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Fragility of Reputation and Clustering of Risk-Taking [*]

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ABSTRACT ——-

Concerns about constructing and maintaining good reputations are known to reduce borrowers' excessive risk-taking. However, I find that the self-discipline induced by these concerns is fragile, and can break down without obvious changes in economic fundamentals. Furthermore, in the aggregate, breakdowns are clustered among borrowers with intermediate and good reputations, which can exacerbate an economy's weakness and contribute to a broad economic crisis. These results come from an aggregate dynamic global game analysis of reputation formation in credit markets. The selection of a unique equilibrium is accomplished by assuming that borrowers have incomplete information about economic fundamentals.

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1 Introduction

The major financial crisis that began in 2008 seems to contradict our understanding of the self-disciplining nature of financial markets. In these markets, borrowers whose actions and profits are not observable tend to take excessive risk when compared to the efficient benchmark in which their actions are observable. The reason is that, by taking risks, borrowers can appropriate most of the extra benefits expected from large successes and, by defaulting, can impose to lenders most of the losses expected from large failures. However, since the 1980s, researchers (Stiglitz and Weiss (1983), Diamond (1989)) have established that borrowers' concerns about constructing and maintaining good reputations restrain their tendency to behave opportunistically. Yet as housing prices shot up over the few years before 2008, financial intermediaries - even well-known, highly rated, reputable ones - began borrowing heavily in order to turn around and lend heavily to high-risk customers for mortgages and other assets. Once the housing bubble burst, that risky behavior led to a collapse in many large financial firms as well as in money markets and eventually in general economic activity.

U.S. Federal Reserve chairman, Ben Bernanke, has pointed to the collapse in the financial market discipline as the main source of the crisis: "Market discipline has in some cases broken down, and the incentives to follow prudent lending procedures have, at times, eroded" (Statement, Board of Governors, December 18, 2007). Why did financial market discipline fail in this way? Why did firms' concerns about gaining and maintaining good reputations apparently disappear and lead to the recent crisis? This paper constitutes a first attempt to understand the relation between the self-discipline of financial markets and the economic fundamentals affecting them.

I argue that firms' concerns about their reputations can have negative as well as positive aggregate effects. These concerns, I show, are, in fact, *fragile* and their breakdown induces a sudden change in risk-taking behavior in response to small and not obvious changes in aggregate fundamentals. Furthermore, these breakdowns are *clustered* among firms with intermediate and good reputations, generating a large change in aggregate risk-taking behavior and a large negative impact on overall economic outcomes, such as corporate rates of success and failures, credit conditions, interest rates, and returns to investors. Historically, then, the loss of reputation concerns may have been an unnoticed amplifier of financial crises characterized by excessive risk-taking, such as the recent one.

I establish this in a way that has not been done before. I begin with a standard model of credit markets in which firms borrow to produce. In this model, all firms can invest in risky projects, while only some of them (*strategic firms*) can also invest in safer projects that increase the probability that the firm *continues* operating - and hence reduce the probability

of loan default - but generate lower profits when the firm continues. A firm's *reputation* is defined in my model as the probability that the firm is strategic. Reputation is updated by lenders after observing the firm's continuation, which is an observable signal correlated with the firm's decisions. Strategic firms want to distinguish themselves from nonstrategic ones so that they can pay lower interest rates in the future. The fear of losing reputation, therefore, leads strategic firms to reduce risk-taking.

In this setting, firms' temptation to take risks varies monotonically with a stochastic aggregate fundamental. As is standard, however, it turns out that when fundamentals are perfectly observable, the model delivers multiple equilibria, the characterization of which varies with lenders' beliefs about firms' behavior. There is a range of this fundamental for which two equilibria coexist. At the one extreme, if lenders believe that all strategic firms play it safe, then firms do that. Firms know that in this case their continuation and loan repayment will be attributed at least partly to their good behavior, thereby improving their reputation. At the other extreme, if lenders believe that all firms take risks, then strategic firms do that. Under these beliefs, firms know that their continuation and repayment will be attributed solely to good luck and won't improve their reputation at all. Reputation concerns clearly reduce risk-taking in the first equilibrium but not in the second.

In order to obtain a unique equilibrium, which is robust to small perturbations of information, I here use techniques from the global games literature. I assume that after negotiating the loan, but before making a decision about risky or safe behavior, firms observe a noisy signal of the fundamental, which becomes part of what determines their decisions. The model thus becomes a nonstandard dynamic global game in which strategic complementarities are not just assumed, but are rather obtained endogenously from the concerns behind reputation formation. Uniqueness of equilibrium is characterized for each kind of reputation by a *cutoff* in signals about fundamentals, below which firms decide to take risks. Fundamentals, that is, do not only affect the temptation but also become a coordination device for risk-taking behavior.

Here equilibrium selection generates the first of two sources of reputation fragility. When, based on its signal about economic fundamentals, a firm believes that similar firms will take risks, it also believes that lenders will assign a low probability of good behavior to all firms, hence playing it safe will not be rewarded with a better reputation and the firm takes risks. If signals about fundamentals are precise, small changes of fundamentals around the cutoff of risk-taking produce a clustering of behavior among firms with the same reputation.

The second source of fragility exists at an aggregate level when firms with different reputation are compared. This source of fragility is independent of the equilibrium selection and depends only on primitive learning properties. Safe projects have higher probabilities of con-

tinuation, which generates two types of incentives for safe behavior. One type, *continuation incentives*, increases with reputation; firms with better reputations face lower rates in the future, have higher expected future profits, and are more afraid to die. The other type, *reputation formation* incentives, is low for extreme reputations and high for intermediate ones; because of learning, priors are harder to change at the extremes. By combining these two types of incentives, we can see that:

- *Poor* reputation firms have no incentives to play it safe because their continuation value is low and, if they survive, their reputation cannot improve much.

- *Intermediate* reputation firms do have incentives to play it safe, not because their continuation value is high; rather, if they survive, they can improve their reputation a lot.

- *Good* reputation firms also have incentives to play it safe, but not because they can improve their reputation if they survive; rather, their continuation value is high and they can lose a lot if they die.

Hence, intermediate and good reputation firms have similar cutoffs for different reasons. They switch to risk-taking under similar conditions, which produces a clustering of behavior change among them. Furthermore, since the distribution of reputation is biased toward intermediate and good reputations, what these firms do strongly affects the aggregate level of risk-taking in the economy.

My work here primarily combines two strands of literature: reputation and global games. With regard to the reputation strand, my model is most closely related to the models of Diamond (1989) and Mailath and Samuelson (2001), who analyze the ability of reputation to deter opportunistic behavior in the presence of both adverse selection and moral hazard. Unlike their work, which is focused on reputation incentives for a single agent living in a state-invariant environment, my work here explicitly introduces a cross section of firms in an environment that evolves stochastically in order to study the interplay between reputation incentives and economic conditions in determining aggregate behavior. As in their work, my model also has multiple equilibria. While Diamond (1989) considers extreme equilibria and Mailath and Samuelson (2001) focus on the most efficient one, I select a unique equilibrium by exploiting the existence of fundamentals as a coordination device. Finally, unlike in the work of Mailath and Samuelson (2001), here firms' behavior affects the probability of their continuation, a signal to update reputation, and unlike Diamond's (1989) model, mine is flexible enough to include the use of additional signals correlated to actions, which breaks the perfect correlation between age and reputation that he obtains.

My work also contributes to the literature of herding behavior generated by reputation concerns. While the work pioneered by Scharfstein and Stein (1990) considers agents that mimic

others and disregard private information, firms in my model cannot observe others' actions and instead use private information to coordinate behavior.

My work relies as well on the dynamic global games literature - such as the work of Morris and Shin (2003), Chassang (2007), and Toxvaerd (2007) - but I exploit novel properties coming from the endogenous reputational generation of strategic complementarities. In particular, the range of fundamentals with multiple equilibria depends on the initial reputation of the firm, and this is useful in characterizing the schedule of cutoffs for different kinds of reputation. My work contributes, as well, to the scarce literature on learning in global games. While most of that literature studies situations in which players learn about a policymaker or a status quo (as in, for example, the 2006 and 2007 work of Angeletos, Hellwig and Pavan), my model deals with the opposite situation, in which the market learns about players' types and so generates coordination problems. My work here is the first to exploit fundamental-driven incentives to create a reputation global game and select a unique equilibrium.

In the next section, I show how, with incomplete information about economic fundamentals, a dynamic global game analysis of reputation formation in credit markets delivers a unique equilibrium. In Section 3, I show that firms' concerns about reputation can have substantial aggregate negative as well as positive effects. In Section 4, I conclude.

2 Selecting a Unique Reputation Equilibrium in Credit Markets

Models of credit markets have been used to analyze the effects of reputation concerns before. However, based on complete information about aggregate economic conditions, their results have not been determined by a unique equilibrium. Introducing small perturbations on the information about those conditions, I am able to select here a unique equilibrium.

2.1 Multiple Equilibria in a Model with Complete Information

First, I demonstrate the standard result that a model with complete information has multiple equilibria.

2.1.1 The Model

Here I describe my basic model of reputation concerns in credit markets, the timing of its events, and the definition of its equilibrium.

a) Description

Credit markets are composed of a continuum of long-lived, risk-neutral firms (with mass 1) and an infinite number of risk-neutral lenders that provide funds to those firms.

Each firm in the model runs a unique project, by selecting either safe (s) or risky (r) production technologies.[1] I assume that firms using safe technologies are more likely to continue operating in the market (c).

Assumption 1 *The selection of a safe technology makes a firm's continuation more likely. This is,* $Pr(c|s) = p_s > Pr(c|r) = p_r$.

If the firm does not continue, or dies, then current and future cash flows are zero. If the firm continues, then current cash flows (Π) depend both on the technology used and on a single-dimensional variable $\theta \in \mathbb{R}$ that represents aggregate economic fundamentals. I assume the relationship between the fundamental and the temptation to play it safe is positive.

Assumption 2 *Safe technologies are more tempting with higher θ:* $\frac{\partial \Pi_s}{\partial \theta} > \frac{\partial \Pi_r}{\partial \theta}$ *for all θ.*

Fundamentals θ are independently and identically distributed over time and distributed with density $v(\theta)$, mean μ, and standard deviation γ at each period. What actually matters in Assumption 2 is not the direction of the inequality, but the monotonic change in incentives to use safe technologies as fundamentals vary.[2]

The model has two types of firms, defined by their access to production technologies. *Strategic* firms $\mathcal{S}$ can choose between safe and risky technologies. *Risky* firms $\mathcal{R}$ can use only risky technologies. Firm's reputation is defined by $\phi = Pr(\mathcal{S})$, the probability of being a strategic firm.[3]

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[1] I will also use these terms interchangeably: playing it safe or taking safe actions (s) and taking risks or taking risky actions (r).

[2] Unlike in other reputation models, in this one incentives differ over the cycle. A good example is a construction firm, in which risky technology is the use of cheap materials and θ is the relative price of cheap materials. Another example of θ is the level of aggregate demand when more structure is imposed into the definition of cash flows. This extension is available in Ordonez (2008). A final example, related to the recent financial crisis, in which the direction of the assumption reverses, is financial institutions whose risky technology is to extend mortgages to risky households and θ is the expected housing price.

[3] The introduction of these two types of firms is based on my (maybe pessimistic) belief that all firms can take risks, but not all of them can play it safe. While all firms can perform trial-error procedures, not all of them have access to well-designed procedures. Particularly in lending markets, saying that some firms are restricted to using inferior technologies rather than superior ones seems to be a better description of reality. An obvious alternative is that nonstrategic firms have access to only safe technologies. In this case the main result of reputation fragility remains unchanged. Another way to rationalize our assumption of types is that some strategic firms have a positive discount factor, while others (risky ones) have a zero discount factor.

To run a project, each firm needs external funds (normalized to one per period), which can be provided by lenders, whose outside option is an alternative investment in a risk-free bond that pays $\overline{R} > 1$.[4] Failure to repay loans (*default*) is characterized by a costly state verification with a bankruptcy process that destroys the value of the firm's output. This is a straightforward way to introduce truth-telling by firms. When cash flows are greater than debts, firms always find it optimal to repay loans and get the positive differential rather than default and file for bankruptcy. I assume that, conditional on continuation, firms can always pay back their loans; hence, default occurs only if a firm dies.[5] Finally, I restrict the analysis to the use of short-term debt, ruling out equity contracts.[6]

b) Timing

This model is repeated during a finite number of periods. The order of events in each period t is the same in all periods. In what follows I focus on the reputational game in a given period t, hence there is no need to use subscripts to denote time. In Section 2.2.2 I study the full-fledged repeated game and introduce explicitly dynamic considerations, denoting periods by the subscript t. The timing in each period is as follows.

- Firms and lenders meet. Each lender observes the reputation ϕ of the firm it is matched to. The firm acquires a loan of 1 at a rate that depends on its reputation, $R(\phi) > 1$.

- Fundamentals θ (that affect short-term cash flows) are realized by firms and lenders.[7]

- Strategic firms decide between using safe (s) or risky (r) technologies. Risky firms just use risky ones (r).

- Production occurs, and the firms either continue or die.

- If the firm dies, it defaults on its loan. If the firm continues, it pays to lenders the negotiated debt $R(\phi) > 1$ and consumes the remaining cash flows.[8]

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[4]Since lenders are the long side of the market, there is no competition for funds. The introduction of such competition makes reputation effects more important and magnifies the results. Other alternative assumptions are that fundamentals do not only affect cash flows (Π), but also the probability of continuation (p_s) and/or the risk-free interest rate ($\overline{R}$). These alternatives do not modify my main results either.

[5]Nothing fundamental changes with this assumption, but it simplifies the notation and eases the exposition. Relaxing it, so that default also exists in case of continuation, does not change the results.

[6]I rule out equity contracts for two reasons. One is that short-term debt not only seems to be widely used in reality but also it better highlights the importance of reputation concerns. The other reason is that the optimal lending contracts make interest rates conditional on fundamentals. When this contract does not eliminate excessive risk-taking completely, reputation concerns still exist and their incentives are also fragile.

[7]The timing in which fundamentals are observed will be relevant later in selecting a unique equilibrium. An alternative, and possibly more realistic, assumption is that a subset of fundamentals is observed before the loan, while another subset is observed after the loan but before production.

[8]Allowing for asset accumulation would introduce not only an additional signal but also an additional decision between asking for the loan or not. This is an interesting extension, but beyond my scope here.

- Lenders do not observe the firm's type, action, or cash flow, just its continuation. They then update the firm's reputation from ϕ to ϕ'.

c) Preliminaries

Before formally defining the equilibrium of this model, I discuss the properties of reputation updating and the definition of the value function that firms maximize.

First, a few preliminaries about the behavior of strategic firms. Define $x(\phi, \theta)$ as the probability that a firm with reputation ϕ that observes fundamentals θ takes risks. I focus on equilibria in *cutoff* strategies, in which a firm with reputation ϕ decides to take risks if it observes fundamentals below a certain cutoff point, $k^*(\phi)$ and play it safe if it observes fundamentals above that cutoff,[9] such that

$$x(\phi, \theta) = \begin{cases} 0 & if \quad \theta > k^*(\phi) \\ 1 & if \quad \theta < k^*(\phi) \end{cases} . \tag{1}$$

i) Reputation Updating

When updating a firm's reputation, lenders have a prior belief about the firm's reputation and an expectation about the firm's behavior.

First, I restrict attention to Markovian strategies, such that the sufficient statistic about the firm's type (strategic or risky) is the lenders' prior belief of the firm's reputation level ϕ. This restriction allows the elimination of many equilibria, standard in models with public signals, that require an implausible degree of coordination between the firm's behavior and the lender's beliefs about the firm's behavior. The restriction also allows the elimination of equilibria in which reputation is not an asset, as we typically observe in reality.[10]

Second, even though I am restricting attention to Markovian strategies, reputation formation still depends on beliefs about the firm's actions. Let $\widehat{x}(\phi, \theta)$ be lenders' beliefs about the probability that a strategic firm with reputation ϕ takes risks when the fundamental is θ. Given cutoff strategies from equation (1), $\widehat{x}(\phi, \theta)$ is pinned down by $\widehat{k}(\phi)$, the lenders' beliefs about the cutoff that firms ϕ follow.

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[9] I restrict attention to this strategy given the monotonicity Assumption 2. In Section 2.1.2, I show that in this family of strategies a multiplicity of equilibria exists when the information about fundamentals is complete. In Section 2.2.1 I show that introducing noise into the observation of fundamentals, the unique equilibrium which survives iterated deletion of dominated strategies (as the noise goes to zero) is a cutoff strategy of this type.

[10] One of these equilibria can be, for example, to play it safe for certain fundamentals until the first bad result happens and then take risks forever afterward. In this particular equilibrium, reputation does not exist as I have interpreted it, and beliefs about firms' behavior require implausible degrees of complexity and coordination. See the discussion in the 2006 work of Mailath and Samuelson.

Using Bayes' rule, we know that after observing a continuing firm, lenders update reputations this way:

$$Pr(\mathcal{S}|c) = \phi'(\phi, \widehat{x}) = \frac{[p_r\widehat{x} + p_s(1 - \widehat{x})]\phi}{[p_r\widehat{x} + p_s(1 - \widehat{x})]\phi + p_r(1 - \phi)}, \tag{2}$$

where ϕ' is the firm's posterior reputation after the observation that the firm has continued.

Note that, for $\phi \in (0, 1)$, $\phi' = \phi$ when $\widehat{x} = 1$ and $\phi' > \phi$ when $\widehat{x} < 1$, with the gap between the new and old reputation $(\phi' - \phi)$ increasing as $\widehat{x}$ goes to 0. Graphically, firms reputation evolves as in Figure 1. Reputation priors ϕ are represented on the horizontal axis; reputation posteriors ϕ' on the vertical axis. For any prior ϕ, we can say three things.

- If lenders believe strategic firms play it safe for sure (that is, if $\widehat{x} = 0$), then the gap $\phi' - \phi$ represents the gains to the firm from continuation, in terms of reputation.

- If lenders believe strategic firms take risks for sure (that is, if $\widehat{x} = 1$), then $\phi' = \phi$ and firms get no gains in terms of reputation.

- Regardless of $\widehat{x}$, the updating is weak when priors are strong (that is, close to $\phi = 0$ or $\phi = 1$). In particular, regardless of $\widehat{x}$, $\phi' = \phi$ for $\phi = 0$ and $\phi = 1$. The maximum gap, $(\phi' - \phi)$ is obtained at some intermediate level, like ϕ_M.

Figure 1: Reputation Updating for Different Beliefs about Risk-Taking ($\widehat{x}$)
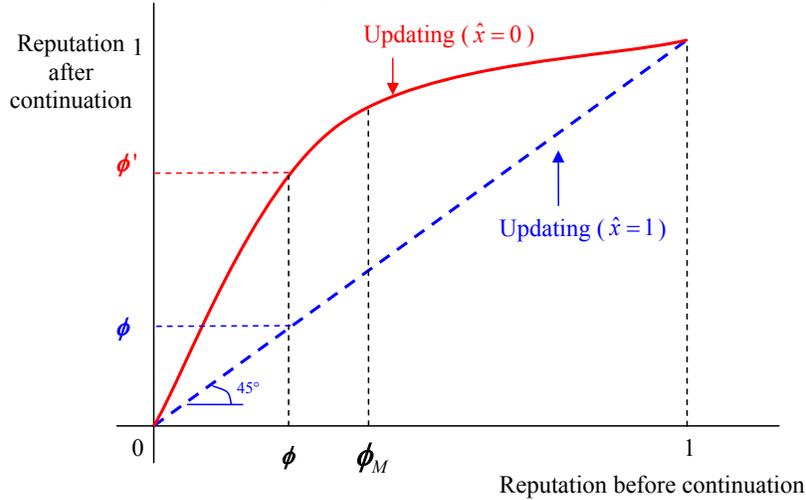
ii) Continuation Values

For now, I will focus on a firm's single period problem, when it faces exogenously fixed expected continuation values $\mathbf{V}(\phi)$, one for each reputation ϕ. These values are characterized by three properties: They are well-defined, they are positive (since profits are bounded below

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by zero), and they are monotonically increasing in the reputation level ϕ (since reputation is a valuable asset).[11]

Total discounted profits for a firm with reputation ϕ, that observes a fundamental θ, conditional on taking risks with probability $x(\phi, \theta)$ and lenders' beliefs about the cutoff $\widehat{k}(\phi)$, are,

$$
\begin{aligned}
\widetilde{V}(\phi, \theta | x, \widehat{k}) = \quad & x \quad [p_r[\Pi_r(\theta) - R(\phi|\widehat{k})] + \beta p_r \mathbf{V}(\phi'_{(\phi, \widehat{x}|\widehat{k})})] \\
& + (1-x)[p_s[\Pi_s(\theta) - R(\phi|\widehat{k})] + \beta p_s \mathbf{V}(\phi'_{(\phi, \widehat{x}|\widehat{k})})],
\end{aligned} \tag{3}
$$

where β is the discount factor. This allows us to define

$$
V(\phi, \theta | \widehat{k}) = \max_{x \in [0,1]} \widetilde{V}(\phi, \theta | x, \widehat{k})
$$

and $\mathbf{V}(\phi'_{(\phi, \widehat{k})}) = \int_{-\infty}^{\infty} V(\phi', \theta' | \widehat{k}') v(\theta') d\theta'$ is the expected continuation value for ϕ', an element of a given stream of expected continuation values $\Upsilon' = \{\mathbf{V}(\phi')\}_{\phi'=0}^{1}$.

d) Equilibrium

Now, I formally define the model's equilibrium in a given period, for any arbitrary stream of expected continuation values. Hence, each variable should have a subscript t. To simplify notation, I am not including it.

Definition 1 *A **single period** Markov perfect equilibrium in cutoff strategies, for a given stream of expected continuation values $\Upsilon' = \{\mathbf{V}(\phi')\}_{\phi'=0}^{1}$, consists of risk-taking cutoffs $k^*(\phi)$, interest rates $R(\phi)$, and posteriors ϕ', for each ϕ, such that*

- *Each firm with reputation ϕ that observes fundamental θ chooses $x^*(\phi, \theta)$ to maximize $\widetilde{V}(\phi, \theta | x, \widehat{k})$ (as in (3)) following a cutoff strategy $k^*(\phi)$ (as in (1)).*

- *Lenders charge $R(\phi)$ to obtain the risk-free rate $\overline{R}$ in expectation.*

- *Posteriors ϕ' are updated using Bayes' rule (using (2)).*

- *A strategy for a firm with reputation ϕ uniquely determines the equilibrium interest rate and the updating rule that lenders must use if their beliefs are to be correct (that is, if $\widehat{k}(\phi) = k^*(\phi)$).*

The equilibrium in a game in which firms live for a finite time T and $\mathbf{V}_{T+1}(\phi) = 0$, for all ϕ, (now explicitly differentiating each period by a subscript t) is as follows:

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[11]I show later, when solving the complete dynamic model, that these properties hold in equilibrium.

Definition 2 *A **finite-horizon** Markov perfect equilibrium in cutoff strategies consists of a sequence of risk-taking cutoffs $\{k_t^*(\phi)\}_{t=0}^T$, interest rates $\{R_t(\phi)\}_{t=0}^T$, posteriors ϕ', and firm continuation values $\{V_t(\phi)\}_{t=0}^T$, for each ϕ, such that*

- *A single period Markov perfect equilibrium exists in each $t \in \{0, 1, ..., T\}$.*

- $V_t(\phi) = \int_{-\infty}^{k_t^*(\phi)} \widetilde{V}_t(\phi, \theta|1, k_t^*) \, d\mathcal{V}(\theta) + \int_{k_t^*(\phi)}^{\infty} \widetilde{V}_t(\phi, \theta|0, k_t^*) \, d\mathcal{V}(\theta).$

2.1.2 Multiple Equilibria

Now I show that in my baseline model, when firms perfectly observe economic fundamentals, a multiplicity of Markovian perfect equilibria exists in monotone cutoff strategies in each period. First I discuss properties of the firms' differential gains from taking safe actions rather than risky ones, which characterize each firm's decisions. Then I show how these properties interact with lenders' beliefs about the firms' actions to create multiplicity of equilibria. I begin analyzing multiplicity in a single period and then I show how it extends to multiplicity in the whole finite horizon game as well.

a) Differential Gains from Safe Actions

Define by $\Delta(\phi, \theta|\widehat{k}) = \widetilde{V}(\phi, \theta|0, \widehat{k}) - \widetilde{V}(\phi, \theta|1, \widehat{k})$ the differential gains to firms from playing it safe rather than taking risks when a firm with reputation ϕ observes a fundamental θ, conditional on cutoff beliefs $\widehat{k}(\phi)$ (and, hence, beliefs $\widehat{x}(\phi, \theta)$ about risk-taking for each θ). Naturally, a firm decides to play it safe if $\Delta(\phi, \theta|\widehat{k}) > 0$ and takes risks if $\Delta(\phi, \theta|\widehat{k}) < 0$.

$$\Delta(\phi, \theta|\widehat{k}) = \overbrace{p_s\Pi_s(\theta) - p_r\Pi_r(\theta)}^{Short-Term} + \overbrace{\beta(p_s - p_r)\mathbf{V}(\phi)}^{Continuation} - \overbrace{(p_s - p_r)R(\phi|\widehat{k})}^{Moral\ Hazard}$$
$$+ \underbrace{\beta[p_s - p_r][\mathbf{V}(\phi'_{(\phi,\widehat{x}|\widehat{k})}) - \mathbf{V}(\phi)]}_{Reputation\ Formation} \tag{4}$$

Equation (4) displays the four essential components of these differential gains:

- *Short-Term* refers to differential gains in expected short-term cash flows to the firm from using a safe technology. This is the only part of the differential gains that depends on θ, and it drives the relative temptation to take risks.

- *Continuation* captures the effect that, taking safe actions increases the probability of the firm's continuation.

- *Moral Hazard* represents the firm's incentives to take risks in order to reduce the probability of having to pay back the debt. This component is the one that generates excessive risk-taking.

- *Reputation Formation* refers to the fact that taking safe actions also increases the probability of reputation improvement.

The following lemma shows that playing it safe is less tempting for firms as fundamentals and lenders' beliefs of safe actions decline:

Lemma 1 *A firm's differential gains from playing it safe, $\Delta(\phi, \theta | \widehat{k})$, is monotonically increasing in fundamentals θ and monotonically not increasing in risk-taking beliefs, $\widehat{k}$ and $\widehat{x}$.*

Proof I divide the proof of Lemma 1 into three steps:

- *Step 1:* $\frac{\partial \Delta(\phi, \theta | \widehat{k})}{\partial \theta} > 0$.

By Assumptions 1 and 2, we know that $\frac{\partial (p_s \Pi_s - p_r \Pi_r)}{\partial \theta} > 0$.

- *Step 2:* $\frac{\partial \Delta(\phi, \theta | \widehat{k})}{\partial \widehat{k}} \leq 0$.

Since loans are negotiated before fundamentals are known, interest rates are defined by the risk-free interest rate $\overline{R}$ divided by the expected probability of firm's continuation. Hence,

$$R(\phi | \widehat{k}) = \frac{\overline{R}}{Pr(c | \phi, \widehat{k})}, \tag{5}$$

where

$$Pr(c | \phi, \widehat{k}) = (1 - \phi) p_r + \phi \left[p_r \mathcal{V}(\widehat{k}) + p_s (1 - \mathcal{V}(\widehat{k})) \right],$$

with $\mathcal{V}(\widehat{k})$ being the cumulative distribution of fundamentals up to $\widehat{k}$ or the ex ante believed probability of risk-taking by firms with reputation ϕ. Since $p_s > p_r$, it is straightforward to show that $\frac{\partial R(\phi | \widehat{k})}{\partial \widehat{k}} \geq 0$.

- *Step 3:* $\frac{\partial \Delta(\phi, \theta | \widehat{k})}{\partial \widehat{x}} \leq 0$.

From equation (2), we know that $\frac{\partial (\phi' - \phi)}{\partial \widehat{x}} \leq 0$ (and strictly negative for $\phi \in (0, 1)$). By assumption (for now),[12] $\mathbf{V}(\phi)$ is monotonically increasing in ϕ, hence, $\frac{\partial (\mathbf{V}(\phi') - \mathbf{V}(\phi))}{\partial \widehat{x}} \leq 0$. Even when I discuss the effects of $\widehat{k}$ and $\widehat{x}$ separately, recall that pessimistic lenders (those with high $\widehat{k}(\phi)$) will not update reputation for a wider range of fundamentals (since $\widehat{x}(\phi, \theta) = 1$ for $\theta < \widehat{k}(\phi)$).

Q.E.D.

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[12]I show later (Section 2.2.2) this assumption holds in equilibrium.

b) Multiplicity

Now I turn to how this relationship between firms' differential gains from playing it safe and lenders' beliefs about their behavior creates a multiplicity of equilibria in this model. But before, I assume *uniform limit dominance*, which defines ranges of fundamentals for which, regardless of beliefs, firms decide to either take risks (fundamentals below a lower bound $\underline{\theta}$) or play it safe (fundamentals above an upper bound $\overline{\theta}$).

Assumption 3 *(Uniform Limit Dominance)*

- *For each ϕ and $\widehat{k}$, there is a lower bound $\underline{\theta}(\phi|\widehat{k})$ such that $\Delta(\phi, \underline{\theta}|\widehat{k}, \widehat{x} = 0) = 0$*

- *For each ϕ and $\widehat{k}$, there is an upper bound $\overline{\theta}(\phi|\widehat{k})$ such that $\Delta(\phi, \overline{\theta}|\widehat{k}, \widehat{x} = 1) = 0$*

A pair of reputation ϕ and cutoff beliefs $\widehat{k}$, determines an interest rate $R(\phi|\widehat{k})$. From equation (4) we know that $\underline{\theta}(\phi|\widehat{k})$ is the level of fundamentals that makes firms indifferent between playing it safe and taking risks when $\widehat{x} = 0$ is artificially assigned to all θ, regardless of $\widehat{k}$. Similarly, $\overline{\theta}(\phi|\widehat{k})$ is the level of fundamentals for firm indifference when $\widehat{x} = 1$ is artificially assigned to all θ. For example, $\underline{\theta}(\phi| - \infty)$ is the indifference θ for firms ϕ given $R(\phi| - \infty) = \frac{\overline{R}}{p_r(1-\phi)+p_s\phi}$ and $\widehat{x} = 0$. Similarly, $\overline{\theta}(\phi|\infty)$ is the indifference θ for firms ϕ given $R(\phi|\infty) = \frac{\overline{R}}{p_r}$ and $\widehat{x} = 1$.

The following lemma characterizes these bounds. The proof follows from inspecting equation (4), using lemma 1 and assumption 3.

Lemma 2   • $\underline{\theta}(\phi|\widehat{k}) \leq \overline{\theta}(\phi|\widehat{k})$ *for all $\widehat{k} \in \mathbb{R}$ (strictly $<$ for $\phi \in (0, 1)$).*

   • $-\infty < \underline{\theta}(\phi| - \infty) \leq \underline{\theta}(\phi|\widehat{k})$ *and $\overline{\theta}(\phi|\widehat{k}) \leq \overline{\theta}(\phi|\infty) < \infty$ for all $\widehat{k} \in \mathbb{R}$.*

The next Proposition formalizes the multiplicity of equilibria in this model. The proof is in the Appendix.

Proposition 1 *(Equilibria Multiplicity)*

For all reputation levels $\phi \in (0, 1)$, there is a continuum of equilibrium strategy cutoffs $k^*(\phi)$. These strategies are located in a range $[\underline{\theta}^*(\phi), \overline{\theta}^*(\phi)]$ where its bounds are given by the fixed-point fundamentals that solve $\Delta(\phi, \underline{\theta}^*|\underline{\theta}^*, \widehat{x} = 0) = 0$ and $\Delta(\phi, \overline{\theta}^*|\overline{\theta}^*, \widehat{x} = 1) = 0$. There is a unique range $[\underline{\theta}^*(\phi), \overline{\theta}^*(\phi)]$ for each ϕ if $v(\theta) < \frac{p_r^2\left[p_s\frac{\partial \Pi_s}{\partial \theta} - p_r\frac{\partial \Pi_r}{\partial \theta}\right]}{\overline{R}(p_s - p_r)^2}$ for all $\theta \in \mathbb{R}$. Only for reputation levels $\phi = 0$ and $\phi = 1$, is there a unique equilibrium cutoff, $k^*(0)$ and $k^*(1)$, respectively.

Figure 2 provides a graphical intuition of this multiplicity. Consider a particular risk-taking cutoff $k^*(\phi)$ for some firm with reputation $\phi \in (0,1)$, such that $k^*(\phi) \in [\underline{\theta}(\phi|k^*), \overline{\theta}(\phi|k^*)]$. Then, the equilibrium differential gain $\Delta(\phi, \theta|k^*)$ for different levels of fundamentals is the bold function with a discrete jump at $k^*(\phi)$. This is an equilibrium because it is a best response for any realization of the fundamental θ such that lenders' beliefs are correct. Playing it safe is optimal for all $\theta > k^*(\phi)$ (since $\Delta(\phi, \theta|k^*, \widehat{x} = 0) > 0$ for all $\theta > k^*(\phi)$), and taking risks is optimal for all $\theta < k^*(\phi)$ (since $\Delta(\phi, \theta|k^*, \widehat{x} = 1) < 0$ for all $\theta < k^*(\phi)$).

We know that in this setup, an arbitrarily small increase in the proposed risk-taking cutoff $k^*(\phi)$ generates an arbitrarily small increase in the interest rate. If interest rates do not change suddenly (which is guaranteed by the sufficient condition in Proposition 1, basically that fundamentals distribution has a variance large enough), then they cannot overcome the discrete jump generated by the effects of changes in reputation that result from sudden changes in beliefs from $\widehat{x} = 1$ to $\widehat{x} = 0$. Hence, we can find equilibrium cutoffs arbitrarily close to each other and, hence, a continuum of equilibria cutoffs. As we move the proposed cutoffs to the right of $k^*(\phi)$, interest rates increase, reducing $\Delta(\phi, \theta|\widehat{k})$ for all θ until $\overline{\theta}^*(\phi)$ is reached. The same is true as we move in the opposite direction, decreasing the proposed cutoffs from k^* toward $\underline{\theta}^*(\phi)$. These extremes are the bounds to equilibrium cutoffs.

Figure 2: Equilibria Multiplicity with Complete Information about Fundamentals
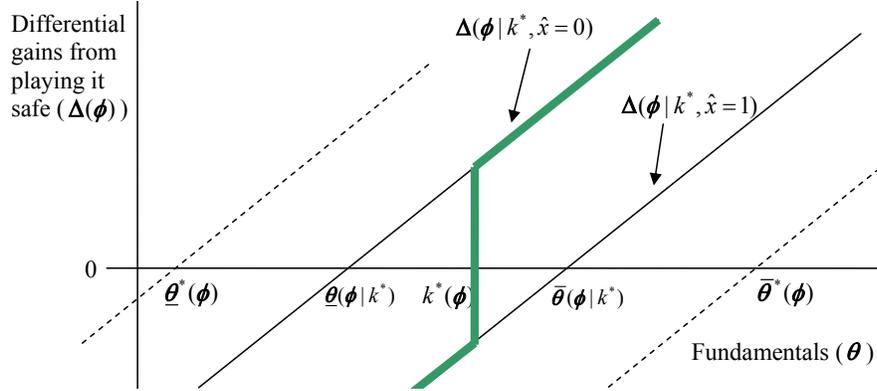
In words, for a fundamental to be a cutoff in equilibrium, it should be the case that two equilibria coexist at exactly that cutoff. At the one extreme, if lenders believe that all strategic firms play it safe, then firms do that. Firms know that in this case their continuation and loan repayment will be attributed at least partly to their good behavior, thereby improving their reputation. At the other extreme, if lenders believe that all firms take risks, then strategic firms do that. Under these beliefs, firms know that their continuation and repayment will be attributed solely to good luck and won't improve their reputation at all. Since the difference of payoffs between these two extremes is strictly positive, there is a continuum of fundamentals that fulfill this condition.

In the finite-horizon game, where continuation values are endogenous, it is straightforward to see that the range of multiple equilibria cutoffs in each period widens. Since multiplicity exists in every single period, multiple streams of future expected continuation values (consistent with multiple equilibria in future periods) can be used to construct $\Delta_t(\phi, \theta)$. By introducing extreme streams of continuation values determined by the highest ($\overline{\Upsilon}'$) and the lowest ($\underline{\Upsilon}'$) probability of risk-taking in all future periods for all reputation levels, we can construct extreme bounds $\underline{\theta}^*(\phi|\overline{\Upsilon}') < \underline{\theta}^*(\phi|\Upsilon')$ and $\overline{\theta}^*(\phi|\underline{\Upsilon}') > \overline{\theta}^*(\phi|\Upsilon')$ such that the region of multiplicity in a given period is wider when considering the multiplicity of equilibria in future periods.

The multiplicity I have described here thwarts attempts to draw conclusions about the effectiveness of reputation to reduce risk-taking. The higher the equilibrium cutoff is, the more likely firms are to take risks and the higher interest rates are. Hence, to perform comparative statics and comparative dynamics is impossible, since there is no explicit theory to guide the selection of equilibrium, leaving a big role to self-fulfilling beliefs and payoff irrelevant sunspots. However, here, as in Morris and Shin (2001), what really creates the multiplicity is the assumption of complete information of fundamentals, which at the same time requires an implausible degree of coordination and prediction of rivals' behavior in equilibrium. This orients us toward what to do next to move toward the selection of a unique equilibrium.

2.2 A Unique Equilibrium with Incomplete Information

What I do is modify the assumption that information about fundamentals is complete. I assume instead that firms observe a noisy signal about the aggregate fundamental before deciding which technology to use (safe or risky). After production occurs, fundamentals are perfectly observed by both firms and lenders.[13] The early signal observed by each firm is not observed by lenders, who can infer it only after observing the true fundamental. Introducing this noise into the observation of fundamentals is what leads to the selection of a unique equilibrium. As I did before, first I show uniqueness in a single period and then I expand the solution into the finitely repeated game.

2.2.1 Uniqueness in a Single Period

Incomplete information about fundamentals allows us to select a unique equilibrium in a single period, assuming exogenous future expected continuation values. The new assumptions about the information structure are as follows:

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[13]The assumption about the timing is important. If interest rates reveal, through the market's aggregation of information, the true fundamental before production occurs, we go back to complete information and a unique equilibrium cannot be pinned down by introducing heterogeneity through signals. See Atkeson (2001).

Assumption 4 *Each firm i observes a signal about the economic fundamentals $z_i = \theta + \sigma\epsilon_i$, where $\epsilon_i \sim F$ (density f and mean 0) is identically and independently distributed across i. The precision of the signal is measured by the inverse of the positive parameter σ.*

Hence, conditional on θ, the distribution of signals z is determined by $F(\frac{z-\theta}{\sigma})$.

Assumption 5 *(Monotone likelihood ratio property). For $a > b$, $\frac{f(a-\theta)}{f(b-\theta)}$ is increasing in θ.*

Assumption (5) means that a firm which receives a signal of the occurrence of high fundamentals assigns a large probability that other strategic firms do too; hence, a large probability that lenders believe strategic firms play it safe. Signals are thus useful not only to estimate θ and cash flows but also to make inferences about the beliefs that lenders will use to update their views about the firm's reputation.

Given this revised, incomplete information structure, the firm uses a cutoff strategy defined over the set of signals rather than over the set of fundamentals, which the firm not longer observes. For a current signal, a strategy of a firm ϕ is a real number $z^*(\phi)$ such that the firm uses safe technologies for $z_i > z^*(\phi)$ and risky ones for $z_i < z^*(\phi)$. In what follows, I eliminate subscripts i for notational simplicity.

The next proposition states that under this incomplete information structure, when signals are precise enough ($\sigma \to 0$), there exists a unique Markovian perfect Bayesian equilibrium in monotone cutoff strategies for each reputation level ϕ. The proof is in the Appendix.

Proposition 2 *(Uniqueness in a Single Period)*

*For a given ϕ, as $\sigma \to 0$, there exists in equilibrium a unique cutoff signal $z^*(\phi)$ such that $\Delta(\phi, z|z^*(\phi)) = 0$ for $z = z^*(\phi)$, $\Delta(\phi, z|z^*(\phi)) > 0$ for $z > z^*(\phi)$, and $\Delta(\phi, z|z^*(\phi)) < 0$ for $z < z^*(\phi)$, where $\Delta(\phi, z|z^*(\phi))$ are the **expected** differential gains to firms from playing it safe if a firm ϕ receives a signal z and lenders believe strategic firms ϕ use a cutoff $z^*(\phi)$ determined by*

$$\Delta(\phi, z^*|z^*) = \int_0^1 \Delta(\phi, \theta(\widehat{x}), \widehat{x}|z^*)d\widehat{x} = 0, \tag{6}$$

where $\theta(\widehat{x}) = z^* - \sigma F^{-1}(\widehat{x})$

Intuitively, relaxing the assumption of complete information about fundamentals and making signals very precise, we can use the approach provided by global games to select a unique equilibrium by iterated deletion of dominated strategies. Assume, for example, that a strategic firm ϕ receives a signal $\underline{\theta}^*(\phi)$. The firm would like to play it safe only if lenders have a

belief $\widehat{x}(\phi) = 0$. However, if fundamentals in fact happen to be $\underline{\theta}^*(\phi)$, lenders believe with some positive probability that the firm observed a signal below $\underline{\theta}^*(\phi)$, in which case it would have taken risks, no matter what. This means $\widehat{x}(\phi)$ cannot be zero. But with $\widehat{x}(\phi) > 0$, the firm would strictly prefer to take risks. Then $\underline{\theta}^*(\phi)$ cannot be an equilibrium cutoff anymore. By continuity, the same reasoning can be applied to signals above $\underline{\theta}^*(\phi)$ and below $\overline{\theta}^*(\phi)$.

As is standard, I require $\sigma \to 0$ so that the firm gives more weight to its private signal than to the public signal given by the prior distribution of θ. In this case, the process of iterated deletion of dominated strategies results in a unique cutoff $z^*(\phi) \in [\underline{\theta}^*(\phi), \overline{\theta}^*(\phi)]$, such that the firm takes risks when $z < z^*(\phi)$ and plays it safe when $z > z^*(\phi)$. Hence, fundamentals become also a coordination device. If a firm observes a low signal, it believes the fundamental is low with high probability, which directly induces the firm to take risks. On top of that, the firm also believes that other similar firms have observed a low signal and will take risks as well. Since lenders will believe with a high probability that a single firm takes risks, there will not be a reputation reward for playing it safe, which indirectly induces the firm to take risks. This is why, fundamentals, through the generation of signals, coordinate firms' expectations about lenders' beliefs and hence coordinate firms' actions.

2.2.2 Uniqueness in a Finite Horizon Model

Now I switch from analyzing a single period to analyzing the full-fledged finite horizon game where future expected continuation values are endogenous. I show also uniqueness in this full model, which is characterized by a unique sequence of equilibrium cutoffs as signals become very precise. Also, as the last period goes to infinity, there is a unique limit to the sequence of perfect Markovian equilibrium for the finite game.

This extension is necessary because the previous sections were based on an assumed profile of well-defined, positive, and monotonically increasing continuation values. Here I confirm these three properties arise in the dynamic equilibrium.

The following Proposition states that, based on the boundary condition $\mathbf{V}_{T+1}(\phi) = 0$ for all ϕ, expected continuation values $\mathbf{V}_t(\phi)$ are well-defined in each period t for all reputation levels ϕ and then, a unique equilibrium exists in the finite horizon game as $\sigma \to 0$. In order to solve this finite dynamic global game, I follow the literature from Morris and Shin (2003), Toxvaerd (2007), Giannitsarou and Toxvaerd (2007), and Steiner (2008).

Proposition 3 *(Uniqueness in a Finite Horizon Game)*

For each reputation ϕ, in each period t, as $\sigma \to 0$, a unique cutoff signal $z_t^*(\phi)$ exists such that the ***expected*** *differential gains from playing it safe $\Delta_t(\phi, z_t | z_t^*(\phi)) = 0$ for $z_t = z_t^*(\phi)$, $\Delta_t(\phi, z_t | z_t^*(\phi)) >$*

0 for $z_t > z_t^*(\phi)$, and $\Delta_t(\phi, z_t | z_t^*(\phi)) < 0$ for $z_t < z_t^*(\phi)$, where $\Delta_t(\phi, z_t | z_t^*(\phi))$ and $z_t^*(\phi)$ (as defined in Proposition 2) depend on $V_{t+1}(\phi)$ and $V_{t+1}(\phi')$. Continuation values $V_t(\phi)$ are well-defined and, given the boundary condition $V_{T+1}(\phi) = 0$, are recursively determined by

$$V_t(\phi) = \int_{-\infty}^{z_t^*(\phi)} p_r[\Pi_r(\theta) - R_t(\phi) + \beta V_{t+1}(\phi)]v(\theta)d\theta \tag{7}$$

$$+ \int_{z_t^*(\phi)}^{\infty} p_s[\Pi_s(\theta) - R_t(\phi) + \beta V_{t+1}(\phi')]v(\theta)d\theta$$

Proof The dynamic game can be solved as a series of static games that deliver a unique equilibrium (that is, a unique cutoff $z_t^*(\phi)$ for each t). In the last period T, the cutoff $z_T^*(\phi)$ is unique (under the condition in Proposition 1) since $\Delta_T(\phi, z_T | \hat{z}_T(\phi))$ is well-defined and $V_{T+1}(\phi) = 0$ for all ϕ; thus here reputation concerns do not generate multiplicity. Once $z_T^*(\phi)$ is determined, the equilibrium interest rate at T for each ϕ is

$$R_T(\phi | z_T^*(\phi)) = \frac{\overline{R}}{Pr(c)_T} = \frac{\overline{R}}{(1 - \phi)p_r + \phi[p_r \mathcal{V}(z_T^*(\phi)) + p_s(1 - \mathcal{V}(z_T^*(\phi)))]}$$

Then we can define expected continuation values in T for each reputation level ϕ. For signals $z_T < z_T^*(\phi)$, firms take risks and for signals $z_T > z_T^*(\phi)$, firms play it safe. As $\sigma \to 0$, expected profits in the last period T are

$$\mathbf{V}_T(\phi) = \int_{-\infty}^{z_T^*(\phi)} p_r[\Pi_r(\theta) - R_T(\phi)]v(\theta)d\theta + \int_{z_T^*(\phi)}^{\infty} p_s[\Pi_s(\theta) - R_T(\phi)]v(\theta)d\theta. \tag{8}$$

Since equilibrium thresholds are well-defined and unique in period T, continuation values $\mathbf{V}_T(\phi)$ are also well-defined and unique for all ϕ.

Now consider the decision of a firm ϕ in the next to last period $T - 1$. The problem is essentially static, since continuation values $\mathbf{V}_T(\phi)$ are well-defined and unique for all ϕ. Then, $\Delta_T(\phi, z_{T-1} | \hat{z}_{T-1}(\phi))$ is also well-defined for all ϕ, thus leading to a unique equilibrium cutoff $z_{T-1}^*(\phi)$ (following Proposition 2) and unique and well-defined $\mathbf{V}_{T-1}(\phi)$ for all ϕ.

By straightforward inductive reasoning, we know that, as $\sigma \to 0$, a unique sequence of cutoffs $\{z_t^*(\phi)\}_{t=0}^T$ and a unique sequence of expected continuation values exist for each reputation level ϕ in each period t, characterized by equation (7). Q.E.D.

The next proposition establishes that under certain conditions, in particular when the variance of the fundamentals distribution is large enough, there is a unique limit to the sequence of perfect Markovian equilibria for the finite game.

Proposition 4 *If $V_T(\phi) \to \overline{V}(\phi)$ as $T \to \infty$ and $\sigma \to 0$, then a cutoff $z^*(\phi)$ exists for each ϕ that is a unique limit to the sequence of cutoffs $\{z_t^*(\phi)\}_{t=0}^T$ of the finite-horizon Markov perfect equilibria described in Proposition 3.*

Proof In the Appendix, I show that the sufficient condition for $\mathbf{V}_T(\phi) \to \overline{\mathbf{V}}(\phi)$ for all ϕ as $T \to \infty$ is $v(\theta) < \frac{1-\beta p_s}{\beta p_s} \frac{p_r^2 \left[p_s \frac{\partial \Pi_s}{\partial \theta} - p_r \frac{\partial \Pi_r}{\partial \theta} \right]}{\overline{R}(p_s - p_r)^2}$ for all $\theta \in \mathbb{R}$. Having shown uniqueness for an arbitrary finite-horizon T, I must show that the same reasoning is extended as $T \to \infty$. First, note that the value of taking safe and risky actions is a bounded and well-behaved monotone function of T when continuation values converge to a fixed point $\overline{\mathbf{V}}(\phi)$ as $T \to \infty$. Second, as defined in equation (4), $\Delta_t(\phi, z_t | z_t^*(\phi))$ also converges to a unique limit as $T \to \infty$. Then $z_t^*(\phi|T) \to z_t^*(\phi|\infty) = z^*(\phi)$ as $T \to \infty$, where $z_t^*(\phi|T)$ is the equilibrium cutoff in a game truncated in T and $z^*(\phi)$ is the equilibrium cutoff in any t in an infinite-horizon game. Q.E.D.

Intuitively, we know that if we solve backward from some T by using as a boundary condition the fixed point $\mathbf{V}_{T+1}(\phi) = \overline{\mathbf{V}}(\phi)$, rather than $\mathbf{V}_{T+1}(\phi) = 0$, then we obtain a unique $z^*(\phi)$ for each ϕ in all periods $t < T$. This matters because, as $\sigma \to 0$, in a period t far enough from the last period $T \to \infty$, unique cutoffs and ex ante probabilities of risk-taking are constant over time for each reputation level ϕ.

3 Establishing Negative Effects of Reputation Concerns

Now I use the unique equilibrium from the last propositions to show how in this model concerns about reputation have potential negative effects as well as the positive effects already understood. Reputation concerns reduce risk-taking, but they are fragile. Small and not obvious changes in economic fundamentals can suddenly break down the discipline induced by those concerns. Furthermore, when they are gone, they are gone for a bunch of different firms with intermediate and good reputation, generating a clustering of risk-taking that can have far-reaching aggregate negative consequences. I demonstrate these results first in a formal abstract analysis and then in an illustrative numerical simulation of the model.

3.1 The Formal Analysis

A formal dynamic analysis of reputation concerns in credit markets establishes both the positive and potential negative effects of reputation at an aggregate level.

3.1.1 The Standard Positive Effects

The next proposition shows why firms are concerned about constructing and maintaining good reputations. A better reputation for a firm implies a lower ex ante probability of taking risks, and hence that it pays lower interest rates and has a higher continuation value. The proof is in the Appendix.

Proposition 5 *(Reputation, Risk-Taking, Lending Rates and Continuation Values)*

As a firm's reputation ϕ improves

i) Its ex ante probability of risk-taking monotonically decreases (that is, $\frac{d\mathcal{V}(z^*(\phi))}{d\phi} < 0$ for all $\phi \in [0, 1]$).

ii) The interest rate it faces monotonically decreases (that is, $\frac{dR(\phi)}{d\phi} < 0$ for all $\phi \in [0, 1]$).

iii) Its continuation value monotonically increases (that is, $\frac{d\mathbf{V}(\phi)}{d\phi} > 0$ for all $\phi \in [0, 1]$).

The next Proposition shows the positive effect of reputation concerns in reducing risk-taking by firms of all kinds of reputation, when compared to an artificial situation in which firms are not concerned about their reputation, simply because the reputation cannot change.[14]

Proposition 6 *(Reputation Concerns Reduce Risk-Taking)*

Define as $\widetilde{z}^*(\phi)$ the risk-taking cutoffs when reputation is not a concern (this is, when reputation cannot change). Reputation concerns reduce the ex-ante probability of risk-taking (that is, $z^*(\phi) < \widetilde{z}^*(\phi)$) for all $\phi \in (0, 1)$ and does not change it (this is $z^*(\phi) = \widetilde{z}^*(\phi)$) for $\phi = \{0, 1\}$.

Proof With reputation concerns, $z^*(\phi)$ is determined by equation (6) in the following way:

$$\int_0^1 \Delta(\phi, \widehat{x}|z^*)d\widehat{x} = 0.$$

Without reputation concerns, $\widetilde{z}^*(\phi)$ is determined just by

$$\Delta(\phi, \widehat{x} = 1|\widetilde{z}^*) = 0,$$

since the restriction that reputation cannot change is exactly the same as assuming $\widehat{x} = 1$. Fixing $\mathbf{V}(\phi)$ and $R(\phi)$ for all ϕ, from equation (4) we know $\Delta(\phi, z^*|z^*, \widehat{x})$ achieves the minimum at $\widehat{x} = 1$. Hence, applying lemma 1, $\widetilde{z}^*(\phi)$ is not lower than $z^*(\phi)$. Since $R(\phi|z^*) \leq R(\phi|\widetilde{z}^*)$ and $\mathbf{V}(\phi|z^*) \geq \mathbf{V}(\phi|\widetilde{z}^*)$ for all ϕ, $\Delta(\phi, \widehat{x}|z^*) \geq \Delta(\phi, \widehat{x}|\widetilde{z}^*)$ for all ϕ and all $\widehat{x}$, which reinforces the previous argument. Hence, from equation (2), $z^*(\phi) < \widetilde{z}^*(\phi)$ for all $\phi \in (0, 1)$ and $z^*(\phi) = \widetilde{z}^*(\phi)$ for $\phi = \{0, 1\}$. Q.E.D.

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[14]For example, credit histories are erased or lenders cannot observe the age of the firm.

3.1.2 The New Potential Negative Effects

Now I can go further and establish that the existence of reputation concerns may also have negative effects by creating sudden changes in aggregate risk-taking behavior without obvious changes in fundamentals. First, we can characterize firms' equilibrium cutoffs $z^*(\phi)$ for different reputation levels ϕ. The next lemma shows that the concerns for reputation formation convexify the schedule of cutoffs. The proof is in the Appendix.
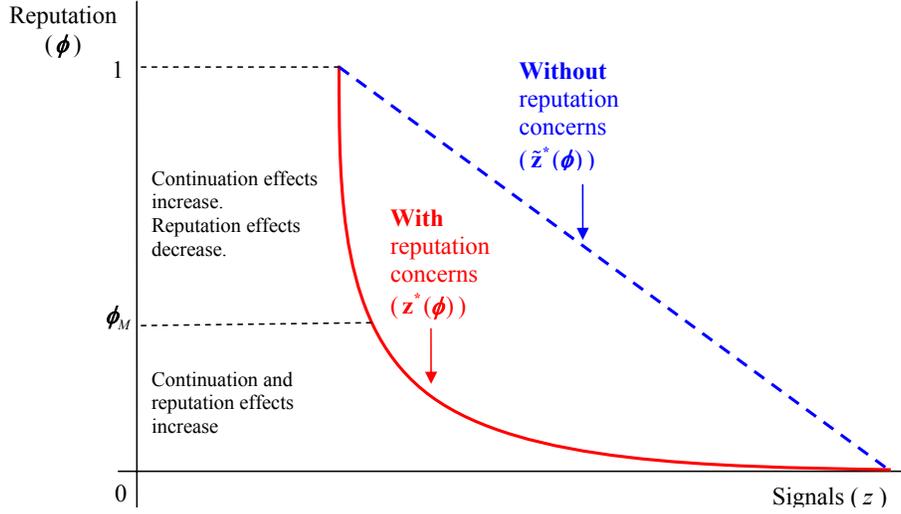
Lemma 3 *Reputation concerns convexify the schedule of cutoffs (that is, $\frac{d^2 z^*(\phi)}{d\phi^2} > \frac{d^2 \widetilde{z}^*(\phi)}{d\phi^2}$ for all $\phi \in [0, 1]$, where $\widetilde{z}^*(\phi)$ are the cutoffs without reputation concerns). Furthermore, there are always signals of the firm's type precise enough ($\frac{p_s}{p_r}$ high enough) such that the schedule of cutoffs is convex (that is, $\frac{d^2 z^*(\phi)}{d\phi^2} > 0$) for all ϕ.*

I use Figure 3 to explain the intuition for Lemma 3. Each firm can be represented by a point on the graph, a combination of levels of reputation ϕ and an observed signal z. Firms follow a cutoff schedule. A firm ϕ that observes a signal below $z^*(\phi)$ will decide to take risks and will play it safe otherwise. Assume, for example, that without reputation concerns, the schedule of cutoffs ($\widetilde{z}^*(\phi)$) is linear in ϕ (as in the figure). As we know from Proposition 6, reputation concerns reduce risk-taking (that is, reduce cutoffs from $\widetilde{z}^*(\phi)$ to $z^*(\phi)$) for all ϕ. However, the strength of this force is not the same across reputation levels. I start by discussing $\phi = 0$ and then gradually raise ϕ.

Firms with reputation $\phi = 0$ cannot change their reputation, which means that the cutoff for risky behavior is the same, with and without reputation concerns ($z^*(0) = \widetilde{z}^*(0)$). As ϕ increases, firms have higher reputation concerns, which rapidly reduces cutoffs. This effect achieves the maximum at ϕ_M, where reputation can increase the most. After this point, further increments in ϕ reduce the role of reputation concerns. At the extreme $\phi = 1$, reputation cannot improve further, so the cutoff is again the same with and without reputation concerns ($z^*(1) = \widetilde{z}^*(1)$). For firms with poor reputation the two types of reputation incentives - continuation and reputation - reinforce each other in reducing risk-taking. For firms with better reputation, while continuation effects increase with ϕ, reputation effects decrease.

Now I can summarize results considering just three types of firms, based on their reputation. *Poor* reputation firms are prone to take risks because their gains from surviving are low (they will have to pay high interest rates in the future) and they cannot change their reputation much. *Intermediate* reputation firms want to take safe actions not because they can lose a lot if they die, but because they can improve their reputation a lot if they survive. *Good* reputation firms want to take safe actions for the reverse reason, not because they can improve their reputation a lot if they survive but because they can lose a lot if they die.

Figure 3: Reputation and Cutoffs for Risk-Taking Behavior

Reputation
(ϕ)

1

Without
reputation
concerns
($\tilde{z}^*(\phi)$)

Continuation effects
increase.
Reputation effects
decrease.

With
reputation
concerns
($z^*(\phi)$)

ϕ_M

Continuation and
reputation effects
increase

0

Signals (z)

This leads us to a final, crucial proposition:

Proposition 7 *(Fragility of reputation and clustering of risk-taking)*

i) Reputation is fragile at a firm level. For highly precise signals about fundamentals ($\sigma \to 0$), small changes in fundamentals θ around the optimal cutoff $z^*(\phi)$ induce a change in risk-taking, clustered among firms with reputation level ϕ.

ii) Reputation is fragile at an aggregate level. For strong reputation concerns ($\frac{p_s}{p_r}$ high enough), small changes in fundamentals θ around the optimal cutoff $z^*(\phi_M)$ for intermediate reputation firms induce a change in risk-taking, clustered among firms with intermediate and good reputation levels.

The first part of this proposition is just a result from global games. If $\theta < z^*(\phi)$, then when the signal noise goes to zero, almost all firms with reputation level ϕ receive a signal $z < z^*(\phi)$ and decide to take risks. Hence, reputation concerns are fragile in the sense that small changes in fundamentals around $z^*(\phi)$ induce sudden changes in risk-taking for firms with reputation ϕ. Our equilibrium selection creates a clustering of risk-taking among firms with the same reputation level.

The second part of Proposition 7 is a corollary of Lemma 3 for a given distribution of reputation levels. When fundamentals are strong enough (high θ), small variations do not induce firms of different kind of reputation to modify their risk-taking behavior. Contrarily, when fundamentals are weak enough (low θ), then small variations may induce firms of different kind of reputation to modify their risk-taking behavior. Changes around weak fundamentals generate clustering of risk-taking among firms with different reputation levels. This fragility

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is generated at an aggregate level by learning primitives. Figure 4 illustrates fragility both at a firm and at an aggregate level.

Figure 4: Individual and Aggregate Clustering of Risk-Taking
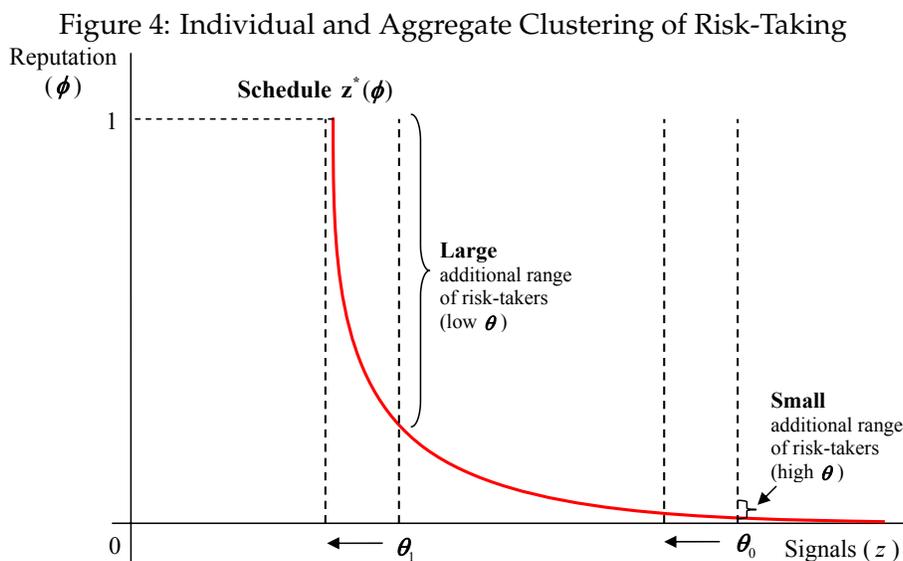
While reputation concerns have a positive effect (reducing excessive risk-taking) they also have a negative effect. Because they are fragile, their breakdown can lead to sudden and isolated clusters of risk-taking, big increases in default probabilities, and huge losses for lenders.

Before illustrating reputation fragility by a simulation exercise (which also shows how to solve the model numerically), let me highlight the fact that clustering depends not only on the convex schedule of cutoffs, but also on the reputation distribution in the economy. In particular, a distribution with a large mass of intermediate and good reputation firms strengthens the results.[15] In the numerical exercise below, I derive numerically the endogenous stationary distribution of firms, reputations and show it is indeed skewed toward intermediate and good reputation levels, where cutoffs are more alike. This reinforces the aggregate effects of clustering among firms with intermediate and good reputation.

3.2 A Numerical Exercise

Here I use a numerical example to illustrate the results just established in the formal analysis above, how a breakdown of reputation concerns may generate large changes in aggregate behavior in response to small changes in aggregate fundamentals.

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[15]To derive theoretically the endogenous reputation distribution is beyond my scope here, however the extension is feasible and interesting. Since cutoffs $z^*(\phi)$ are independent of the distribution, the reputation distribution depends primarily on assumptions about the birth and reputation priors of new firms, which can be pinned down assuming entry of firms at a given cost

3.2.1 The Exercise

For this exercise I must assign the model's parameters some reasonable values. In particular I require these values to fulfill four conditions.

- Risk-taking is almost never efficient.[16] It happens for fundamentals that occur ex ante with a probability of only 0.001%.

- Short-term cash flows for any level of fundamental are higher than interest rates in equilibrium, hence, firms can always pay back debts if they continue.

- Without reputation concerns, interest rates are not convex in ϕ, so I can show the forces of reputation concerns in convexifying them.

- Conditions for uniqueness of bounds (from Proposition 1) and convergence of continuation values to a fixed point (from Proposition 4) are fulfilled.

I assume that profits from safe actions (Π_s) are constant and profits from risky actions (Π_r) decrease with fundamentals. Hence, as in the model, risky actions are more tempting for low values of θ. In particular $\Pi_r = \Pi_s + K - \psi\theta$ with $\psi > 0$, where $\Pi_s = 1.5$, $K = 0.4$, and $\psi = 0.2$. I also assume that the probability of continuation is $p_s = 0.9$ when playing it safe and $p_r = 0.7$ when taking risks. Finally, I assume the discount factor is $\beta = 0.95$, the risk-free interest rate is $\overline{R} = 1$ and fundamentals are distributed as a standard normal (this is, $\theta \sim \mathcal{N}(0,1)$).

For this exercise, I also add to the basic model a set of signals correlated to actions, that lenders observe after the firm continues. One reason is that it shows the flexibility of the model to include additional signals such that they allow lenders to make better inferences about the firm's type. Another reason is that, in the basic setup of my model, firm reputation only increases with age since the only potential positive signal is continuation. These additional signals increase the importance of reputation formation. Naturally, I assume the probability of generating a good signal after a firm continues is greater if playing it safe ($\alpha_s > \alpha_r$). For this exercise in particular I assume $\alpha_s = 0.8$ and $\alpha_r = 0.4$.

Using these parameters I compare the results of the basic model (with reputation concerns) with an artificial situation in which the firm's reputation cannot change (without reputation concerns), and identify the positive and negative effects of reputation concerns, as described in the formal analysis.

I proceed in two steps. First I compute the limit of the schedule of cutoffs for different reputation levels as $T \to \infty$, with and without reputation concerns. These cutoffs are the same

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[16]Risk-taking is efficient when firms decide to take risks if lenders observe their actions and charge a corresponding high interest rate.

in every period of the model. Then I simulate realizations of fundamentals for 100 periods and I aggregate the risk-taking behavior of firms that follow those cutoffs. The computational procedure is described in the Appendix.

3.2.2 Positive Effects

Figure 5 shows, for each reputation ϕ, cutoffs with and without reputation concerns ($z^*(\phi)$ and $\widetilde{z}^*(\phi)$, respectively) and ex ante probabilities of risk-taking ($\mathcal{V}(z^*(\phi))$ and $\mathcal{V}(\widetilde{z}^*(\phi))$) that apply to any period when $T \to \infty$. As stated in Proposition 5, the probability of risk-taking decreases with reputation. Furthermore, as stated in Proposition 6, for all reputation levels, the probability that firms take risks is lower with reputation concerns than without them. For example, the ex ante probability that a firm with a reputation level $\phi = 0.4$ takes risks is only 4% with reputation concerns but 50% without them. Recall that, by construction, risk-taking is almost never efficient (it is only efficient for $\theta < -4$, which happens with a probability of only 0.001%). Hence, the gap between the two curves in the second plot of Figure 5 shows how reputation concerns reduce ex ante probabilities of inefficient risk-taking.
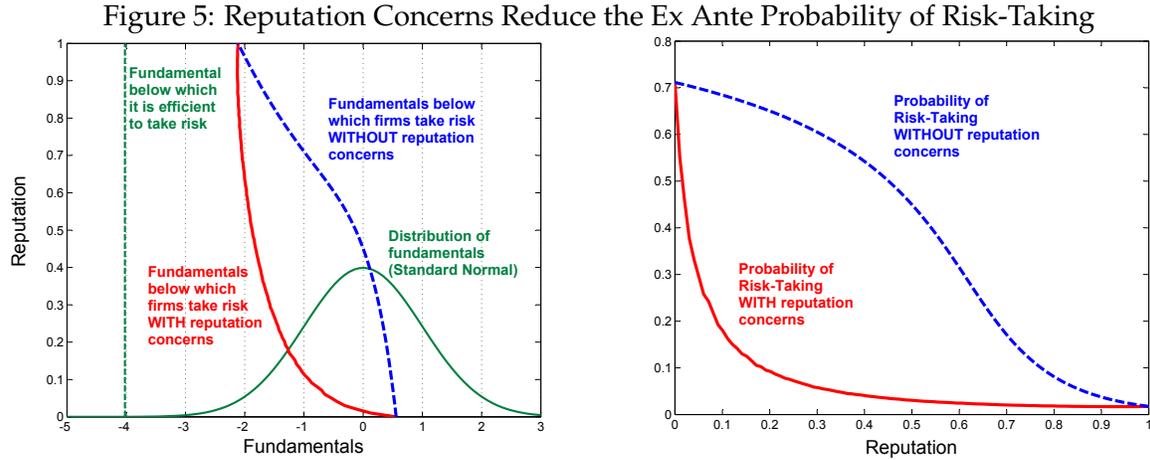
Figure 5: Reputation Concerns Reduce the Ex Ante Probability of Risk-Taking

Figure 6 shows the expected continuation values and lending rates for different reputation levels ϕ. Also as stated in Proposition 5, continuation values increase and lending rates decrease with reputation. Furthermore, firms have higher expected continuation values and pay lower interest rates when having reputation concerns.

3.2.3 Negative Effects

Here I simulate fundamental shocks realization for 100 periods and aggregate the risk-taking behavior in the economy. In this way I can show how the fragility of reputation concerns can

Figure 6: Reputation Concerns Increase Continuation Values and Reduce Lending Rates
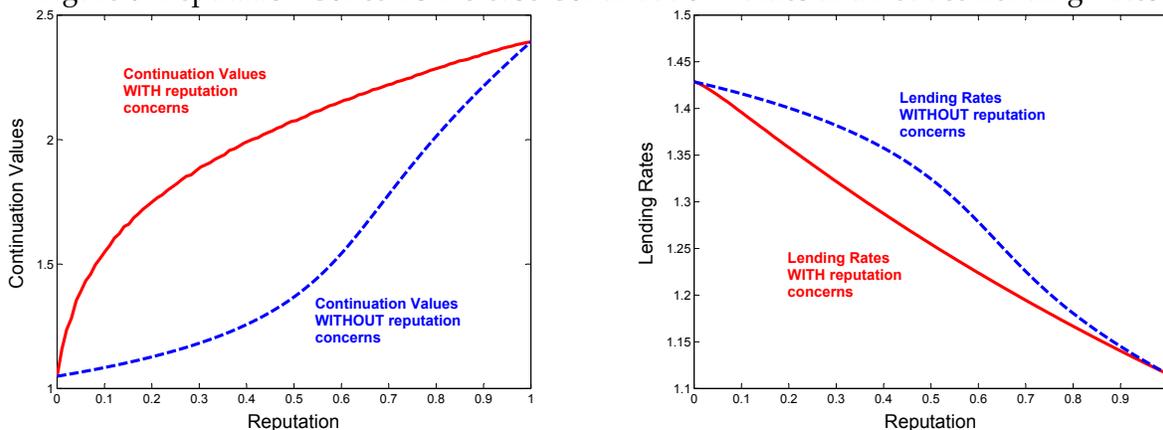
create sudden and isolated events of clustering of risk-taking, with spikes in aggregate loan defaults and lenders' losses, without obvious changes in aggregate fundamentals.

Three characteristics of this simulation exercise are worthwhile to note. First, I'm aggregating only strategic firms, since these are the ones whose behavior changes with fundamentals. If all of them play safe, the aggregate probability of loan default is 10% (since $p_s = 0.9$). If all of them take risks, the aggregate probability of loan default is 30% (since $p_r = 0.7$). Second, I aggregate over firms assuming an invariant uniform reputation distribution in all periods first and an evolving reputation distribution later. Finally, I choose 100 periods where risk-taking is never efficient (that is, no fundamental realization is below -4).

The first plot of Figure 7 shows aggregate loan default probabilities in the 100 periods for a fixed uniform reputation distribution. Without reputation concerns, aggregate default closely follows changes in fundamentals. With reputation concerns, aggregate default is less sensitive to those changes in fundamentals. It is low in general and spikes when conditions weaken enough. This comparison reveals that reputation concerns in general reduce inefficient risk-taking (a positive aggregate effect) but also creates large spikes in default rates without obvious changes in fundamentals (a negative aggregate effect). The main reason can be seen in Figure 5. Reputation effects are stronger for intermediate reputation levels than for extreme ones, convexifying the schedule of cutoffs and generating clustering of risk taking among a bunch of firms with intermediate and good reputations, when fundamentals go below values around -2.

In the second plot of Figure 7, I allow the reputation distribution to evolve over time, starting from a uniform distribution in the initial period. I assume that new firms enter to replace dead ones with a reputation prior $\phi = 0.5$. As I show later, this distribution evolves biasing towards good reputations. Recognizing this bias reinforces not only the positive effects of reputation

concerns (less inefficient aggregate default at most periods), but also the negative ones, since spikes in default rates are even more isolated and sensible to small changes in fundamentals. This is mainly because the market is mostly composed by firms with intermediate and good reputations, which are the ones that cluster when fundamentals weaken enough.

Figure 7: Simulated Default Probability with Fixed and Evolving Reputation Distribution
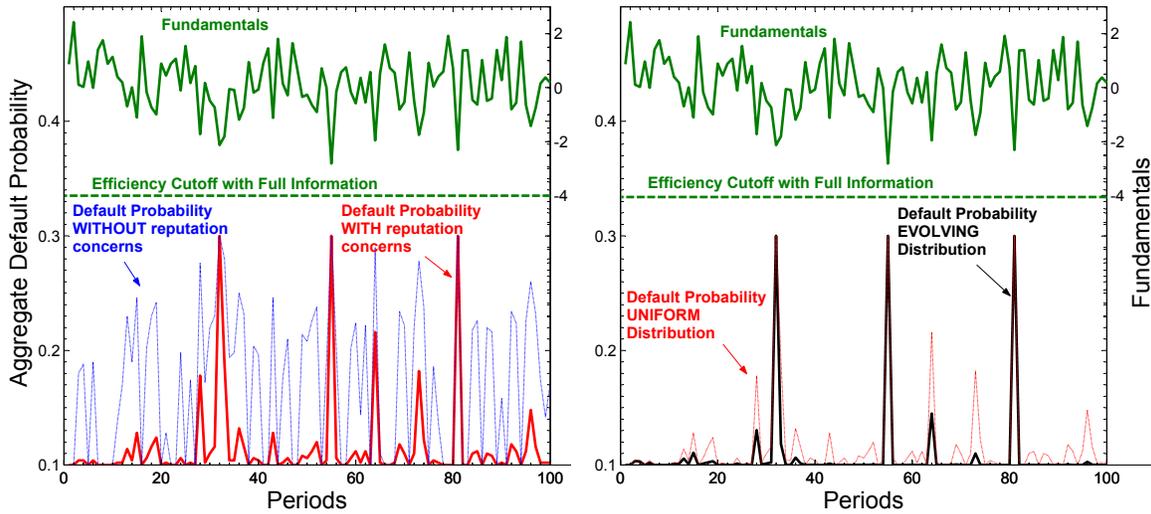

Figure 8 shows the stationary expected distribution of reputation in the market and the evolution of firms with reputation 0.01, 0.99 and the assigned prior 0.5, as a fraction of the total of firms. The fraction of firms with poor reputation tends to disappear while the fraction of firms with good reputation grows over time toward a stationary distribution. However, this evolution is not monotonic. When a spike of risk-taking occurs, good firms die at a higher rate than in normal times, and are replaced by new firms with an intermediate reputation of $\phi = 0.5$. Hence, in those periods of high risk-taking, there is a decline in the average quality of firms in the market. This is relevant because a bad enough shock in fundamentals does not only magnify crisis, but also makes it persistent.

Finally, Figure 9 shows the model's view of aggregate net returns to lenders.[17] When fundamentals weaken enough, we see that returns decline catastrophically, since most firms, regardless of their reputations, take risks. Since lenders charge low rates to good reputation firms, sudden losses are large. With reputation concerns, lending rates are lower, hence lenders losses are greater when they rarely occur. This matters because reputation concerns reduce the frequency of crises, but they magnify lenders' losses when crises do occur.

This exercise has illustrated that reputation concerns have negative as well as positive effects.

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[17]First I obtained individual net returns for each reputation level (computed by the lending rate charged to ϕ multiplied by the true probability of no default minus the risk-free rate). Then I calculated the weighted sum of individual returns to obtain aggregate net returns.

Figure 8: Stationary Reputation Distribution and Initial Evolution of Certain Reputations
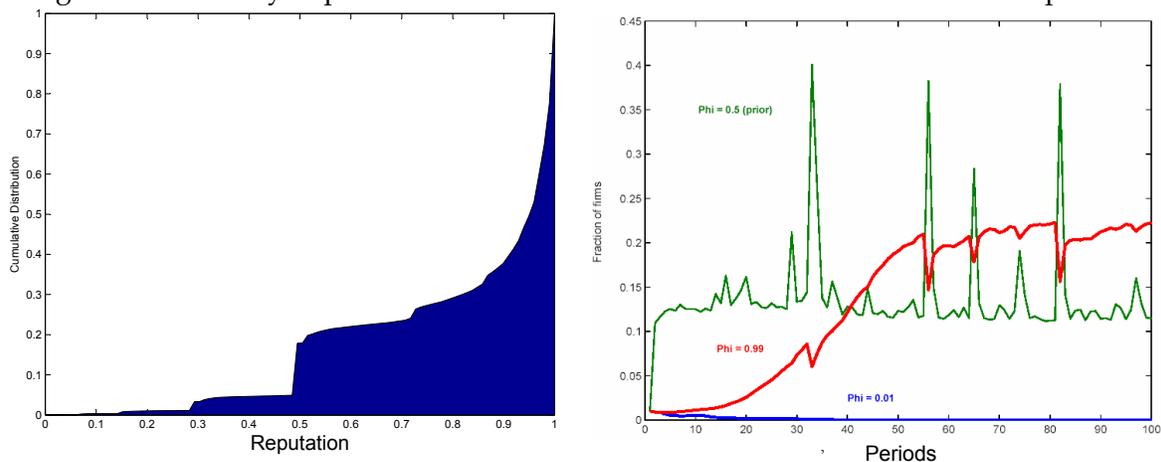
Reputation concerns deter excessive risk-taking in general, but also generate sudden clusters of risk-taking behavior, characterized by more default and large losses to lenders, even without noticeable changes in fundamentals. Even when the exercise is based on arbitrary parameters, results are highly robust to variations in them, as long as reputation concerns remain important and conditions for uniqueness and convergence are guaranteed.

4   Summary and Implications

Firms' concerns about their reputation reduce excessive risk-taking. This positive effect of reputation is widely accepted on both formal and informal grounds. Here I have studied the effects of reputation concerns from an aggregate perspective, when incentives to take risk vary with the state of the economy. My main finding is that reputation concerns may have negative as well as positive aggregate effects. These concerns are in fact fragile, and may suddenly disappear, leading to large changes in aggregate risk-taking as well-known and reputable firms cluster in response to small or not obvious changes in fundamentals.

In my model, reputations can eventually be constructed, destroyed, and managed. However, this desirable feature comes with a cost in terms of equilibria multiplicity. To overcome this problem I have interpreted the reputation model extended with fundamentals as a non-standard dynamic global game in which strategic complementarities arise endogenously from reputation formation. This allows me to select a unique equilibrium, robust to perturbations in information about fundamentals, which become a coordination device for risk-taking.

Two clear testable implications from the model have been supported empirically. One is that, if risk-taking is more tempting during recessions, then reputation should evolve less and with

Figure 9: Simulated Aggregate Net Returns to Lenders
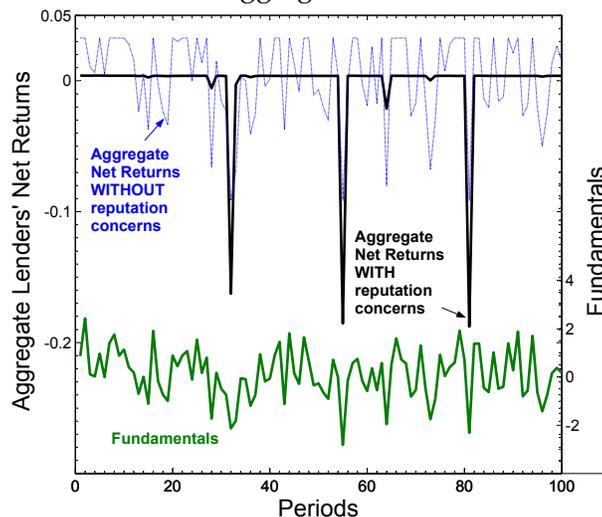
more difficulty in bad times than in good times.[18] The other supported implication is that risk-taking and defaults cluster in time, especially around recessions, to a large degree that cannot be explained just from a weakening in fundamentals.[19]

The model's results also have several new policy implications, all of which may tell us something about the recent financial crisis. One implication is that reputation can be thought as a self-disciplining screening device provided by the market, which makes unnecessary and inefficient a regulatory intervention to deter excessive risk-taking under certain fundamentals. However, I argue that this device is fragile and may suddenly disappear, eventually making regulation necessary to prevent crises. This suggests that, to be efficient, regulation should be cyclical, operating only when economic fundamentals move toward regions in which risk-taking becomes more tempting. For the recent crisis this suggests, for example, that when the housing bubble was growing, the intervention of a regulatory body, responsible for overseeing mortgage paperwork, controlling loan risks, and restricting debt ratios of financial institutions might have helped prevent the subsequent crisis.

Another implication is that, during periods of clustering of risk-taking, credit ratings are likely to be uninformative about default probabilities. In certain times, firms with AAA bond ratings may have incentives to undertake risky projects, exactly like firms with not-so-good ratings. In these periods, reputation loses its meaning. This argues against the use of Basel II regulations, which rely on ratings to determine the level of capital that banks should hold.

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[18]This is supported empirically by Nickell, Perraudin, and Varotto (2000), Bangia et al. (2002), Altman and Rijken A. (2006), and Ordonez (2008) using data on corporate credit-rating transitions over the business cycle. This pattern has been also observed during the latest financial crisis. The Moody's Credit Policy report of July 2008 shows that the rating volatility decreased significantly, almost 50% with respect to historical averages.

[19]See the work of Campbell et al. (2001) and Das et al. (2007).

Since ratings may lose information content exactly when they are most needed, they may in fact spread a loss of confidence throughout the financial system. In fact, part of the failure that led to the wider crisis has been assigned to a breakdown in the bond rating system (*The Economist*, August 16, 2007).

Finally, a third implication is that policies which promote credit bureaus can be double-edged. They do improve learning and increase reputation incentives in domestic financial systems, so they may be effective in reducing excessive risk-taking. However, they may also have the potential to exacerbate credit crises.

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A Appendix

A.1 Proof of Proposition 1

First we will prove two lemmas that describe single crossing properties and allow us to identify a unique cutoff in the set of fundamentals (θ) and in the set of beliefs ($\widehat{x}$) when fixing an interest rate (this is, a fixed $\widehat{k}$).

Lemma 4 *(Fundamental single crossing) For every reputation level $\phi \in (0,1)$ and cutoff belief $\widehat{k}$, fix a $\widehat{x} \in [0,1]$ for all θ. There exists a unique $\theta^* \in [\underline{\theta}(\phi|\widehat{k}), \overline{\theta}(\phi|\widehat{k})]$ such that $\Delta(\phi, \theta|\widehat{k}, \widehat{x}) < 0$ for $\theta < \theta^*$, $\Delta(\phi, \theta|\widehat{k}, \widehat{x}) = 0$ for $\theta = \theta^*$, and $\Delta(\phi, \theta|\widehat{k}, \widehat{x}) > 0$ for $\theta > \theta^*$. For $\phi = \{0,1\}$, $\theta^* = \underline{\theta}(\phi|\widehat{k}) = \overline{\theta}(\phi|\widehat{k})$ for any $\widehat{x}$. Furthermore, θ^* is non-decreasing in $\widehat{k}$ and $\widehat{x}$.*

Proof By Lemma 1 $\Delta(\phi, \theta|\widehat{k})$ is monotonically increasing in θ, hence there is a unique θ^* such that $\Delta(\phi, \theta^*|\widehat{k}, \widehat{x}) = 0$. By assumption 3, and since $\widehat{x} \in [0,1]$, $\theta^* \in [\underline{\theta}(\phi|\widehat{k}), \overline{\theta}(\phi|\widehat{k})]$. Also by Lemma 1, $\Delta(\phi, \theta|\widehat{k})$ is non-increasing in $\widehat{k}$ and $\widehat{x}$, then θ^* is non-decreasing in $\widehat{k}$ and $\widehat{x}$. If beliefs of risk-taking increase, the firm will weakly prefer to play risky at the previous θ^*, requiring θ increasing to recover the indifference.   Q.E.D.

Lemma 5 *(Belief single crossing) For every reputation level $\phi \in (0,1)$ and cutoff belief $\widehat{k}$, fix a $\theta \in [\underline{\theta}(\phi|\widehat{k}), \overline{\theta}(\phi|\widehat{k})]$. There exists a unique $\widehat{x}^* \in [0,1]$ such that $\Delta(\phi, \theta|\widehat{k}, \widehat{x}) > 0$ for $\widehat{x} < \widehat{x}^*$, $\Delta(\phi, \theta|\widehat{k}, \widehat{x}) = 0$ for $\widehat{x} = \widehat{x}^*$ and $\Delta(\phi, \theta|\widehat{k}, \widehat{x}) < 0$ for $\widehat{x} > \widehat{x}^*$. For $\phi = \{0,1\}$, any $\widehat{x} \in [0,1]$ delivers $\Delta(\phi, \theta|\widehat{k}, \widehat{x}) = 0$. Furthermore, $\widehat{x}^*$ is increasing in θ.*

Proof By Lemma 1 $\Delta(\phi, \theta|\widehat{k})$ is monotonically decreasing in $\widehat{x}$ when $\phi \in (0,1)$, hence there is a unique $\widehat{x}^* \in [0,1]$ such that $\Delta(\phi, \theta|\widehat{k}, \widehat{x}^*) = 0$. Also by Lemma 1, $\Delta(\phi, \theta|\widehat{k})$ is increasing in θ, and so is $\widehat{x}^*$. If fundamentals improve, the firm will strictly prefer to play safe at $\widehat{x}^*$, requiring an increase in the beliefs that the firm will play risky ($\widehat{x}$) to recover the indifference. Finally, for the special cases of $\phi = 0$ and $\phi = 1$, from equation (2) and assumption 3, $\underline{\theta}(\phi|\widehat{k}) = \overline{\theta}(\phi|\widehat{k})$ and by definition any $\widehat{x} \in [0,1]$ supports indifference.   Q.E.D.

The first part of the Proposition follows directly from Lemmas 4 and 5. A cutoff $k^*(\phi)$ is an equilibrium strategy only if it is a best response for any realization of the fundamental θ. Take a cutoff $k^*(\phi)$ such that $k^*(\phi) \in (\underline{\theta}(\phi|k^*), \overline{\theta}(\phi|k^*))$. Such a $k^*(\phi)$ is guaranteed by Lemmas 2 and 4. Furthermore we know that $x(\phi, \theta) = 0$ for all $\theta > k^*(\phi)$ and $x(\phi, \theta) = 1$ for all $\theta < k^*(\phi)$. From Lemma 5, at $\theta = k^*(\phi)$, indifference occurs at some $0 < x^*(\phi, k^*) < 1$. The cutoff $k^*(\phi)$ is an equilibrium because, for all $\theta > k^*(\phi)$, $\Delta(\phi, \theta|k^*) > 0$, and hence it is optimal for the firm to play safe (i.e., $x(\phi, \theta) = 0$). Similarly, for all $\theta < k^*(\phi)$, $\Delta(\phi, \theta|k^*) < 0$, and hence it is optimal for the firm to take risks (i.e., $x(\phi, \theta) = 1$).

Finally, since the conditions for equilibrium are both $\Delta(\phi, k^*|k^*, \widehat{x} = 0) > 0$ and $\Delta(\phi, k^*|k^*, \widehat{x} = 1) < 0$ and since $\Delta(\phi, \theta|\widehat{k})$ is monotonically not increasing in $\widehat{k}$ (from lemma 1), an arbitrarily close fundamental $k^* + \varepsilon$ (with $\varepsilon \to 0$) is also an equilibrium cutoff.

The bounds of the equilibrium cutoffs $[\underline{\theta}^*(\phi), \overline{\theta}^*(\phi)]$ are determined in the following way: $\underline{\theta}^*(\phi)$ is the indifference fundamental when beliefs are $\widehat{k} = \underline{\theta}^*$ and $\widehat{x} = 0$. This is the lowest fundamental that can be sustained in equilibrium fully considering reputation gains and the lowest possible interest rate $R(\phi|\underline{\theta}^*)$. Lemmas 2 and 4 guarantee this bound exists and is finite. Similarly, $\overline{\theta}^*(\phi)$ is determined by the fundamental that solves $\Delta(\phi, \theta|\overline{\theta}^*, \widehat{x} = 1) = 0$.

To show the sufficient condition under which these bounds are unique for each ϕ, fix a $\widehat{x}$ for all θ and fix a belief $\widehat{k}$ (i.e, a given $R(\phi|\widehat{k})$). Then k^* solves $p_s\Pi_s(k^*) - p_r\Pi_r(k^*) + \beta(p_s - p_r)\mathbf{V}(\phi'(\phi, \widehat{x})) = (p_s - p_r)R(\phi|\widehat{k})$ (from equalizing equation (4) to zero). Taking derivatives with respect to $\widehat{k}$, $\left[p_s\frac{\partial\Pi_s}{\partial k^*} - p_r\frac{\partial\Pi_r}{\partial k^*}\right]\frac{\partial k^*}{\partial\widehat{k}} = (p_s - p_r)\frac{\partial R(\phi)}{\partial\widehat{k}}$. There will be a unique best response to $\widehat{k}$ when $\frac{\partial k^*}{\partial\widehat{k}} < 1$. Considering $\frac{\partial R(\phi)}{\partial\widehat{k}} = \frac{\overline{R}(p_s - p_r)\phi v(\widehat{k})}{Pr(c|\phi, \widehat{k})^2}$, the condition for uniqueness is therefore,

$$p_s\frac{\partial\Pi_s}{\partial k^*} - p_r\frac{\partial\Pi_r}{\partial k^*} > \frac{\overline{R}(p_s - p_r)^2\phi v(\widehat{k})}{Pr(c|\phi, \widehat{k})^2}. \tag{9}$$

The worst situation to fulfill the condition happens when $\phi = 1$ and $Pr(c|\phi, \widehat{k}) = p_r$. Since this should be true for all fundamentals θ, the sufficient condition can be written as

$$v(\theta) < \frac{p_r^2\left[p_s\frac{\partial\Pi_s}{\partial\theta} - p_r\frac{\partial\Pi_r}{\partial\theta}\right]}{\overline{R}(p_s - p_r)^2}, \qquad \text{for all } \theta \in \mathbb{R}. \tag{10}$$

This condition basically requires a variance of fundamentals large enough such that interest rates do not jump suddenly with changes in beliefs. For example, if fundamentals are normally distributed, $\theta \sim N(\mu, \gamma^2)$, $v(\theta) \leq \frac{1}{\gamma\sqrt{2\pi}}$ for all θ and the sufficient condition can be expressed in terms of the standard deviation as $\gamma > \frac{\sqrt{2\pi}\overline{R}(p_s - p_r)^2}{p_r^2\left[p_s\frac{\partial\Pi_s}{\partial\theta} - p_r\frac{\partial\Pi_r}{\partial\theta}\right]}$ for all θ.

For $\phi = 1$, and from Lemmas 2 and 5, $k^*(1) = \underline{\theta}^*(1) = \overline{\theta}^*(1)$, since $\Delta(\phi, \theta|\widehat{k}, \widehat{x} = 0) = \Delta(\phi, \theta|\widehat{k}, \widehat{x} = 1)$ for all $\widehat{k}$ (i.e., $\widehat{x}$ does not play a role in generating multiplicity). Under the condition in equation (10), $k^*(1)$ is unique. Using the same logic, $k^*(0)$ is always a unique equilibrium for $\phi = 0$. To see this, just replace $\phi = 0$ in equation (9).

A.2   Proof of Proposition 2

Proof To prove this proposition, I proceed in four steps. First, I derive the posterior density and distribution of θ given a signal z. Second, I prove there is a unique signal $z^*(\phi)$ that makes a strategic firm ϕ indifferent in expectation between taking risk or not. Third, I show that, for any σ, using $z^*(\phi)$ is a best response when the prior about θ follows a uniform distribution on the real line and lenders believe $z^*(\phi)$ is the equilibrium cutoff. Finally, I show that, as $\sigma \to 0$, the best response in a game with any prior distribution of θ uniformly converges to following the unique cutoff $z^*(\phi)$ when lenders believe $z^*(\phi)$ is the equilibrium cutoff.

● **Step 1: Distributions of fundamentals conditional on signals**

Lemma 6 *The posterior density $f_{\theta|z}$ and distribution $F_{\theta|z}$ of θ given a signal z are given by*

$$f_{\theta|z}(\eta|z) = \frac{v(\eta)f(\frac{z-\eta}{\sigma})}{\int_{-\infty}^{\infty} v(\theta)f(\frac{z-\theta}{\sigma})d\theta}, \tag{11}$$

$$F_{\theta|z}(\eta|z) = \frac{\int_{-\infty}^{\eta} v(\theta)f(\frac{z-\theta}{\sigma})d\theta}{\int_{-\infty}^{\infty} v(\theta)f(\frac{z-\theta}{\sigma})d\theta} = \frac{\int_{\frac{z-\eta}{\sigma}}^{\infty} v(z-\sigma u)f(u)du}{\int_{-\infty}^{\infty} v(z-\sigma u)f(u)du}. \tag{12}$$

Proof By Bayes' rule,

$$f_{\theta|z}(\theta|z) = \frac{v(\theta)f_{z|\theta}(z|\theta)}{f_z(z)}, \tag{13}$$

where f_z and $f_{z|\theta}$ are the densities of z and $z|\theta$, respectively. Since z is the sum of θ and $\sigma\epsilon$, its density is given by the convolution of their densities, i.e., v and $f_{\sigma\epsilon}$. Considering that $F_{\sigma\epsilon}(\eta) = F(\eta/\sigma)$, $f_{\sigma\epsilon}(\eta) = \frac{f(\eta/\sigma)}{\sigma}$, then f_z can be defined as

$$f_z(z) = \sigma^{-1}\int_{-\infty}^{\infty} v(\theta)f\left(\frac{z-\theta}{\sigma}\right)d\theta. \tag{14}$$

We can obtain the distribution of the observed signal z after observing a fundamental θ,

$$F_{z|\theta}(\eta|\theta) = Pr(z \leq \eta|\theta) = F\left(\frac{\eta-\theta}{\sigma}\right),$$

$$f_{z|\theta}(\eta|\theta) = \frac{dF_{z|\theta}(\eta|\theta)}{dz} = \sigma^{-1}f\left(\frac{\eta-\theta}{\sigma}\right). \tag{15}$$

Plugging equations (15) and (14) in (13), we obtain equation (11). The posterior distribution is obtained by integrating over the density,

$$F_{\theta|z}(\eta|z) = \int_{-\infty}^{\eta} f_{\theta|z}(\theta|z)d\theta = \frac{\int_{-\infty}^{\eta} v(\theta)f(\frac{z-\theta}{\sigma})d\theta}{\int_{-\infty}^{\infty} v(\theta)f(\frac{z-\theta}{\sigma})d\theta},$$

and the expression in equation (12) follows from variable transformation $u = \frac{z-\theta}{\sigma}$.   Q.E.D.

• **Step 2: Unique equilibrium cutoff $z^*(\phi)$.**

Lemma 7 *There is a unique cutoff signal for each reputation ϕ such that $\Delta(\phi, z|z^*) = 0$ for $z = z^*$, $\Delta(\phi, z|z^*) > 0$ for $z > z^*$, and $\Delta(\phi, z|z^*) < 0$ for $z < z^*$, where $\Delta(\phi, z|z^*)$ are the expected differential gains from playing safe for a firm ϕ that observes z when lenders believe the cutoff is z^*.*

This cutoff z^* is obtained using Laplacian beliefs over the probability the firm plays risky when the fundamental is θ, where $\widehat{x} = F\left(\frac{z^*-\theta}{\sigma}\right)$, such that

$$\Delta(\phi, z^*|z^*) = \int_0^1 \Delta(\phi, \theta(\widehat{x}), \widehat{x}|z^*)d\widehat{x} = 0. \tag{16}$$

Proof When fundamentals θ are not observed directly, differential gains Δ turn into **expected** differential gains conditional on the signal. When the firm observes a signal z and lenders believe firms use a cutoff $\widehat{z}$, expected gains from playing safe are

$$\Delta(\phi, z|\widehat{z}) \equiv E[\Delta(\phi, \theta|\widehat{z})|z]. \tag{17}$$

Introducing noise in the observation of fundamentals allows us to pin down the believed probability of risk taking $\widehat{x}$ as a function of cutoff beliefs $\widehat{z}$ and observed fundamentals θ.

$$\widehat{x} = F\left(\frac{\widehat{z} - \theta}{\sigma}\right). \tag{18}$$

Developing the expectation from equation (17),

$$\Delta(\phi, z|\widehat{z}) = \int_{-\infty}^{\infty} \Delta(\phi, \theta, \widehat{x}(\theta)|\widehat{z}) dF_{\theta|z}(\theta|z).$$

Note that $\theta = \widehat{z} - \sigma F^{-1}(\widehat{x})$. From equation (12), define

$$\Psi(\widehat{x}|z, \widehat{z}) = F_{\theta|z}(\widehat{z} - \sigma F^{-1}(\widehat{x})|z, \widehat{z}) = \frac{\int_{\frac{z-\widehat{z}}{\sigma} + F^{-1}(\widehat{x})}^{\infty} v(z - \sigma u) f(u) du}{\int_{-\infty}^{\infty} v(z - \sigma u) f(u) du}.$$

Changing variables from θ to $\widehat{x}$,

$$\Delta(\phi, z|\widehat{z}) = \int_{0}^{1} \Delta(\phi, \theta(\widehat{x}), \widehat{x}|\widehat{z}) d\Psi(\widehat{x}|z, \widehat{z}).$$

Laplacian beliefs arise from

$$\Psi(\widehat{x}|z, \widehat{z}) = Pr(\theta < \widehat{z} - \sigma F^{-1}(\widehat{x})|z) = F[\frac{z - \widehat{z}}{\sigma} + F^{-1}(\widehat{x})].$$

In equilibrium $\widehat{z} = z^*$. Evaluating the expectation at $z = z^*$, $\Psi(\widehat{x}|z^*, z^*) = \widehat{x}$.

$$\Delta(\phi, z^*|z^*) = \int_{0}^{1} \Delta(\phi, \theta(\widehat{x}), \widehat{x}|z^*) d\widehat{x} = 0.$$

By Lemmas 4 and 5, we know there is a unique solution $z^*(\phi)$ to this equation.   Q.E.D.

● **Step 3: Best response with uniform priors over fundamentals**

Now we need to verify that a firm ϕ playing risky if $z < z^*(\phi)$ and safe if $z > z^*(\phi)$ indeed constitutes an equilibrium. Signals z allow firms to have an idea not only about the fundamental but also about the signal that lenders believe the firm has observed. Following Toxvaerd (2007), I first assume θ is drawn from a uniform distribution on the real line, hence an improper distribution with infinite probability mass. This assumption allows us to normalize the prior distribution assuming $v(\theta) = 1$, simplifying the density to $f_{\theta|z}(\theta|z) = \sigma^{-1} f(\frac{z-\theta}{\sigma})$ and the distribution to $F_{\theta|z}(\theta|z) = F(\frac{z-\theta}{\sigma})$. We will denote $\widetilde{\Delta}(\phi, z|\widehat{z})$ the expected differential

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gains from safe actions for the special case in which the prior of fundamentals is uniform,

$$\widetilde{\Delta}(\phi, z|\widehat{z}) = \int_{-\infty}^{\infty} \Delta\left(\phi, \theta, F\left(\frac{\widehat{z} - \theta}{\sigma}\right)|\widehat{z}\right) \sigma^{-1} f\left(\frac{z - \theta}{\sigma}\right) d\theta.$$

Changing variables introducing $m = \frac{\theta - \widehat{z}}{\sigma}$,

$$\widetilde{\Delta}(\phi, z|\widehat{z}) = \int_{-\infty}^{\infty} \Delta\left(\phi, \theta, F(-m)|\widehat{z}\right) \sigma^{-1} f\left(\frac{z - \widehat{z}}{\sigma} - m\right) d\theta.$$

We can rewrite it more conveniently, defining $\widehat{\Delta}$

$$\widetilde{\Delta}(\phi, z|\widehat{z}) = \widehat{\Delta}(\phi, z, z'|\widehat{z}) = \int_{-\infty}^{\infty} B(z', m|\widehat{z}) D(z, m|\widehat{z}) dm,$$

where $B(z', m|\widehat{z}) = \Delta\left(\phi, z', F(-m)|\widehat{z}\right)$ (renaming θ as z') and $D(z, m|\widehat{z}) = \sigma^{-1} f\left(\frac{z - \widehat{z}}{\sigma} - m\right)$. As shown in Athey (2002), because of the monotone likelihood property, $\widehat{\Delta}(\phi, z, z'|\widehat{z})$ inherits the single crossing property of $\Delta(\phi, \theta|\widehat{z})$. This means there exists a $z^*(\phi, \widehat{z}, z')$ such that $\widehat{\Delta}(\phi, z, z'|\widehat{z}) > 0$ if $z > z^*(\phi, \widehat{z}, z')$ and $\widehat{\Delta}(\phi, z, z'|\widehat{z}) < 0$ if $z < z^*(\phi, \widehat{z}, z')$. Assuming $z < z'$ and $\widehat{\Delta}(\phi, z, z|\widehat{z}) = 0$,

$$\widehat{\Delta}(\phi, z', z'|\widehat{z}) \geq \widehat{\Delta}(\phi, z, z'|\widehat{z}) \geq \widehat{\Delta}(\phi, z, z|\widehat{z}) = 0, \qquad (strict > for \ \phi \in (0, 1)).$$

The first inequality comes from the state monotonicity and the second from the single crossing property. A symmetric argument holds for $z > z'$. Hence, there exists a best response $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \widetilde{\Delta}(\phi, z|\widehat{z}) > 0 \quad &if \quad z > \chi(\widehat{z}) \\ \widetilde{\Delta}(\phi, z|\widehat{z}) = 0 \quad &if \quad z = \chi(\widehat{z}) \\ \widetilde{\Delta}(\phi, z|\widehat{z}) < 0 \quad &if \quad z < \chi(\widehat{z}) \end{aligned}$$

There exists a unique z^* that solves

$$\widetilde{\Delta}(\phi, z^*|z^*) = \int_0^1 \widetilde{\Delta}(\phi, \theta(\widehat{x}), \widehat{x}|z^*) d\widehat{x} = 0. \tag{19}$$

Hence, $\chi(z^*(\phi)) = z^*(\phi)$, showing that there is a unique equilibrium in cutoff strategies for each ϕ such that

$$x^*(\phi, z) = \begin{cases} 0 & if \quad z > z^*(\phi) \\ 1 & if \quad z < z^*(\phi) \end{cases}. \tag{20}$$

Q.E.D.

• **Step 4: Best response with general priors over fundamentals**

Lemma 8 $\Delta(\phi, z|\widehat{z}) \to \widetilde{\Delta}(\phi, z|\widehat{z})$ *uniformly, when* $\widehat{z} = z - \sigma\xi$, *as* $\sigma \to 0$.

Proof First, $\Delta(\phi, z|z - \sigma\xi) \to \widetilde{\Delta}(\phi, z|z - \sigma\xi)$ continuously as $\sigma \to 0$, this is,

$$\Psi(\widehat{x}|z, z - \sigma\xi) = \frac{\int_{\xi+F^{-1}(\widehat{x})}^{\infty} v(z - \sigma u) f(u) du}{\int_{-\infty}^{\infty} v(z - \sigma u) f(u) du} \to 1 - F\left(\xi + F^{-1}(\widehat{x})\right) \equiv \widetilde{\Psi}(\widehat{x}|z, z - \sigma\xi).$$

As in Toxvaerd (2007), we show convergence with respect to the uniform convergence norm, which implies uniform convergence. Uniformity ensures that the equivalence between the games with the two different assumptions about the prior distributions is not the result of a discontinuity at $\sigma = 0$.

Pick $\underline{z}(\phi) < \underline{\theta}^*(\phi)$ and $\overline{z}(\phi) > \overline{\theta}^*(\phi)$ and restrict attention to the compact sets $Z \equiv [\underline{z}(\phi), \overline{z}(\phi)]$ and $Z_\sigma \equiv [\underline{z}(\phi) - \sigma\xi, \overline{z}(\phi) + \sigma\xi]$. Hence, $\Delta(\phi, z|\widehat{z})$ maps into a compact set.

Define the uniform convergence norm as

$$\| \Delta(\phi) \| \equiv sup_{z, \widehat{z}}\{|\Delta(\phi, z|\widehat{z})|\}.$$

We can show continuity with respect to the Euclidean metric. Fix z' and $\widehat{z}'$ such that

$$\forall \epsilon_1 > 0, \exists \delta_1 \, |z - z'| < \delta_1 \Rightarrow |\Delta(\phi, z|\widehat{z}) - \widetilde{\Delta}(\phi, z'|\widehat{z})| < \epsilon_1, \forall \widehat{z}$$
$$\forall \epsilon_2 > 0, \exists \delta_2 \, |\widehat{z} - \widehat{z}'| < \delta_2 \Rightarrow |\Delta(\phi, z|\widehat{z}) - \widetilde{\Delta}(\phi, z|\widehat{z}')| < \epsilon_2, \forall z$$

This implies

$$\sqrt{(z - z')^2 + (\widehat{z} - \widehat{z}')^2} < \sqrt{\delta_1^2 + \delta_2^2}.$$

By the triangle inequality,

$$\begin{aligned} |\Delta(\phi, z|\widehat{z}) - \Delta(\phi, z'|\widehat{z}')| &= |\Delta(\phi, z|\widehat{z}) - \Delta(\phi, z'|\widehat{z}) + \Delta(\phi, z'|\widehat{z}) - \Delta(\phi, z'|\widehat{z}')| \\ &\leq |\Delta(\phi, z|\widehat{z}) - \Delta(\phi, z'|\widehat{z}')| + |\Delta(\phi, z'|\widehat{z}) - \Delta(\phi, z'|\widehat{z}')| \\ &\leq \epsilon_1 + \epsilon_2. \end{aligned}$$

Hence, $\Delta(\phi, z|\widehat{z})$ belongs to the space of continuous functions on $Z \times \widehat{Z}$.

Uniform convergence is equivalent to

$$\| \Delta(\phi) - \widetilde{\Delta}(\phi) \| = sup_{z, \widehat{z}}\{\Delta(\phi, z|\widehat{z}) - \widetilde{\Delta}(\phi, z|\widehat{z})\} \to 0$$

with respect to the uniform convergence norm, as $\sigma \to 0$, after substituting for the functions and taking limits. Q.E.D.

A.3 Conditions for Proposition 4

In this section we discuss the conditions for $\mathbf{V}_t(\phi) \to \overline{\mathbf{V}}(\phi)$ as $T \to \infty$ (i.e., by backward induction, continuation values converge to a fixed point for all ϕ and periods t far enough

from T). These fixed points are the bounded limits required to show that there is an infinite horizon equilibrium that is a unique limit of the finite horizon Markov perfect equilibrium.[20]

In short, the condition for convergence is that the variance of fundamentals is large enough. I will prove this by steps. First, I discuss the case without reputation formation as a benchmark in which reputation levels do not interact and obtain sufficient conditions for convergence. Then, I introduce reputation formation and show that those conditions are also sufficient.

Step 1: No reputation formation:

This is an artificial and expositional convenient case in which a firm is born with a given reputation and cannot change it (because age cannot be observed, for example). First assume safe actions deliver higher expected continuation values. That is, if commitment were feasible, firms would choose to take safe actions rather than risky actions, regardless of their reputation. This assumption makes sense in our context, since the focus is on the case in which safe actions are almost always the efficient behavior.

From Proposition 3 and without reputation formation (i.e., $\mathbf{V}_{t+1}(\phi') = \mathbf{V}_{t+1}(\phi)$),

$$
\mathbf{V}_t(\phi) = \mathcal{V}(z^*(\phi))[\beta p_r \mathbf{V}_{t+1}(\phi) - p_r R_t(\phi)] + \int_{-\infty}^{z_t^*(\phi)} p_r \Pi_r(\theta) v(\theta) d\theta
$$

$$
+ (1 - \mathcal{V}(z^*(\phi)))[\beta p_s \mathbf{V}_{t+1}(\phi) - p_s R_t(\phi)] + \int_{z_t^*(\phi)}^{\infty} p_s \Pi_s(\theta) v(\theta) d\theta.
$$

Applying the envelope theorem,

$$
\frac{\partial \mathbf{V}_t(\phi)}{\partial \mathbf{V}_{t+1}(\phi)} = \mathcal{V}(z^*(\phi)) \beta p_r + (1 - \mathcal{V}(z^*(\phi))) \beta p_s - \frac{\partial R(\phi|z^*)}{\partial z^*} \frac{\partial z^*}{\partial \mathbf{V}_{t+1}(\phi)} \left[p_s - \mathcal{V}(z^*(\phi))(p_s - p_r) \right].
$$

The cutoff z^* is determined by $p_s \Pi_s(z^*) - p_r \Pi_r(z^*) - (p_s - p_r) R(\phi|z^*) = -\beta(p_s - p_r)\mathbf{V}_{t+1}(\phi)$, since there is no reputation formation. Taking derivatives with respect to $\mathbf{V}_{t+1}(\phi)$,

$$
\frac{\partial z^*}{\partial \mathbf{V}_{t+1}(\phi)} = -\frac{\beta(p_s - p_r)}{\left[p_s \frac{\partial \Pi_s}{\partial z^*} - p_r \frac{\partial \Pi_r}{\partial z^*} \right]} < 0.
$$

Also

$$
\frac{\partial R(\phi|z^*)}{\partial z^*} = \frac{\phi \overline{R}(p_s - p_r)}{Pr(c)^2} v(z^*) > 0.
$$

Recall $\frac{\partial \mathbf{V}_t(\phi)}{\partial \mathbf{V}_{t+1}(\phi)} > 0$ and $\mathbf{V}_t(\phi) > 0$ when $\mathbf{V}_{t+1}(\phi) = 0$. Hence, convergence to a fixed point $\overline{\mathbf{V}}(\phi)$ happens if $\frac{\partial \mathbf{V}_t(\phi)}{\partial \mathbf{V}_{t+1}(\phi)} < 1$. It is clear this is the case for $\phi = 0$ (since $\frac{\partial R(\phi|z^*)}{\partial z^*} = 0$). At the other extreme, when $\phi = 1$, imposing the worst combination of parameters to fulfill the requirement ($\mathcal{V}(z^*(\phi)) = 0$ and $Pr(c) = p_r$) and considering all fundamentals θ, the sufficient condition for convergence is

[20]I have not yet examined the broader issue of what other equilibria there might be in the infinite horizon game.

$$v(\theta) < \frac{1 - \beta p_s}{\beta p_s} \frac{p_r^2 \left[p_s \frac{\partial \Pi_s}{\partial \theta} - p_r \frac{\partial \Pi_r}{\partial \theta} \right]}{\overline{R}(p_s - p_r)^2}, \qquad \qquad \textit{for all } \theta \in \mathbb{R}. \qquad (21)$$

In words, the variance of fundamentals should be large enough (or the density low enough) to have convergence in continuation values for all reputation levels when reputation cannot be modified. First, recall that this is a really stringent sufficient condition, since the worst combination of parameters are not jointly consistent. For example, if $\phi = 1$ and $\mathcal{V}(z^*(\phi)) = 0$, $Pr(c)$ is not p_r but p_s, hence convergence conditions are effectively more relaxed. Second, note the sufficient condition for convergence is more stringent than the sufficient condition for uniqueness when $\beta p_s > 0.5$.

Step 2: Reputation formation:

Assume the sufficient condition expressed in equation (21) is met. Then, there is a unique $\overline{\mathbf{V}}(\phi)$ for all ϕ such that considering reputation formation

$$\overline{\mathbf{V}}(\phi) = \frac{\beta p_s - \mathcal{V}(z^*(\phi))\beta p_s}{1 - \mathcal{V}(z^*(\phi))\beta p_r} \overline{\mathbf{V}}(\phi') - \frac{1 - \mathcal{V}(z^*(\phi))p_s + \mathcal{V}(z^*(\phi))p_r}{1 - \mathcal{V}(z^*(\phi))\beta p_s} R(\phi)$$

$$+ \frac{1}{1 - \mathcal{V}(z^*(\phi))\beta p_r} \left[\int_{-\infty}^{z_t^*(\phi)} p_r \Pi_r(\theta)v(\theta)d\theta + \int_{z_t^*(\phi)}^{\infty} p_s \Pi_s(\theta)v(\theta)d\theta \right].$$

Taking derivatives to consider a greater continuation value for playing safe obtained from a higher reputation

$$\frac{\partial \overline{\mathbf{V}}(\phi)}{\partial \overline{\mathbf{V}}(\phi')} = \frac{\beta p_s - \mathcal{V}(z^*(\phi))\beta p_s}{1 - \mathcal{V}(z^*(\phi))\beta p_r} - \frac{\partial R(\phi|z^*)}{\partial z^*} \frac{\partial z^*}{\partial \overline{\mathbf{V}}(\phi')} \left[\mathcal{V}(z^*(\phi))p_r + (1 - \mathcal{V}(z^*(\phi)))p_s \right].$$

It is straightforward to see $\frac{\partial \overline{\mathbf{V}}(\phi)}{\partial \overline{\mathbf{V}}(\phi')} > 0$. It is also possible to check monotonicity of continuation values, since $\frac{\partial \overline{\mathbf{V}}(\phi)}{\partial \overline{\mathbf{V}}(\phi')} < 1$ when the sufficient condition expressed in equation (21) is fulfilled. With and without reputation formation extreme continuation values, $\overline{\mathbf{V}}(0)$ and $\overline{\mathbf{V}}(1)$), are the same. Since reputation generates a convex combination between unique values in a compact set, the resulting continuation values $\overline{\mathbf{V}}(\phi)$ with reputation formation are also unique.

A.4   Proof of Proposition 5

Proof As a first step, assume convergence has been achieved (Proposition 4). Then $z^*(\phi)$ is determined by equation (6) in the following way:

$$\int_0^1 \Delta(\phi, z^*|z^*, \widehat{x})d\widehat{x} = p_s \Pi_s(z^*) - p_r \Pi_r(z^*) + (p_s - p_r) \left[\beta \int_0^1 \mathbf{V}(\phi'|\phi, \widehat{x})d\widehat{x} - R(\phi, z^*) \right] = 0.$$

Taking derivatives with respect to ϕ,

$$\frac{dz^*(\phi)}{d\phi} = -\frac{\int_0^1 \frac{\partial \Delta(\phi, z^*|\widehat{x})}{\partial \phi} d\widehat{x}}{\frac{\partial \Delta(\phi, z^*)}{\partial z^*}}, \tag{22}$$

where

$$\frac{\partial \Delta(\phi, z^*|z^*, \widehat{x})}{\partial \phi} = (p_s - p_r) \left[\beta \frac{\partial \mathbf{V}(\phi')}{\partial \phi'} \frac{\partial \phi'}{\partial \phi}\Big|_{\widehat{x}} - \frac{\partial R}{\partial \phi} \right],$$

and

$$\frac{\partial \Delta(\phi, z^*|z^*, \widehat{x})}{\partial z^*} = p_s \frac{\partial \Pi_s}{\partial z^*} - p_r \frac{\partial \Pi_r}{\partial z^*} - (p_s - p_r) \frac{\partial R}{\partial z^*}, \qquad \text{for all } \widehat{x}.$$

From equation (5),

$$\frac{dR(\phi)}{d\phi} = \frac{\partial R}{\partial \phi} + \frac{\partial R}{\partial z^*} \frac{dz^*}{d\phi}, \tag{23}$$

where $\frac{\partial R}{\partial \phi} = -\frac{\overline{R}(p_s - p_r)(1 - \mathcal{V}(z^*))}{Pr(c)^2} < 0$ and $\frac{\partial R}{\partial z^*} = \frac{\overline{R}(p_s - p_r)\phi v(z^*)}{Pr(c)^2} > 0$ for all ϕ.

Finally, from equation (7), using the envelope theorem

$$\frac{d\overline{\mathbf{V}}(\phi)}{d\phi} = \frac{\partial \overline{\mathbf{V}}}{\partial \phi} + \frac{\partial \overline{\mathbf{V}}}{\partial z^*} \frac{dz^*}{d\phi}, \tag{24}$$

where

$$\frac{\partial \overline{\mathbf{V}}}{\partial \phi} = \beta \left[\mathcal{V}(z^*) p_r \frac{\partial \overline{\mathbf{V}}(\phi)}{\partial \phi} + (1 - \mathcal{V}(z^*)) p_s \frac{\partial \overline{\mathbf{V}}(\phi')}{\partial \phi'} \frac{\partial \phi'}{\partial \phi}\Big|_{\widehat{x}=0} \right] - (p_s - \mathcal{V}(z^*)(p_s - p_r)) \frac{\partial R(\phi)}{\partial \phi},$$

and

$$\frac{\partial \overline{\mathbf{V}}}{\partial z^*} = \beta v(z^*) \left[p_s \left(\int_0^1 \overline{\mathbf{V}}(\phi'|\widehat{x}) d\widehat{x} - \overline{\mathbf{V}}(\phi'|\widehat{x}=0) d\widehat{x} \right) - p_r \left(\int_0^1 \overline{\mathbf{V}}(\phi'|\widehat{x}) d\widehat{x} - \overline{\mathbf{V}}(\phi) d\widehat{x} \right) \right].$$

To determine the sign of each derivative, I solve backward from the last period T. To do this, we simply introduce period subscripts in all equations, particularly replacing $\overline{\mathbf{V}}(\phi)$ by $\mathbf{V}_t(\phi)$ and $\mathbf{V}_{t+1}(\phi)$ where needed.

At period T, $\frac{\partial \Delta_T(\phi, z_T^*|\widehat{x})}{\partial \phi} > 0$ for all $\widehat{x}$ (since $\mathbf{V}_{T+1} = 0$ for all ϕ) and $\frac{\partial \Delta_T(\phi, z_T^*)}{\partial z_T^*} > 0$ (from condition in Proposition 1). Hence, $\frac{dz_T^*}{d\phi} < 0$. From equation (23), $\frac{dR_T(\phi)}{d\phi} < 0$. Finally, from equation (24) (since $\mathbf{V}_{T+1} = 0$ for all ϕ), $\frac{d\mathbf{V}_T(\phi)}{d\phi} > 0$.

At period $T - 1$ we additionally have the effects coming from $\mathbf{V}_T$. From Bayesian learning $\frac{\partial \phi'}{\partial \phi}\Big|_{\widehat{x}} = \frac{p_r[\widehat{x}p_r + (1-\widehat{x})p_s]}{p_r + (p_s - p_r)(1-\widehat{x})\phi} > 0$ for all $\widehat{x}$ and all ϕ. From results at T $\int_0^1 \frac{\partial \mathbf{V}_T(\phi')}{\partial \phi'} \frac{\phi'}{\phi}\Big|_{\widehat{x}} d\widehat{x} > 0$. Hence, $\int_0^1 \frac{\partial \Delta_{T-1}(\phi, z_{T-1}^*|\widehat{x})}{\partial \phi} d\widehat{x} > 0$ and $\frac{dz_{T-1}^*}{d\phi} < 0$. From equation (23), $\frac{dR_{T-1}(\phi)}{d\phi} < 0$. Finally, it follows that $\frac{d\mathbf{V}_{T-1}(\phi)}{d\phi} > 0$ from equation (24) and from the fact that $\frac{\partial \mathbf{V}_T(\phi)}{\partial \phi} > 0$ for all ϕ and that $\int_0^1 \mathbf{V}_T(\phi'|\widehat{x}) d\widehat{x}$ can be written as a convex combination between $\mathbf{V}_T(\phi)$ and $\mathbf{V}_T(\phi'|\widehat{x}=0)$.

Following the same steps and solving backward until convergence, $\frac{d\mathcal{V}(z^*(\phi))}{d\phi} < 0$ ($\frac{dz^*(\phi)}{d\phi} < 0$),

$\frac{dR(\phi)}{d\phi} < 0$ and $\frac{d\overline{\mathbf{V}}(\phi)}{d\phi} > 0$ for all $\phi \in [0, 1]$.

<div align="right">Q.E.D.</div>

A.5 Proof of Lemma 3

Proof Differentiating equation (22) with respect to ϕ, we get

$$\frac{d^2 z^*}{d\phi^2} = -\frac{1}{\frac{\partial \Delta}{\partial z^*}} \left[\frac{\partial^2 \Delta}{\partial \phi^2} + 2 \frac{\partial^2 \Delta}{\partial \phi \partial z^*} \frac{dz^*}{d\phi} + \frac{\partial^2 \Delta}{\partial z^{*2}} \left(\frac{dz^*}{d\phi} \right)^2 \right]. \tag{25}$$

In what follows, I assume a linear relation between payoffs and fundamentals, so the shape of the cutoffs is not just an artifice of the shape of the payoffs. Under this assumption, we have

$$\frac{\partial^2 \Delta}{\partial \phi^2} = (p_s - p_r) \left[\beta \int_0^1 \left(\frac{\partial \overline{\mathbf{V}}}{\partial \phi'} \frac{\partial^2 \phi'}{\partial \phi^2} |_{\widehat{x}} + \frac{\partial^2 \overline{\mathbf{V}}}{\partial \phi'^2} \frac{\partial \phi'^2}{\partial \phi} |_{\widehat{x}} \right) d\widehat{x} - \frac{\partial^2 R}{\partial \phi^2} \right], \tag{26}$$

$$\frac{\partial^2 \Delta}{\partial \phi \partial z^*} = -(p_s - p_r) \frac{\partial^2 R}{\partial \phi, \partial z^*}$$

and

$$\frac{\partial^2 \Delta}{\partial z^{*2}} = -(p_s - p_r) \frac{\partial^2 R}{\partial z^{*2}}.$$

From equation (24)

$$\frac{\partial^2 \overline{\mathbf{V}}}{\partial \phi^2} |_{\widehat{x}} = \mathcal{V}(z^*) p_r \left[\beta \frac{\partial^2 \overline{\mathbf{V}}(\phi)}{\partial \phi^2} - \frac{\partial^2 R(\phi)}{\partial \phi^2} \right] + (1 - \mathcal{V}(z^*)) p_s \left[\beta \left(\frac{\partial \overline{\mathbf{V}}}{\partial \phi'} \frac{\partial^2 \phi'}{\partial \phi^2} |_{\widehat{x}} + \frac{\partial^2 \overline{\mathbf{V}}}{\partial \phi'^2} \frac{\partial \phi'^2}{\partial \phi} |_{\widehat{x}} \right) - \frac{\partial^2 R(\phi)}{\partial \phi^2} \right], \tag{27}$$

where

$$\frac{\partial^2 R}{\partial \phi^2} = \frac{2 \overline{R} (p_s - p_r)^2 (1 - \mathcal{V}(z^*))^2}{Pr(c)^3} > 0.$$

I will proceed in three steps. First, as a benchmark, I solve backward from T when reputation cannot be updated. Then, I show how reputation formation convexifies the schedule of cutoffs.

Step 1: No reputation formation: This is an artificial and expositionally convenient case in which a firm is born with a given reputation and cannot change it (because age cannot be observed, for example). I call the cutoffs in this case $\widetilde{z}^*$. In this case, beliefs $\widehat{x}$ do not play any role, $\frac{\partial \phi'}{\partial \phi} = 1$ and $\frac{\partial^2 \phi'}{\partial \phi^2} = 0$ for all ϕ. Hence, equations (26) and (27) can be rewritten as

$$\frac{\partial^2 \Delta_t}{\partial \phi^2} = (p_s - p_r) \left[\beta \frac{\partial^2 \mathbf{V}_{t+1}}{\partial \phi^2} - \frac{\partial^2 R_t}{\partial \phi^2} \right],$$

and

$$\frac{\partial^2 \mathbf{V}_t}{\partial \phi^2} = (\mathcal{V}(\widetilde{z}_t^*) p_r + (1 - \mathcal{V}(\widetilde{z}_t^*)) p_s) \left[\beta \frac{\partial^2 \mathbf{V}_{t+1}(\phi)}{\partial \phi^2} - \frac{\partial^2 R_t(\phi)}{\partial \phi^2} \right].$$

At period T, since $\mathbf{V}_{T+1} = 0$ for all ϕ, $\frac{\partial^2 \Delta_T}{\partial \phi^2} < 0$ and $\frac{\partial^2 \mathbf{V}_T}{\partial \phi^2} < 0$. However, these signs do not guarantee that equation (25) is positive. The sufficient condition for $\frac{d^2 \widetilde{z}_T^*}{d\phi^2} > 0$ is the following $\left| \frac{d\widetilde{z}_T^*}{d\phi} \right| > \frac{v(\widetilde{z}_T^*)}{\phi} \frac{[p_r - (p_s - p_r)(1 - \mathcal{V}(\widetilde{z}_T^*))\phi]}{[Pr(c)v'(\widetilde{z}_T^*) + 2(p_s - p_r)v^2(\widetilde{z}_T^*)]}$ or, more generally, the variance of fundamentals is large enough.[21] The condition is more difficult to be fulfilled for low values of ϕ.

At period, $T - 1$, $\frac{\partial^2 \Delta_{T-1}}{\partial \phi^2} < \frac{\partial^2 \Delta_T}{\partial \phi^2} < 0$ and $\frac{\partial^2 \mathbf{V}_{T-1}}{\partial \phi^2} < \frac{\partial^2 \mathbf{V}_T}{\partial \phi^2} < 0$. This means $\frac{d^2 \widetilde{z}_{T-1}^*}{d\phi^2} > 0$ for a higher range of ϕ values. The same analysis hold until convergence. In this case, $\frac{\partial^2 \overline{\mathbf{V}}}{\partial \phi^2} = -\frac{\psi(\widetilde{z}^*)}{1 - \beta \psi(\widetilde{z}^*)} \frac{\partial^2 R}{\partial \phi^2}$ and $\frac{\partial^2 \Delta}{\partial \phi^2} = -\frac{(p_s - p_r)}{1 - \beta \psi(\widetilde{z}^*)} \frac{\partial^2 R}{\partial \phi^2}$, with $\psi(\widetilde{z}^*) = \mathcal{V}(\widetilde{z}^*)p_r + (1 - \mathcal{V}(\widetilde{z}^*))p_s$. Still it is not clear that without reputation concerns $\frac{d^2 \widetilde{z}^*}{d\phi^2} > 0$ for all ϕ, being more difficult at lower reputation levels.

Step 2: Reputation formation: Consider now the full model with reputation formation. This leads to convexity, since it relates continuation values of different reputation levels. We consider again equations (26) and (27).

At period T, as in step 1, $\frac{\partial^2 \Delta_T}{\partial \phi^2} < 0$, $\frac{\partial^2 \mathbf{V}_T}{\partial \phi^2} < 0$, and $\frac{d^2 z_T^*}{d\phi^2} = \frac{d^2 \widetilde{z}_T^*}{d\phi^2}$.

At period $T - 1$, since $\frac{\partial \phi'}{\partial \phi} = \frac{p_r(p_s(1 - \widehat{x}) + p_r \widehat{x})}{[p_r + (p_s - p_r)(1 - \widehat{x})\phi]^2} > 0$ and $\frac{\partial^2 \phi'}{\partial \phi^2} = -\frac{2p_r(p_s(1 - \widehat{x}) + p_r \widehat{x})(p_s - p_r)(1 - \widehat{x})}{[p_r + (p_s - p_r)(1 - \widehat{x})\phi]^3} < 0$ for all $\widehat{x} \in [0, 1]$), $\frac{\partial^2 \Delta_{T-1}}{\partial \phi^2} < \frac{\partial^2 \Delta_T}{\partial \phi^2} < 0$ and $\frac{\partial^2 \mathbf{V}_{T-1}}{\partial \phi^2} < \frac{\partial^2 \mathbf{V}_T}{\partial \phi^2} < 0$, exactly as in step 1. Furthermore, $\int_0^1 \frac{\partial \mathbf{V}_T}{\partial \phi'} \frac{\partial^2 \phi'}{\partial \phi^2} |_{\widehat{x}} d\widehat{x} < 0$ and $\int_0^1 \frac{\partial^2 \mathbf{V}_T}{\partial \phi'^2} \frac{\partial \phi'}{\partial \phi}^2 |_{\widehat{x}} d\widehat{x} < \frac{\partial^2 \mathbf{V}_T}{\partial \phi^2}$, which means $\frac{\partial^2 \Delta_{T-1}}{\partial \phi^2}$ and $\frac{\partial^2 \mathbf{V}_{T-1}}{\partial \phi^2}$ are lower than their counterparts without reputation concerns, derived in step 1. This implies that $\frac{d^2 z_{T-1}^*}{d\phi^2} > \frac{d^2 \widetilde{z}_{T-1}^*}{d\phi^2}$ for all ϕ.

Solving backward until convergence, reputation formation introduces pressure for concavity of continuation values and hence the convexity of the schedule of cutoffs and interest rates at all reputation levels, leading to $\frac{d^2 z^*}{d\phi} > \frac{d^2 \widetilde{z}^*}{d\phi}$ for all ϕ.

Even more importantly, as reputation formation becomes easier (i.e., signals are more precise), for $\frac{p_r}{p_s} \to 0$, $\frac{\partial \phi'}{\partial \phi}|_{\phi=0} \to \infty$ and $\frac{\partial^2 \phi'}{\partial \phi^2}|_{\phi=0} \to \infty$, hence $\frac{d^2 z_{T-1}^*}{d\phi^2} > 0$ for all ϕ (since it always convexifies the schedule of cutoffs for low reputation levels, which are the levels of reputation where convexity was more difficult to obtain without reputation formation). Hence, for any reputation ϕ, there is always a $\overline{\frac{p_r}{p_s}}(\phi) \in (0, 1]$ such that $\frac{d^2 z_{T-1}^*}{d\phi^2} = 0$. Furthermore, from the condition in step 1, $\overline{\frac{p_r}{p_s}}(\phi)$ is weakly increasing in ϕ. $\hfill$ Q.E.D.

A.6 Computational Procedure

We solve the model following the next procedure.

- Set a large grid of $\phi \in [0, 1]$.

- Solve full information (FI) environment (efficiency).

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[21]This condition requires some algebra that is available upon request.

- Guess a $\mathbf{V}_{FI,0} = 0$.
- Obtain $\theta_{FI,0}^*$ from $\Delta(\theta)_{FI} = p_s\Pi_s(\theta) - p_r\Pi_r(\theta) + \beta(p_s - p_r)\mathbf{V}_{FI,0} = 0$.
- Obtain $\mathbf{V}_{FI,1} = \frac{\int_{-\infty}^{\theta_{FI,0}^*}[p_r\Pi_r(\theta) - \overline{R}]v(\theta)d\theta + \int_{\theta_{FI,0}^*}^{\infty}[p_s\Pi_s(\theta) - \overline{R}]v(\theta)d\theta}{1 - \beta(p_s - \mathcal{V}(\theta_{FI,0}^*)(p_s - p_r))}$.
- Use $\mathbf{V}_{FI,1}$ as the new guess and iterate until $\mathbf{V}_{FI,i} - \mathbf{V}_{FI,i-1} < \varepsilon$.

- Solve the environment without (wo) reputation formation.

  - Guess $\mathbf{V}(\phi)_{wo,0} = 0$ and $\theta^*(\phi)_0 = \theta_{FI}^*$ for all ϕ.
  - Obtain $\theta^*(\phi)_1$ from $\Delta(\phi, \theta^*(\phi)_1) = 0$, where

$$\Delta(\phi, \theta) = p_s\Pi_s(\theta) - p_r\Pi_r(\theta) + (p_s - p_r)[\beta\mathbf{V}(\phi)_{wo,0} - R(\phi|\theta^*(\phi)_0)].$$

  - For each ϕ, obtain

$$\mathbf{V}(\phi)_{wo,1} = \frac{\int_{-\infty}^{\theta^*(\phi)_1} p_r[\Pi_r(\theta) - R(\phi|\theta^*(\phi)_1)]v(\theta)d\theta + \int_{\theta^*(\phi)_1}^{\infty} p_s[\Pi_s(\theta) - R(\phi|\theta^*(\phi)_1)]v(\theta)d\theta}{[1 - \beta(p_r + \mathcal{V}(\theta^*(\phi)_1)(p_s - p_r))]}.$$

  - Use $\mathbf{V}(\phi)_{wo,1}$ and $\theta^*(\phi)_1$ as new guesses and iterate until $\mathbf{V}(\phi)_{wo,i} - \mathbf{V}(\phi)_{wo,i-1} < \varepsilon_1$ and $\theta^*(\phi)_i - \theta^*(\phi)_{i-1} < \varepsilon_2$ for all ϕ.

- Solve the environment with reputation formation.

  - Guess a $\mathbf{V}(\phi)_0 = 0$ and $z^*(\phi)_0 = \theta^*(\phi)$ for all ϕ.
  - Using $\mathbf{V}(\phi)_0$, for each belief $\widehat{x} \in [0, 1]$ from a large grid of size N_x, obtain

$$\Delta(\phi, z, \widehat{x}|z^*(\phi)_0)_0 = E_z[p_s\Pi_s(\theta) - p_r\Pi_r(\theta)] + (p_s - p_r)[\beta\mathbf{V}(\phi'|\widehat{x})_0 - R(\phi|z^*(\phi)_0)].$$

  Recall that for $\sigma \to 0$, this expression can be well approximated by

$$\Delta(\phi, z, \widehat{x}|z^*(\phi)_0)_0 = p_s\Pi_s(z) - p_r\Pi_r(z) + (p_s - p_r)[\beta\mathbf{V}(\phi'|\widehat{x})_0 - R(\phi|z^*(\phi)_0)].$$

  - Solve for $z^*(\phi)_1$ from $\frac{\sum \Delta(\phi, z, \widehat{x}|z^*(\phi)_0)_0}{N_x} = 0$.
  - For all $\theta < (>)z^*(\phi)_1$, $x(\phi, \theta)_1 = 1(= 0)$.
    * $R(\phi|z^*(\phi)_1)$ follows from $z^*(\phi)_1$.
    * ϕ' follows from $x(\phi, \theta)_1$.
  - Obtain $\mathbf{V}(\phi)_1$ as

$$\mathbf{V}(\phi)_1 = \int_{-\infty}^{z^*(\phi)_1} p_r[\Pi_r(\theta) - R(\phi|z^*(\phi)_1) + \beta\mathbf{V}(\phi)_0]v(\theta)d\theta$$
$$+ \int_{z^*(\phi)_1}^{\infty} p_s[\Pi_s(\theta) - R(\phi|z^*(\phi)_1) + \beta\mathbf{V}(\phi')_0]v(\theta)d\theta.$$

  - Use $\mathbf{V}(\phi)_1$ and $z^*(\phi)_1$ as new guesses and iterate until $\mathbf{V}(\phi)_i - \mathbf{V}(\phi)_{i-1} < \varepsilon_1$ and $z^*(\phi)_i - z^*(\phi)_{i-1} < \varepsilon_2$ for all ϕ.