general framework than the usual mean-variance protfolio model. In the states of the world model a complete market of contingent claims guarantees that the investor can generate any desired pattern of returns. While firm decisions may alter the opportunity set of investors, they do not alter the space spanned by available assets. Under this circumstance as long as firms are valued as the contingent claims they implicitly represent, market value maximization is optimal for all stockholders.1/

The states of the world model suffers from the problem in defining exactly what the finite number of states are.2/ At best it would seem that the finite number of states are only used as an approximation to an infinite number of states. The question immediately arises whether it is a good approximation to a world with an infinite number of states. In this paper the states of the world is extended to allow for uncountably infinitely many states.3/ We will then see under what circumstances a finite number and a small finite number of contingent claims can be used to approximate a complete market.

1/ See Reference 4.

2/ See Reference 5, p. 158.

3/ Peter A. Diamond has examined the case of a continuum of states. However he either constrains individuals' return functions to be linear combinations of firm return functions, or allows a continuum of contingent claims. See Reference 2.

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The Model

Let Ω be a set the elements of which are states of the world. Let ξ be a σ algebra of sets in Ω and P_i be probability measures of ξ for individuals $i=1,\ldots,N$. (Ω,ξ,P_i) are probability spaces. Suppose Ω has cardinal number less than or equal to the cardinal of the continuum. Further let X be a 1-1 mapping of Ω into [0,1] which is measurable in ξ. X is a random variable.

For each state in Ω the return of each of K firms assumed by a given individual is a fixed real number. Let these firm return functions assumed by individuals be the same for all individuals. Using X the return functions can be written as functions with domain [0,1] and the distribution functions F_i of X for each P_i over measurable sets in [0,1] can be used in the place of P_i and ξ. Further let $r_j(s)$ be the return of firm j assumed by all individuals to occur if $X=s$, $j=1,\ldots,K$, $s\in[0,1]$. In order for expected returns to be defined $r_j(s)$ must be measurable functions (random variables). Also assume that $r_j(s)$ is a bounded function of s for given j. The economy can pay off a limited amount in any state.

If there were a complete market, a contingent claim for each of the possibly uncountably infinite number of states, any appropriately bounded return function $v_i(s)$ could be chosen by individual i. However, the economy cannot produce an infinite number of claims. The question is, then, can $v_i(s)$ be approximated (in an appropriate sense) by a finite number of contingent claims? The usual assumption made is that individuals are expected utility maximizers, and that utility functions are continuous and strictly increasing in wealth. For expected utility of $v_i(s)$ to exist $v_i(s)$ must be measurable. Further $v_i(s)$ must be bounded as limited payoffs are available. Let u_i be the i^{th} individual's utility of income function with the above ascribed attributes. Then $v_i(s)$ and, therefore, $u_i(v_i(s))$ are square integrable.4/

For simplicity of exposition the subscript i will be deleted from the u_i, v_i, and F_i functions in the following discussion. We need to find a return function $v^*(s)$ which is a linear combination of a finite number of contingent claims that approximates $v(s)$ in an appropriate sense. $v(s)$ was chosen to maximize expected utility under constraints. An appropriate requirement is that $v^*(s)$ approximate $v(s)$ in expected utility, that $\int_0^1 \{u[v(s)]-u[v^*(s)]\}\, dF(s)$ be small. A sufficient condition for this clearly is that $\int_0^1 \{u[v(s)]-u[v^*(s)]\}^2\, dF(s)$ be small.

We have already assumed enough to guarantee that there exist a $v^*(s)$ satisfying the above criterion. $u[v(s)]$ is a square integrable function. Therefore there exist functions $h_n(s)$, each taking on a finite number of values, such that they converge to $u[v(s)]$ in mean square, $\lim_{n\to\infty} \int_0^1 \{u[v(s)]-h_n(s)\}^2\, dF(s) = 0$ [Reference 8, Vol. 2 p. 85]. $h_n(s)$ is a good approximation to $u[v(s)]$ for large n. There has been shown that an approximation, in mean square, to $u[v(s)]$ exists, but not that the desired approximation to $v(s)$ exists. However, this follows immediately as $h_n(s)$ is equal to $u[v_n^*(s)]$ where $v_n^*(s)$ is also some function taking on a finite number of values. It remains to show that $v_n^*(s)$ is a linear combination of a finite number of contingent claims. Suppose $v_n^*(s)$ takes on values $y_1,\ldots,y_M$ on $T_1,\ldots,T_M$ respectively, where $\bigcup_{i=1}^{M} T_j = [0,1]$ and $T_i \cap T_j = \emptyset, i\neq j$, T_i measurable. Then $v_n^*(s) = \sum_{i=1}^{M} y_i\, \chi T_i(s)$ where χT_i is the

4/ Integrable and bounded imply square integrable.

characteristic function $\chi^{T}_{i}(s) = \{ \begin{matrix} 1, s \in T_i \\ 0, \text{otherwise.} \end{matrix}$ $v^{*}_{n}(s)$ is a linear combination of a finite number of contingent claims that pay one if s belongs to a measurable subset of [0,1] and zero otherwise.5/

If contingent claims are written on sets of states rather than states each individual can get arbitrarily close to his desired return function using a finite number of claims. Therefore with a finite number of individuals an approximately complete market can be achieved with a finite number of contingent claims.6/

In general the more claims available, the larger M, the better the approximation. The ability to approximate a complete market is of little interest if the number of claims markets necessary for a good approximation is large. The smaller the variation in individuals' desired return functions the fewer claims are necessary to achieve a given closeness of approximation. It is impossible to determine a priori the degree of variation in desired return functions. The number of claims necessary to achieve a given closeness of approximation is an empirical question, not a theoretical one. However, by imposing additional assumptions, we can rule out the possibility that as the number of individuals in a market grows the necessary number of claims increases without bound. Suppose $\{u_i[v_i(s)]\}$ form, or can be approximated by, a piecewise equicontinuous family of founctions.7/ Then all desired return functions can be approximated with a set of claims on subintervals of equal size. Even if there are an infinite number of individuals, a given closeness of approximation to each

5/ It follows from the Luzin C-property and the boundedness of $u[v(s)]$ that subintervals can be used. See Reference 8, Vol. II p. 42.

6/ Similar analysis follows for the states of the world model with K goods in states $\alpha \in E^1$ to be allocated between N consumers. See Reference 1.

7/ See Reference 8 Vol. I p. 54.

$v_i(s)$ can be achieved with a finite number of claims. The greater the "degree" of equicontinuity, the fewer claims are necessary for a given closeness of approximation. In the appendix conditions sufficient for such equicontinuity are presented.

In the preceding discussion boundedness was imposed on the return functions of firms. This boundedness guarantees that all the moments of the return functions exist. However empirical evidence tends to refute the existence of second and higher moments of common stock.[8/] Fortunately this does not present a serious problem for the model. The concern is with the approximation of expected utility, that is to say with the first moments of the utility functions. These first moments are assumed to exist, $E\{u[v(s)]\}$ exist. Therefore if $\gamma^B = \{s \mid u[v(s)] > B\}$, $E\{u[v(s)]\} = \int_{[0,1]/\gamma^B} u[v(s)]\, dF(s) + \int_{\gamma^B} u[v(s)]\, dF(s)$ so that

$\lim_{B\to\infty} \int_{\gamma^B} u[v(s)]\, dF(s) = 0$. $u[v(s)]$ can be truncated to form the function

$$w^B[v(s)] = \begin{cases} u[v(s)] & \text{if } u[v(s)] \leq B \\ B & \text{otherwise} \end{cases}$$

$E\{w^B[v(s)]\} \simeq E\{u[v(s)]\}$ for large B, and $w^B[v(s)]$ can be approximated by the techniques presented above. Alternatively, of course, one could simply restrict oneself to bounded utility functions. The boundedness of $r_j(s)$ was used only to guarantee boundedness of $u[v(s)]$.

We assumed that the states of the world were mapped into [0,1]. The results are easily extended to the case where the mapping is into the unit cube I^n, in other words X is a vector random variable. Further the restriction to [0,1] or I^n rather than $\overline{E^1}$ or $\overline{E^n}$ is not important as they are homeomorphic. The states of the world can be considered as n real valued variables, an interpretation with more intuitive appeal in application to the real world.

8/ See Reference 6.

Summary

The case of n states of the world is generalized to n real valued variables. In such a market profolios of the claims to the returns of a finite number of firms are not complete, or even approximately so. However, a complete market of contingent claims can be approximated by a finite number of contingent claims written on sets of states. This approach obviates the problem of exactly what a "state of the world" is.

It should be stressed that an approximately complete market does not in general imply that firms are approximately market value maximizers. As with complete markets, it is also necessary that individuals assume the same return functions for firms.[9/] The results of this paper suggest that significantly non-market value maximizing behavior of firms may be better explained by differences in assumed return functions that by incomplete markets.

9/ See Reference 2 p. 765.

Appendix I: Equi-Risk Aversion and an Infinite Number of Individuals

In the text it is mentioned that if $\{u_i[v_i(s)]\}$ can be approximated by a piecewise equicontinuous family of functions, a finite number of claims can be used to approximate a complete market, even if there are an infinite number of individuals. Clearly the greater the "degree" of equicontonuity (the larger ε for given δ), the fewer claims are necessary for a given closeness of approximation. In this appendix conditions are presented which insure equicontinuity holds.

The discussion is limited to equicontinuity as the extention to piecewise equicontinuity is obvious. Let $p(s)$ be the price of a claim to a dollar in state s. That is to say, the market price of any asset is the integral of the product of its return function and $p(s)$. If $r_j(s)$ and $\sum_i [v_i(s)]$ are continuous (supply and demand) and $p(s) \leq 1$ (one can hold a dollar) price will be continuous. It may be that $p(s)$ in the complete market differs from any price function used in an incomplete market. This does not matter, as the more closely a market approximates the complete market the more closely a price function used in this approximately complete market approximates $p(s)$. Conditions will be given under which $p(s)$ continuous implies that $v_i(s)$ can be approximated by an equicontinuous family of functions.

Equi-Risk Aversion Assumption: For all i and $X_1, X_2 \in [0, \sup_{i,s} [v_i(s)])$ and all $\lambda \in (0,1)$,

$u_i[\lambda X_1 + (1-\lambda) X_2] \geq \lambda u_i(X_1) + (1-\lambda) u_i(X_2) + q[\lambda(1-\lambda), |X_2-X_1|]$ where q is a continuous real valued function on RxR non decreasing in its second argument such that $q(x,y) > 0$ if $x, y > 0$ and $q(x, y) = 0$ if $x=0$ or $y=0$.

Theorem If (i) individuals are equi-risk averse, (ii) $\{u_i(\gamma)\}$, $\{dF_i(s)/ds\}$, $p(s)$ form (almost everywhere) equicontinuous families of functions with (iii) $dF_i(s)/ds$ and $p(s)$ uniformly (essentially) bounded away from zero, then $\{v_i(s)\}$ can be approximated by an equicontinuous family of functions.

Proof:

The proof proceeds by providing a method to generate a counter example to the denial of the assertion.

Assume the theorem is false. Consider families of simple functions[1/] $\{g_i(s)\}$ approximating $\{v_i(s)\}$ to a desired degree of accurancy; for all i, $\left| \int_0^1 \{u_i[v_i(s)]-u_i[g_i(s)]\}\, dF_i(s) \right| < \rho_1$, $\left| \int_0^1 [v_i(s)-g_i(s)]\, p(s)\, ds \right| < \rho_2$ for given ρ_1, $\rho_2 > 0$. This can be done by the measureability of u_i and v_i. Then it must be true for all such families of simple functions that for some $\delta > 0$ and any $\varepsilon > 0$ there exist s_1, $s_2 \in [0,1]$ such that $|s_1 - s_2| < \varepsilon$ but $|g_i(s_1) - g_i(s_2)| \geq \delta$ for some i (criterion 1).

Consider such a δ, ε and i. Assume that the largest and smallest values taken on by some $g_i(s)$ in an open ε' interval ξ occur at s_1, s_2 where s_1, $s_2 \in [0,1]$, $|s_1 - s_2| < \varepsilon'$ for $\varepsilon' > \varepsilon$ and $|g_i(s_1) - g_i(s_2)| \geq \delta$ (criterion 2). Let F_1 and F_2 be the i^{th} probability measure of sets in ξ such that $g(s) = g(s_1) \equiv \gamma_1$, $g(s) = g(s_2) \equiv \gamma_2$ respectively. Take the smaller of F_1 and F_2 (F_1, say) and shrink the set around s_2 until it has probability measure F_1. Call the resulting sets

$$\Omega_1 = \{s \mid g_i(s) = \gamma_1,\ s \in \xi\}, \Omega_2 = \{s \mid g_i(s) = \gamma_2,\ s \in \xi,\ \int_{\Omega_2} dF_i(s) = F_1 = \int_{\Omega_1} dF_i(s).\}$$

Then $\displaystyle\int_{\Omega_1 \cup \Omega_2} u_i[g_i(s)]\, dF_i(s) = F_1\, u(\gamma_1) + F_1\, u(\gamma_2)$.

1/ Functions taking on a countable number of values.

Consider $\hat{\gamma} = \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2$. Then

$$2 F_1 u(\hat{\gamma}) \geq F_1 u(\gamma_1) + F_1 u(\gamma_2) + 2 F_1 q(\tfrac{1}{4}, \delta)$$

Next consider the point

$$\tilde{\gamma} = \frac{\int_{\Omega_1} p(s)ds}{\int_{\Omega_1 U \Omega_2} p(s)ds}\gamma_1 + \frac{\int_{\Omega_2} p(s)ds}{\int_{\Omega_1 U \Omega_2} p(s)ds}\gamma_2$$

If $g_i(s)$ is replaced by $\tilde{\gamma}$ over the set $\Omega_1 U \Omega_2$ the resulting function

$$h_i(s) = \begin{cases} \tilde{\gamma}, & s \in \Omega_1 U \Omega_2 \\ g_i(s), & \text{otherwise} \end{cases}$$

will cost the same as $g_i(s)$:

$$\int_0^1 h_i(s)\ p(s)ds = \int_0^1 g_i(s)\ p(s)ds.$$

By the uniform continuity of $p(s)$ and $u_i(\gamma)$ if ε' is small enough $|\tilde{\gamma} - \hat{\gamma}|$ is small enough so that $|u_i(\tilde{\gamma}) - u_i(\hat{\gamma})| < \alpha$, for some α such that $0 < \alpha < q(\frac{1}{4}, \delta)$. This implies that:

$\int_0^1 u_i[h_i(s)]\ dF_i(s) \geq \int_0^1 u_i[g_i(s)]\ dF_i(s) + 2 F_1 [q(\frac{1}{4}, \delta) - \alpha]$. As $v_i(s)$ maximizes utility for a given cost, $h_i(s)$ is a closer approximation to $v_i(s)$ than is $g_i(s)$.

$q(\frac{1}{4}, \delta)$ is independent of ε'. By the equicontinuity of $\{u_i(\gamma)\}$, $p(s)$, ε can be chosen small enough so that for some $\varepsilon' > \varepsilon$ for any approximating $g_i(s)$ and any s_1, s_2 satisfying criterion 2 for $g_i(s)$ the related $\tilde{\gamma}$, $\hat{\gamma}$ satisfy $|u_i(\hat{\gamma}) - u_i(\tilde{\gamma})| < \alpha$.

Pick any i. For s_1, s_2 satisfying criterion 2 for given $g_i(s)$ perform the above described improvement technique. Then apply it again on the newly generated approximating function,... etc. Each time the improvement technique is applied to generate a new approximating function expected utility increases by $2\ F_{1'}[q(\tfrac{1}{4},\ \delta)-\alpha]$. However the expected utility of the generated series of approximating functions is bounded above by $E\{u_i\,[v_i(s)]\}$. As $[q(\tfrac{1}{4},\ \delta)-\alpha]$ is a positive constant the probability measure of available s satisfying criterion 2 must approach zero as more and more improvements made (ΣF_1 "$\to$" 0). As the probability measure of available s approaches zero, the linear measure must also approach zero by the uniform lower bound on $\{dF_i(s)/ds\}$.

Continue the improvement technique (k times, say) until the linear measure of s satisfying criterion 2 is less than $\varepsilon' - \varepsilon$,

$$\mu\ [\{s\ |s \in [0,1] \text{ and } \exists\ s' \in [0,1] \ni |s-s'| < \varepsilon' \text{ and } |h_i^k(s) - h_i^k(s')| \geq \delta\}] < \varepsilon' - \varepsilon$$

From the resulting improved approximating simple function $h_i^k(s)$ delete the values in the range corresponding to all s satisfying criterion 2. Then in each gap put the value of the function $h_i^k(s)$ over an adjacent remaining interval to form $h_i^*(s)$. For example choose

$$h_i^*(s) = \begin{cases} h_i^k(s) & \text{s does not meet criterion 2} \\ h_i^k(s') & \text{otherwise} \\ \text{where } s'<s,\ s' \text{ does not meet} & \\ \text{criterion 2 and for any } s'' & \\ \text{not meeting criterion 2 such that} & \\ s' < s'' < s,\ h_i^k(s') = h_i^k(s'') & \end{cases}$$

If $|s_1 - s_2| < \varepsilon$ then $|h_i^*(s_1) - h_i^*(s_2)| < \delta$

Further h_i^* costs at most an arbitrarily small amount more than $h_i^k(s)$ and yields at least an arbitrarily smaller amount of expected utility, while $h_i^k(s)$ costs the same as $g_i(s)$ and has higher expected utility. Therefore $h_i^*(s)$ is a good approximation to $v_i(s)$. For each i and any $g_i(s)$ form $h_i^*(s)$. Then $\{h_i^*(s)\}$ is a family of simple functions approximating $v_i(s)$ such that for each i for all $s_1, s_2 \in [0,1]$ $|s_1 - s_2| < \varepsilon$ implies $|h_i^*(s_1) - h_i^*(s_2)| < \delta$ Thus $\{h_i^*(s)\}$ violates criterion 1 for each i.

The preceding proof depends upon the (essential) lower bound and (almost everywhere) uniform continuity of p(s). This in turn depends upon the continuity of $r_j(s)$. However, as mentioned in the text $r_j(s)$ may be unbounded, and therefore not continuous on [0,1]. However we may proceed, as in the text, by truncation. The first moments and cost of $r_j(s)$ are assumed to exist. Further the equicontinuity of $\{dF_i(s)/ds\}$ guarantees a uniform bound on $dF_i(s)/ds$. For a given closeness of approximation $r_j(s)$ and p(s) can be truncated over the same subset of [0,1] for all individuals.

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