Interpreting Economic Time Series

Thomas J. Sargent

April 1980

Staff Report #58

University of Minnesota
and
Federal Reserve Bank of Minneapolis

This paper explores some of the implications for econometric practice of the principle that people's observed behavior will change when their constraints change. In dynamic contexts, a proper definition of people's constraints includes among them laws of motion that describe the evolution of the taxes they must pay and the prices of the goods that they buy and sell. Changes in agents' perceptions of these laws of motion (or constraints) will in general produce changes in the schedules that describe the choices they make as a function of the information that they possess. Until very recently, received dynamic econometric practice ignored this principle. The practice of dynamic econometrics should be changed so that it is consistent with the principle that people's rules of choice are influenced by their constraints. This is a substantial undertaking, and involves major adjustments in the ways that we formulate, estimate, and simulate econometric models.

Introduction

This paper explores some of the implications for econometric practice of a single principle from economic theory. This principle is that people's observed behavior will change when their constraints change. In dynamic contexts, a proper definition of people's constraints includes among them laws of motion that describe the evolution of the taxes they must pay and the prices of the goods that they buy and sell. Changes in agents' perceptions of these laws of motion (or constraints) will in general produce changes in the schedules that describe the choices they make as a function of the information that they possess. Until very recently, received dynamic econometric practice ignored this principle, and routinely deduced policy conclusions by assuming that people's rules of choice would not vary, for example, with the government's choices of laws of motions for variables such as tax rates, government purchases and so on. These variables are supposed to have their effects precisely because they influence the constraints of some private agents.

The practice of dynamic econometrics should be changed so that it is consistent with the principle that people's rules of choice are influenced by their constraints. This is a substantial undertaking, and involves major adjustments in the ways that we formulate, estimate, and simulate econometric models. Foremost, we need a stricter definition of the class of parameters that can be regarded as "structural". The body of doctrine associated with the "simultaneous equations" model in econometrics properly directs the attention of the researcher beyond reduced form parameters to the parameters of "structural equations" which presumably describe those aspects of the behavior of people that prevail

across a range of hypothetical environments. Estimates of the
parameters of structural equations are needed in order to analyze
an interesting class of policy interventions. Most often, however,
included in a prominent way among the "structural equations" have
been equations describing the rules of choice for private agents. Con-
sumption functions, investment schedules, demand functions for assets,
and agricultural supply functions are all examples of such rules of choice.
In dynamic settings, regarding the parameters of these rules of choice as
structural or invariant under interventions violates our simple principle
from economic theory.

This paper describes methods for interpreting economic time series
in a manner consistent with the principle that people's constraints
influence their behavior. For the most part, I shall restrict things
so that the dynamic economic theory is of the equilibrium variety,
with optimizing agents and cleared markets. However, many of the
principles described here will pertain to other types of dynamic economic
theories, such as "disequilibrium" models with optimizing agents. The
line of work I shall describe has diverse antecedents, of which major
ones are contributions of Muth [46, 47], Lucas [33, 34], Lucas and
Prescott [37], Nerlove [50], and Graves and Telser [17].[1/] The works
of Granger [16] and Sims [57] have provided key technical econometric
foundations.

The basic idea is to interpret a collection of economic time
series as resulting from the choices of private agents interacting
in markets assumed to be organized along well specified lines. The
private agents are assumed to face nontrivial dynamic and stochastic
optimization problems. This is an attractive assumption because
the solutions of such problems are known to imply that the chosen
variables (e.g. stocks of factors of production or financial assets)
can exhibit serial correlation and cross-serial correlation. Since

time series of economic data usually have the properties of high
own-serial correlation and various patterns of cross-serial correlation,
it seems that there is potential for specifying dynamic preferences,
technologies, constraints and rules of the market game that roughly
reproduce the serial correlation and cross-correlation patterns in
a given collection of time series measuring market outcomes. If
this can be done in such a fashion that the free parameters of
preferences, technologies, and constraints are identifiable
econometrically, it is then possible to interpret the collection
of time series as the outcome of a well specified dynamic, stochastic
equilibrium model. This paper is intended as a nontechnical summary
of some of the econometric and theoretical issues involved in
interpreting data in this way.

But why should anybody _want_ to interpret time series data as
representing the results of interactions of private agents' opti-
mizing choices? The answer is not that this way of modeling is
aesthetically pleasing, although it is, nor that modeling in this
way guarantees an analysis that implies no role for government
intervention, which it doesn't. The reason for interpreting time
series in this way is practical: it potentially offers the analyst
the ability to predict how agents' behavior and the random behavior
of market-determined variables will each change when there are
policy interventions or other changes in the environment that alter
some of the agents' dynamic constraints. There is a general
presumption that private agents' behavior and the random behavior
of market outcomes both will change whenever agents' constraints
change, as when policy interventions or other changes in the environ-
ment occur. The most that can be hoped for is that the parameters
of agents' preferences and technologies will not change in the

face of such changes in the environment. If the dynamic econometric model is formulated explictly in terms of the parameters of preferences, technologies and constraints, it will in principle be possible for the analyst to predict the effects on observed behavior of changes in the stochastic environment.

Past dynamic econometric studies should usually be regarded as having been directed at providing ways of summarizing the observed behavior of interrelated variables, without attempting to infer the objectives, opportunities, and constraints of the agents whose decisions determine those variables. Most existing studies can be viewed, at best, as having estimated parameters of agents' decision rules for setting chosen variables as functions of the information they possess. Most of the better studies of consumption, investment, asset demand, and agricultural supply functions must be interpreted as having estimated such decision rules. Dynamic economic theory implies that these decision rules cannot be expected to remain invariant in the face of policy interventions that take the form of changes in some of the constraints facing agents. This means that there is a theoretical presumption that historical econometric estimates of such decision rules will provide poor predictions about behavior in a hypothetically new environment. This was Robert E. Lucas's [34] critique of econometric policy evaluation procedures as they existed in 1973.

Some readers of Lucas [34] have interpreted the message as a call to evaluate policies by using existing econometric models differently.2/ However, one implication of Lucas's argument, and of dynamic economic theory generally, is that the formulation, identification, and estimation of the models much each be approached in substantially new and different ways. Most existing models simply cannot be saved by simulating them a little more shrewdly.3/

Formulating and estimating "rational expectations" models and dynamic equilibrium models of economic time series involves a variety of important conceptual and econometric issues, some of which I try to summarize in this paper. Among the issues to be treated are the following:

(i) <u>Identification criteria</u>. Prior identifying information of the Cowles commission variety, i.e., mainly exclusion restrictions, play a much smaller role in dynamic equilibrium models. Nonlinear cross-equation restrictions implied by dynamic theory are used extensively. This shift involves important modifications of past ways of thinking about identification and estimation.

(ii) <u>Models of error terms</u>. The dynamic equilibrium modeling strategy virtually forces the researcher to think about the sources and interpretations of the error terms in the stochastic equations that he fits. The explicitly stochastic nature of the theorizing makes it difficult to "tack on" error terms <u>after</u> the theorizing is done, a usual procedure in the past.

(iii) The role of <u>Granger-causality</u>. Granger causality turns out to be a critical concept in the formulation of dynamic economic models, as it is coincident with the condition for the appearance as an information variable in an agent's decision rule of a variable <u>not</u> otherwise in the agent's criterion function or constraints.

(iv) <u>Bayesian analysis</u>: Bayesian econometric techniques provide a means of mixing prior theoretical information about parameters with information from the data. Such procedures are widely used by applied time series econometricians, although often no formal Bayesian justification is given. Dynamic economic theory provides no justification for one widely imposed class of prior restrictions which can be viewed as restrictions directly on decision rules. Instead, dynamic economic theory suggests that prior information about agents' criterion functions

and constraints is what should be used in estimation. This feature of dynamic economic theory has implications for the proper implementation both of formal Bayesian procedures and of less formal procedures for constraining parameter estimates.

I shall organize my discussion around a pair of examples. In section 1, I describe a linear-quadratic version of Lucas and Prescott's model of investment under uncertainty. I shall use this example for discussing the econometric implications of dynamic equilibrium models. In section 2, I describe a very simple model of interrelated corn and hog markets. Though it is overly simple, this model illustrates how many of the points addressed in section 1 generalize, and also illustrates what I believe to be some essential principles that would be involved in building a useful econometric model of the "corn-hog cycle". In the Lucas-Prescott example, state variables which help the firm predict the future prices of inputs appear in the representative firm's decision rule. However, the laws of motion of these input prices are taken as given from outside the model. In actuality, the prices of these inputs are usually thought to be determined by trades in "another" market, one source of demand for which stems from the industry being modeled by Lucas and Prescott. What modifications to the analysis occur if this "other" market is modeled explicitly? The corn-hog example is designed to study this question. The corn-hog model goes beyond the Lucas-Prescott example by permitting the output of one industry, corn, to be an input into the production of the other industry, hogs. Since hog producers have an incentive to forecast future corn prices, it follows that "state variables" that appear in the laws of motion for the total output and price of corn appear in the decision rule of the representative hog producer. Also, because the corn producer has an incentive to forecast the price of corn,

which depends partly on the demand for corn from hog producers, the "state variables" that appear in the laws of motion for the total hog output and the price of hogs appear in the optimal decision rule of the representative corn producer. Hence, each industry inherits the state variables of the other. Further, the equilibria in the two industries are inextricably interrelated. The corn-hog example illustrates an important general phenomenon: that rational expectations equilibria of industries that are related only because they trade with one another technically become equivalent to the solution of "interrelated factor demand" problems in which quantities and prices for different goods interact pervasively.

I have adopted linear-quadratic setups because they simplify both the theoretical and econometric discussions, while illustrating many of the salient methodological implications of dynamic decision theory. Linear-quadratic optimum problems deliver difference equations that are linear in the variables, and so match up nicely with much existing dynamic econometric theory. The reader familiar with Lucas and Prescott [37] will recognize how the corn-hog model of section 2 can be generalized to incorporate more general specifications for the technologies, preferences, and constraints. That increased generality would make the econometric implications harder to extract than with the present setup, without altering the basic message.[4/]

1. Investment Under Uncertainty

This section describes a linear-quadratic version of Lucas and Prescott's model of investment, and uses it as a vehicle for expositing a variety of conceptual and econometric issues. The model describes the mutual determination over time of the capital stock, output, and market price of a single industry. The model can be generalized to handle multiple factors of production, at the cost of what are really only technical complications. In section 2, I indicate how the model can be extended to analyze two or more interdependent industries, such as the "hog and corn" industries. Finally, I mention that it is straightforward to modify the model to incorporate much richer dynamics by generalizing the nature of the adjustment costs.

I define the following variables:

y_t = output of the representative firm

n = number of firms in the industry, assumed constant over time

Y_t = ny_t = total output of industry

P_t = price of output

D_{1t} = a $(p_1 \times 1)$ vector of random variables appearing in the industry demand schedule, $p_1 \geq 1$

D_{2t} = a $(p - p_1) \times 1$ vector of random variables which help predict future values of the collection of variables D_{1t}, $p \geq p_1$

D_t = $\begin{bmatrix} D_{1t} \\ D_{2t} \end{bmatrix}$

w_t = rental rate on capital

W_t = a $(q \times 1)$ vector whose first element is w_t; the remaining elements of W_t are variables that help predict future w_t's.

u_t = a random shock to demand

ε_t = a random shock in the production function

k_t = stock of capital of the representative firm

K_t = nk_t = total capital stock in industry

The subscript t indexes the date to which the variable corresponds.

I further define the following polynomials in the lag operator L:

$$\delta_u(L) = 1 - \sum_{j=1}^{r_u} \delta_{uj} L^j, \quad \text{where} \quad \delta_{uj} \quad \text{is a scalar}$$

$$\delta_D(L) = I_p - \sum_{j=1}^{r_D} \delta_{Dj} L^j, \quad \text{where} \quad \delta_{Dj} \quad \text{is a } p \times p \text{ matrix and } I_p \text{ is the } p \times p \text{ identity matrix.}$$

$$\delta_w(L) = I_q - \sum_{j=1}^{r_w} \delta_{wj} L^j, \quad \text{where} \quad \delta_{wj} \quad \text{is a } (q \times q) \text{ matrix and } I_q \text{ is the } (q \times q) \text{ identity matrix.}$$

$$\delta_\varepsilon(L) = 1 - \sum_{j=1}^{r_\varepsilon} \delta_{\varepsilon j} L^j, \quad \text{where} \quad \delta_{\varepsilon j} \quad \text{is a scalar.} [5]$$

The industry consists of n identical competitive firms, each of which uses a single factor of production, capital, to produce a single output. Output of the representative firm y_t is governed by

$$(1.1) \qquad y_t = fk_t + n^{-1}\varepsilon_t , \qquad\qquad f > 0$$

where k_t is the representative firm's stock of capital at t, and ε_t is a random error in the technology. The firm knows $\{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$, but does not know with certainty future values of the shock ε_t. The error ε_t is known to follow the r_ε-th order Markov process

$$(1.2) \qquad \delta_\varepsilon(L)\varepsilon_t = V_t^\varepsilon$$

where V_t^ε is a "fundamental" white noise error term for ε_t.[6]/

The firm is assumed to know $\delta_\varepsilon(L)$ and $E(V_t^\varepsilon)^2$ with certainty.

The demand curve for output is given by [7]/

$$(1.3) \qquad P_t = A_0 - A_1 Y_t + A_2 D_{1t} + u_t \qquad\qquad A_0, A_1 > 0$$

where D_{1t} is a $(p_1 \times 1)$ vector of "demand shifters", A_2 is a $(1 \times p_1)$ vector of constants, and u_t is a random shock to the demand curve. The random term u_t obeys the r_u^{th} order Markov process

$$(1.4) \qquad \delta_u(L)u_t = V_t^u$$

where V_t^u is a fundamental white noise for u_t. The $(p_1 \times 1)$ vector of demand shifters D_{1t} consists of the first p_1 rows of the $p \times 1$ vector D_t, which follows the r_D^{th} order vector autoregressive process

$$(1.5) \qquad \delta_D(L)D_t = V_t^D$$

where V_t^D is a $(p \times 1)$ vector white noise that is fundamental for the process D_t. The representative firm is assumed to know $\delta_u(L)$, $\delta_D(L)$, A_0, A_1, A_2, and the second moments of V_t^u and V_t^D with certainty.

At time t, total output is given by

$$(1.6) \qquad Y_t = n y_t = f K_t + \varepsilon_t$$

The representative firm's problem is to choose a contingency plan for k_{t+j} to maximize the criterion

(1.7)
$$E_0 \sum_{t=0}^{\infty} \beta^t \{P_t y_t - w_t k_t - \frac{d}{2}(k_{t+1} - k_t)^2\}$$

subject to k_0 given. In (7), E_t is the mathematical expectation operator, conditional on information known to the firm at time t. This information set will shortly be specified precisely. In (7), d is a positive constant. The term $\frac{d}{2}(k_{t+1} - k_t)^2$ is intended to represent the notion that there are costs internal to firm of adjusting the capital stock, and that these rise at an increasing rate with the absolute value of the change in capital. We assume that the rental on capital w_t is the first element of the $(q \times 1)$ vector random process W_t that obeys the r_W^{th} order vector autoregression

(1.8)
$$\delta_W(L) W_t = V_t^W$$

where V_t^W is a $(q \times 1)$ vector white noise that is fundamental for W_t. The firm is supposed to know $\delta_W(L)$ and the second moment matrix of V_t^W with certainty.

At time t, the firm chooses k_{t+1}, given the information that it has available at t. However, the maximization problem (7) is not yet well posed, since we have not completely spelled out the dynamic constraints with respect to which the maximization is supposed to occur. To complete the problem (7), we begin by substituting $(f\, k_t + n^{-1} \varepsilon_t)$ for y_t, and $[A_0 - A_1 f K_t - A_1 \varepsilon_t + A_2 D_{1t} + u_t]$ for P_t to get

(1.9)
$$E_0 \sum_{t=0}^{\infty} \beta^t \{[A_0 - A_1 f K_t - A_1 \varepsilon_t + A_2 D_{1t} + u_t][f\, k_t + n^{-1} \varepsilon_t]$$
$$- w_t k_t - \frac{d}{2}(k_{t+1} - k_t)^2\}.$$

In order that the problem of maximizing (9) with respect to a contingency plan for $\{k_{t+j}\}$ be well posed, it is necessary to attribute to the firm precise views about the laws of motion of the random variables that it cannot control, but whose values influence the best choice of its own stocks of capital. For problem (9), these "uncontrollable" variables about which the representative firm cares are K_t, D_{1t}, u_t and w_t. The firm cares about the present and future behavior of the variables (K_t, D_{1t}, u_t) because they influence the present and future behavior of the market price through the demand relationship $P_t = A_0 - A_1 f K_t + A_2 D_{1t} + u_t$. The firm cares about the evolution of the rental process w_t because it influences its costs. We have already completely described our assumptions about the firm's views of the laws of motion of D_{1t}, u_t and w_t, namely, that the firm knows the Markov laws (4), (5), and (8) that govern them, and at time t knows D_t, $D_{t-1}, \ldots, u_t, u_{t-1}, \ldots,$ and W_t, $W_{t-1}, \ldots,$. To complete the specification requires that we specify the firm's views about the evolution of the aggregate capital stock K_t. We assume that the representative firm believes that the aggregate capital stock evolves according to the law

(1.10)
$$K_{t+1} = H_0 + H_w(L)W_t + H_D(L)D_t + H_\varepsilon(L)\varepsilon_t$$
$$+ H_u(L)u_t + H_1 K_t$$

where H_0 and H_1 are scalars and

$$H_w(L) = \sum_{j=0}^{r_w-1} H_{wj} L^j, \quad \text{where } H_{wj} \text{ is } (1 \times q)$$

$$H_D(L) = \sum_{j=0}^{r_D-1} H_{Dj} L^j, \quad \text{where } H_{Dj} \text{ is } (1 \times p)$$

$$H_\varepsilon(L) = \sum_{j=0}^{r_\varepsilon - 1} H_{\varepsilon j} L^j, \quad \text{where} \quad H_{\varepsilon j} \quad \text{is a scalar}$$

$$H_u(L) = \sum_{j=0}^{r_u - 1} H_{uj} L^j, \quad \text{where} \quad H_{uj} \quad \text{is a scalar.}$$

The representative firm is assumed to know all of the parameters of the linear law of motion (10) with certainty. The reason that we have chosen the form (10) as the firm's perceived law of motion for K will shortly become apparent.

With these specifications, the maximization of (9) is now well posed. Summarizing the setup, we have that the representative firm maximizes

(1.9)
$$E_0 \sum_{t=0}^{\infty} \beta^t \{ [A_0 - A_1 f K_t - A_1 \varepsilon_t + A_2 D_{1t} + u_t][f\, k_t + n^{-1} \varepsilon_t]$$
$$- w_t k_t - \frac{d}{2}(k_{t+1} - k_t)^2 \}$$

subject to the laws of motion[8/]

(1.10)
$$K_{t+1} = H_0 + H_w(L)W_t + H_D(L)D_t + H_\varepsilon(L)\varepsilon_t$$
$$+ H_u(L)u_t + H_1 K_t$$

(1.8)
$$\delta_w(L)W_t = v_t^w$$

(1.4)
$$\delta_u(L)u_t = v_t^u$$

(1.5)
$$\delta_D(L)D_t = v_t^D$$

(1.2)
$$\delta_\varepsilon(L)\varepsilon_t = v_t^\varepsilon$$

and subject to the information set at time t, [9]

$$\{K_t, \ k_t, \ W_t, \ W_{t-1}, \ldots, W_{t-r_w+1}, \ D_t, \ D_{t-1}, \ldots, D_{t-r_D+1},$$

$$\varepsilon_t, \ \varepsilon_{t-1}, \ldots, \varepsilon_{t-r_\varepsilon+1}, \ u_t, \ u_{t-1}, \ldots, u_{t-r_u+1}\}.$$

The firm maximizes (9) taking the laws of motion (8), (4), (5), (2) and (10) as given and beyond its control. The firm is assumed to behave competitively and to act as if it has no control over the aggregate capital stock K. This is a reasonable assumption if n is large. The firm is assumed to know the δ's and H's with certainty, and to know the first and second moments of the V_t's.[10] We further restrict the problem so that the solution is a linear contingency plan.[11] For this to be true, it is sufficient that the least squares predictors of future W, D, ε, and u's be linear in the conditioning variables. This will be true if V_t^ε, V_t^u, V_t^D, and V_t^W obey normal probability laws. Alternatively, the analyst can simply assume that the industry is operating under optimal linear rules. In either case, the _solution_ of the representative firm's problem is a linear contingency plan of the form [12]

(11) $$k_{t+1} = h_0 + h_w(L)W_t + h_D(L)D_t + h_\varepsilon(L)\varepsilon_t$$

$$+ h_u(L)u_t + h_1 K_t + h_2 k_t$$

where h_0, h_1, and h_2 are scalars, and

$$h_w(L) = \sum_{j=0}^{r_w-1} h_{wj}L^j, \quad \text{where } h_{wj} \text{ is } (1 \times q)$$

$$h_D(L) = \sum_{j=0}^{r_D-1} h_{Dj}L^j, \quad \text{where } h_{Dj} \text{ is } (1 \times p)$$

$$h_\varepsilon(L) = \sum_{j=0}^{r_\varepsilon-1} h_{\varepsilon j}L^j, \quad \text{where } h_{\varepsilon j} \text{ is a scalar}$$

$$h_u(L) = \sum_{j=0}^{r_u-1} h_{uj}L^j, \quad \text{where } h_{uj} \text{ is a scalar}..$$

The h's of (11) are in general functions both of the parameters in the criterion function (9), the parameters of δ_w, δ_u, δ_D, and δ_ϵ appearing in (8), (4), (5), and (2), and the H's of the perceived law of motion for capital (10). The mapping giving the h's as functions of these other parameters is defined implicitly by standard formulas in linear optimal control theory, as exposited for example by Bertsekas [3] and Kwakernaak and Sivan [29]. For present purposes, it is enough to note the existence of this mapping without exploring its nature in detail. The economic content of the mapping from the δ's, H's and objective function parameters to the h parameters of the firm's decision rule is easy to understand, since it captures the notion that the firm's rule of choice depends both on its objective and its perceived constraints (10), (8), (4), (5) and (2).

Multiplying both sides of the firm's decision rule (11) by n and using $K_t = nk_t$ gives

(1.12)
$$K_{t+1} = nh_0 + nh_w(L)W_t + nh_D(L)D_t + nh_\epsilon(L)\epsilon_t$$
$$+ nh_u(L)u_t + (nh_1 + h_2)K_t .$$

Equation (12) is the actual law of motion for aggregate capital that results from the behavior of the representative firm. The representative firm's optimization problem in effect induces a mapping from the firm's perceived law of motion for aggregate capital (10) to the actual law of motion (12). For each possible particular perceived law of motion of the form (10), there is an implied actual law of motion for aggregate capital of the form (12). The notion of rational expectations is that the representative firm's perceptions of (10) are correct. In effect, a rational expectations equilibrium is a fixed point of the mapping that

the representative firm's optimization problem induces from (10) to
(12). Formally, we define a <u>rational expectations equilibrium</u> as a
perceived law of motion (10) and an implied actual law of motion (12)
which are identically equal. In a rational expectations equilibrium,
firms' perceptions about the law of motion for aggregate capital turn
out to be confirmed by the aggregate of the choices made by firms.
Upon comparing (10) with (12) it is evident that necessary and sufficient
conditions for a rational expectations equilibrium are

$$H_0 = nh_0$$

$$H_w(L) = nh_w(L)$$

$$H_D(L) = nh_D(L)$$

$$H_\varepsilon(L) = nh_\varepsilon(L)$$

$$H_u(L) = nh_u(L)$$

$$H_1 = (nh_1 + h_2)$$

Implicit in the above definition of a rational expectations equilibrium
are the following elements: (a) market clearing, (b) optimization of the
firm's expected present value, and (c) correct perceptions on the part of
firms of the laws of motion of variables affecting their present value
but beyond their control.

We begin our analysis of the model by briefly describing aspects
of the optimization problem solved by the firm. Among the first-
order necessary conditions for the maximization of (9) is the following
system of stochastic "Euler equations", which are derived by

differentiating (9) with respect to k_t for $t = 1,2,\ldots;$

$$(1.13) \qquad \beta d k_{t+1} - d(1+\beta)k_t + d k_{t-1} = \beta w_t$$

$$-\beta f [A_0 - A_1 f K_t - A_1 \epsilon_t + A_2 D_{1t} + u_t]$$

or

$$k_{t+1} - (\tfrac{1}{\beta} + 1)k_t + \tfrac{1}{\beta} k_{t-1} = \tfrac{1}{d} w_t - \tfrac{f}{d} P_t$$

In addition to the system of Euler equations, a transversality condition is among the first order necessary conditions. The transversality condition can be derived by methods described in Sargent [54]. The transversality condition for the present problem in effect requires that the solution possess the property $\lim\limits_{j \to \infty} E_t \beta^{t+j} k_{t+j} = 0.$

Using the lag operator, the preceding Euler equation can be rewritten as[13]

$$(1 - (\tfrac{1}{\beta} + 1)L + \tfrac{1}{\beta} L^2)k_{t+1} = \tfrac{1}{d} w_t - \tfrac{f}{d} P_t$$

Using the factorization

$$(1 - (\tfrac{1}{\beta} + 1)L + \tfrac{1}{\beta} L^2) = (1 - \tfrac{1}{\beta} L)(1-L),$$

the above Euler equation can be written as

$$(1 - \tfrac{1}{\beta} L)(1-L) k_{t+1} = \tfrac{1}{d} w_t - \tfrac{f}{d} P_t .$$

Noting that $(1 - \tfrac{1}{\beta} L) = -\beta^{-1} L(1-\beta L^{-1})$ and operating on both sides of the above equation with $[-\beta^{-1} L(1-\beta L^{-1})]^{-1}$ gives the solution[14]

$$(1-L)k_{t+1} = \frac{-d^{-1}\beta L^{-1}}{1-\beta L^{-1}} w_t + \frac{\beta f d^{-1} L^{-1}}{1-\beta L^{-1}} P_t$$

or equivalently

$$(1.14) \qquad (1-L)k_{t+1} = -d^{-1}\beta \sum_{i=0}^{\infty} \beta^i w_{t+i+1} + \beta f d^{-1} \sum_{i=0}^{\infty} \beta^i P_{t+i+1}$$

It can be verified that (14) satisfies both the Euler equations and the transversality condition. Equation (14) would give the appropriate rule for setting k_{t+1} if the firm had perfect foresight about the entire future paths of the rental w_t and the output price P_t. When the firm does not have perfect foresight, the correct decision rule can be derived by replacing the future values on the right side of (14) with the corresponding mathematical expectations conditional on information the firm does have. This leads to the decision rule [15]/

$$(1.15) \qquad (1-L)k_{t+1} = -d^{-1}\beta \sum_{i=0}^{\infty} \beta^i E w_{t+i+1}|\Omega_t + \beta f d^{-1} \sum_{i=0}^{\infty} \beta^i E P_{t+i+1}|\Omega_t$$

Here Ω_t is defined as the information set $\Omega_t = \{W_t, W_{t-1}, \ldots, u_t, u_{t-1}, \ldots, D_t, D_{t-1}, \ldots, \varepsilon_t, \varepsilon_{t-1}, \ldots, K_t\}$. The conditional mathematical expectations are assumed to be computed using the laws of motion (10), (8), (4), (5), and (2) for K, W, u, D, and ε, respectively, as well as the demand relationship $P_t = A_0 - A_1(fK_t + \varepsilon_t) + A_2 D_{1t} + u_t$, which is used to deduce the law of motion for P_t. Once these conditional mathematical expectations are explicitly calculated in terms of the parameters of (10), (8), (4), (5), (2), and the demand curve (3), they can be substituted into equation (15) to deduce the optimum decision rule (11) for the representative firm. The decision rule (11) is linear in all of the "information variables" that appear on the right side. However, as the above method of calculating the parameters h of the decision rule (11) suggests, the parameters h are themselves complicated nonlinear functions of the underlying parameters of the model: the parameters A_0, A_1, A_2 of the demand curve, the parameters f and d

of the technology; and the parameters $\delta_u(L)$, $\delta_\epsilon(L)$, $\delta_w(L)$, and $\delta_D(L)$ of the laws of motion of the random processes given from outside the model.[16/] The h's are also nonlinear functions of the H's of the law of motion of aggregate capital (10), which are not given from outside but are to be determined from the analysis. The nature of these nonlinearities has been characterized by Hansen and Sargent [23], and will be alluded to further below.

Equation (15), which was derived by purely formal manipulations, has the virtue of indicating clearly that the firm has an incentive to forecast future realizations of the rental w and the output price P . As a result, any state variables that the firm sees at t, and that help predict either future P's or future w's, will appear in the firm's decision rule for k_{t+1}, given by equation (11). That the h's of (11) are nonlinear functions of the parameters $\{A_0,\ A_1,\ A_2,\ f,\ d,\ \beta,\ \delta_u,\ \delta_\epsilon,\ \delta_w,\ \delta_D,\ H_u,\ H_\epsilon,\ H_w,\ H_D,\ H_0\ \text{and}\ H_1\}$, stems from the nonlinear way in which the conditional mathematical expectations of future w's and P's are functions of these parameters.

In practice, to compute a rational expectations equilibrium it is not necessary ever to calculate the right side of (15). Indeed, it is never necessary explicitly to calculate the h's that determine the decision rule (11) of the representative firm. Instead, the H's of the equilibrium law of motion for the industry can be calculated directly as follows.[17/] First, multiply both sides of equation (13) by n, then use $K_t = nk_t$, and collect all terms in K on the left side to get

(1.16) $\qquad \beta d K_{t+1} - [d(1+\beta)+A_1 f^2 \beta n]K_t + dK_{t-1} = \beta n w_t$

$$- \beta n f A_0 + A_1 \beta f n \epsilon_t - \beta f n A_2 D_{1t} - \beta f n u_t$$

It is of some interest that (16) is itself the Euler equation for the "social planning" problem of maximizing [18]

$$(1.17) \quad E_0 \sum_{t=0}^{\infty} \beta^t [\{A_0(fK_t + \epsilon_t) - \tfrac{1}{2} A_1(fK_t + \epsilon_t)^2 + [fK_t + \epsilon_t]A_2 D_{1t}$$

$$+ [fK_t + \epsilon_t]u_t \} - w_t K_t - \tfrac{1}{2} n^{-1} d(K_{t+1} - K_t)^2]$$

subject to the laws of motion (8), (4), (5), and (2) for w_t, u_t, D_{1t}, and ϵ_t, [19] and subject to K_0 given.

The term in braces is the area under the demand curve, since

$$\int_0^{Y_t} [A_0 - A_1 x + A_2 D_{1t} + u_t] dx$$

$$= A_0 Y_t - \tfrac{1}{2} A_1 Y_t^2 + Y_t A_2 D_{1t} + Y_t u_t \quad .$$

Thus (17) is the discounted area under the demand curve minus the total costs of production. Dividing each side of (16) by βd, the Euler equation can be written

$$(1.18) \quad \begin{aligned} K_{t+1} - [1 + \tfrac{1}{\beta} + \frac{A_1 f^2 n}{d}] K_t + \tfrac{1}{\beta} K_{t-1} \\ = \tfrac{n}{d} w_t - \frac{n f A_0}{d} + \frac{A_1 f n}{d} \epsilon_t \\ - d^{-1} f n A_2 D_{1t} - \frac{f n}{d} u_t \end{aligned}$$

It can easily be proved that there exists a λ such that

$$[1 - (1 + \beta^{-1} + A_1 f^2 n d^{-1})L + \beta^{-1} L^2] = (1 - (\lambda\beta)^{-1} L)(1 - \lambda L)$$

where $|\lambda| < 1/\sqrt{\beta}$. [20] Using $-(\lambda\beta)^{-1} L(1 - \lambda\beta L^{-1}) = (1 - (\lambda\beta)^{-1} L)$, we have that the Euler equation (18) can be written as

$$-(\lambda\beta)^{-1}L(1 - \lambda\beta L^{-1})(1 - \lambda L)K_{t+1} = \frac{-nfA_0}{d} + \frac{n}{d} w_t$$

$$+ \frac{A_1 fn}{d} \varepsilon_t - \frac{fn}{d} A_2 D_{1t} - \frac{fn}{d} u_t$$

A solution of the Euler equation that also satisfies the transversality condition for the social planning problem is

$$(1.19) \qquad (1 - \lambda L)K_{t+1} = \frac{+\lambda\beta nfA_0}{d}(1 - \lambda\beta)^{-1} - \frac{n\lambda\beta}{d}\frac{L^{-1}}{1-\lambda\beta L^{-1}} w_t$$

$$- \frac{A_1 fn\lambda\beta}{d}\frac{L^{-1}}{1-\lambda\beta L^{-1}} \varepsilon_t + \frac{fn\lambda\beta d^{-1}L^{-1}}{1-\lambda\beta L^{-1}} A_2 D_{1t}$$

$$+ \frac{fn\lambda\beta d^{-1}L^{-1}}{1-\lambda\beta L^{-1}} u_t$$

Recall, for example, that $(1 - \lambda\beta L^{-1})^{-1}w_t = \sum_{j=0}^{\infty} (\lambda\beta)^j w_{t+j}$. Then it can be recognized that equation (19) is the perfect foresight solution of the planning problem that the rational expectations competitive equilibrium implicitly solves. Thus, equation (19) expresses the aggregate capital stock K_t as a linear function of K_t and all future values of w_t, ε_t, D_{1t}, and u_t .

By using the methods of Hansen and Sargent [23, especially appendix A], equation (19) can be converted to the "realizable" law for K that satisfies the Euler equations and transversality conditions, and which expresses K_{t+1} as a function only of information known at time t.[21/] This involves replacing the terms w_{t+i}, ε_{t+i}, D_{1t+i}, and u_{t+i} in (19) by the corresponding mathematical expectations conditioned on Ω_t. The resulting equilibrium law of motion for K can be shown to be

(1.20) $K_{t+1} = H_0 + H_w(L)W_t + H_D(L)D_t + H_\varepsilon(L)\varepsilon_t$

$$+ H_u(L)u_t + H_1 K_t$$

where $H_0 = \dfrac{+\lambda\beta nfA_0}{d(1-\lambda\beta)}$

$H_1 = \lambda$

(1.21) $H_w(L) = \dfrac{-n\lambda\beta}{d}\, \phi_w \left[\dfrac{L^{-1}(I-\delta_w(\lambda\beta)^{-1}\delta_w(L))}{1-\lambda\beta L^{-1}}\right]$

$H_\varepsilon(L) = \dfrac{-A_1 fn\lambda\beta}{d}\left[\dfrac{L^{-1}(1-\delta_\varepsilon(\lambda\beta)^{-1}\delta_\varepsilon(L))}{1-\lambda\beta L^{-1}}\right]$

$H_D(L) = +fn\lambda\beta d^{-1}A_2\phi_D\left[\dfrac{L^{-1}(I-\delta_D(\lambda\beta)^{-1}\delta_D(L))}{1-\lambda\beta L^{-1}}\right]$

$H_u(L) = +d^{-1}fn\lambda\beta\left[\dfrac{L^{-1}(1-\delta_u(\lambda\beta)^{-1}\delta_u(L))}{1-\lambda\beta L^{-1}}\right]$

Here ϕ_w is a $1 \times q$ vector with 1 in the first position, followed by $(q - 1)$ zeroes, and ϕ_D is a $p_1 \times p$ matrix with a $(p_1 \times p_1)$ identity matrix as the first p_1 columns and zeroes elsewhere. Notice that $w_t \equiv \phi_w W_t$ and $D_{1t} \equiv \phi_D D_t$. It is convenient at this point to recall the laws of motion assumed for w_t, u_t, ε_t and D_{1t}, namely

(8) $\delta_w(L)W_t = V_t^W$

(4) $\delta_u(L)u_t = V_t^u$

(5) $\delta_D(L)D_t = V_t^D$

(2) $\delta_\varepsilon(L)\varepsilon_t = V_t^\varepsilon$

Equation (20) expresses the equilibrium law for the industry-wide capital stock K_{t+1} as a linear function of K_t, and current and past values of W, D, ε, and u. Current and past values of W appear in (20) because they help predict future values of the rental rate w_t, while current and past values of D, ε, and u appear because they are used by agents to predict the future course of the market price P. The number of lagged values of W, D, ε, and u in (20) are r_w-1, r_D-1, $r_\varepsilon-1$, and r_u-1, as expressions (21) can be used to show.[22/] Thus, the numbers of lagged values of these "information variables" W, D, ε, and u in (20) are entirely inherited from the specifications of the actual laws of motion for W, D, ε, and u in (8), (5), (2) and (4).

Notice that the appearance of w_t and D_{1t} in the objective function of the representative firm (9) (or equivalently in the objective function of the fictitious social planner (17)) gives rise to the appearance in (20) of the entire blocks of variables W_t and D_t that help predict w and D_{1t}, respectively. Thus any variables that help predict w and D_{1t}, and which agents have information on, belong in the equilibrium law of motion for industry wide capital. The property that the remaining variables in W (or D) help predict future values of w (or D_1) is said to be the property that the remaining variables in W (or D) <u>Granger cause</u> w (or D_1). The notion of Granger causality thus turns out to be coincident with the criterion for whether random variables that don't themselves appear in the agent's criterion function nevertheless end up in the equilibrium law of motion or decision rule, essentially because they appear in the agents' constraints as information variables that help

predict variables that do appear in the criterion function. It is mainly for this reason that the concept of Granger causality has played an important role in work with rational expectations models.[23/]

Equations (20) and (21) reveal explicitly how the parameters of the equilibrium law of motion for industry wide capital are themselves nonlinear functions of the underlying parameters $\{A_0, A_1, A_2, f, d, \beta, n, \delta_w(L), \delta_u(L), \delta_\varepsilon(L), \delta_D(L)\}$. The nonlinearity has two sources. First, there is the fact that λ is a nonlinear function of β and $(A_1 f^2 n d^{-1})$ via the factorization defining λ,

$$(1-(\lambda\beta)^{-1}L)(1-\lambda L) = [1-(1+\beta^{-1} + A_1 f^2 n d^{-1})L + \beta^{-1}L^2].$$ Second, given λ, the formulas for $H_w(L)$, $H_u(L)$, $H_\varepsilon(L)$, and $H_D(L)$ in (21) are nonlinear in the parameters of $\delta_w(L)$, $\delta_u(L)$, $\delta_\varepsilon(L)$ and $\delta_D(L)$. Nonlinear cross-equation restrictions of the kind illustrated by (20) and (21) are the hallmark of rational expectations models. Such cross-equation restrictions are largely absent from "pre-rational expectations" dynamic econometric models.[24/] The presence of these restrictions impinges on a variety of fundamental econometric and conceptual issues, including identification, the analysis of interventions, models of "error terms", and the role of "prior information". I now turn to discussing each of these issues, using (20) and (21) as an instrument.

Analysis of Interventions

At this point, it is useful to remind ourselves of the principal

reason that an economist might want to construct a dynamic econometric

model of an industry along the lines of our example. It is to be able

to make quantitative predictions about the effects on the industry

that various hypothetical interventions or "changes in the environment"

will have. In the present context, a hypothetical "intervention" or

"change in the environment" means a change in one of the polynomials

$\delta_w(L)$, $\delta_u(L)$, $\delta_D(L)$, or $\delta_\varepsilon(L)$ that describe respectively, the

stochastic processes for W, u, D, and ε, that impinge on the market.[25/]

Several interesting examples of such interventions can be given, including

the following:

(a) Suppose that there is a specific tax imposed on sales of

the product. Such a specific tax can be modeled as a component of (A_2D_{1t}).

Since the behavior of the tax through time will be described by an element

of the vector Markov law $\delta_D(L)D_t = V_t^D$, changes in the "rule" for

setting the specific tax amount to changes in one of the rows of $\delta_D(L)$.

(b) Suppose that there is a specific tax on the use of the

factor of production. This tax can be modeled as an addition to the

rental w_t. A change in the rule for setting this tax can be modeled

as a change in a row of $\delta_w(L)$.

(c) Suppose there is a change in the structure of the process

governing the "pre-tax" part of the rental. Again this can be modeled

as a change in one row of $\delta_w(L)$. With a little imagination, the

effects of a change in the organization of the industry[26/] supplying the

factor might be modeled in this way.

The model leading to (20) and (21) provides a way of predicting

quantitatively the effects of such changes, once agents have caught

on to them. The effect of interventions in the sense described here

is to change the <u>function</u> (20) describing the evolution of industry

capital, in a way predicted by the formulas given in (21). Since

interventions of this class change the law of motion (20), it is

necessary to have analytic methods which use the "cross-equation"

restrictions (21) to predict how the H's of the K-law of motion

(20) will change if there is a hypothetical intervention operating

on one or more of the δ's.

In order to evaluate policy interventions in this way, it is

essential that the H's of (20) should not be viewed as being among

the "free parameters" of the model. Instead, the model's free

parameters are to be regarded as the deeper parameters $\{A_0, A_1, A_2,$

$f, d, \delta_w, \delta_u, \delta_\epsilon, \delta_D\}$. The researcher needs to know these

parameters in order to be able to use the formulas (21) to predict

the consequences of hypothetical changes in the functions δ .[27/]

From the dynamic economic theory leading to (20) and (21), it

is evident that a given numerical version of (20), estimated from

historical data, cannot be used to evaluate the consequences of

<u>arbitrary</u> input sequences for $\{W_t\}$, $\{D_t\}$, $\{\epsilon_t\}$, and $\{u_t\}$. That is, a

fixed law of motion of the form (20) with given numerical values for the

H's cannot be used to investigate the consequence of arbitrarily specified

numerical sequences for the W, D, ϵ, and u's. In effect, a

particular version of (20) can be expected to hold up only for W,

D, ϵ, and u sequences drawn from a restricted domain: namely,

sequences obeying the probability laws (8), (4), (5), and (2).[28/]

However, until Lucas [34] wrote in 1973, *evaluating the effects*
of interventions in this inappropriate way was the accepted procedure
in both the macroeconometric and the microeconometric literatures.
Regrettably, to this day it remains the procedure used in the overwhelming
majority of analyses of policy interventions. It should be emphasized
once again that from the viewpoint of the dynamic decision theory
described above, the question of how agents will respond to "arbitrary
sequences" of "forcing variables" W, u, ε, and D is not well posed.
In effect, unless the researcher specifies precisely the perceived
laws of motion for the "forcing variables", he has not specified the
constraints subject to which decision makers are thought to be
acting.$\frac{29}{}$/

Thus, in order to be able to evaluate interventions operating
on the δ's, it is necessary to formulate and estimate the model in
terms of the parameters of preferences (A_0, A_1, A_2), technology
(f, and d), and the constraints (the δ's). The argument in favor
of formulating and estimating the dynamic model at the level of the
deep parameters $\{A_0, A_1, A_2, d, f, \delta_w, \delta_u, \delta_\epsilon, \delta_D\}$ is in much the
same spirit as the usual justification for estimating "structural"
parameters rather than reduced form parameters. As Marschak [40]
argued, the researcher wants to estimate those objects which
will permit him to analyze an interesting class of changes in the
environment. Dynamic economic theory has forced us to re-examine
whether objects long thought to be "structural", including the parameters
of decision rules such as consumption, investment, and portfolio
balance schedules, are correctly taken to be invariant with respect
to changes in the environment. Once agents' behavior is modeled in
terms of genuinely dynamic optimization problems, it becomes apparent
that the parameters of observed decision rules should not be viewed as

structural (see Muth [47], Lucas and Prescott [37], Lucas [33], and Merton [42]).

The Neglect of Learning

At this point it is worthwhile to discuss a modification of the preceding kind of setup which several economists have apparently had in mind.[30] For this purpose it is sufficient to consider the problem of maximizing the social welfare criterion subject to the given laws of motion (8), (4), (5) and (2) for W_t, u_t, D_t, and ε_t. By relabeling and reinterpreting the variables, we can think of this as a choice problem faced by a single private agent. In posing this problem, it was assumed that the "agent" solving the problem knows the true values of the parameters of the objective function (17) and the true values of the polynomials in the lag operator $\delta_\varepsilon(L)$, $\delta_u(L)$, $\delta_D(L)$, and $\delta_w(L)$. The observation has been made that this setup fails to incorporate a model of how the "agent" optimally learns about the δ's from observations on past realizations of the forcing variables ε, u, D, and W. Presumably, if the "agent" has only finite histories of observations on ε, u, D, and W at his disposal, then at each point in time he is uncertain about the parameters of the polynomials δ. Why not modify the preceding setup to include uncertainty about the δ's and a model of optimal learning about the δ's? There seem to be three reasons for why such extensions have not as yet successfully been incorporated into rational expectations models.

The first is as follows. A general model of optimal learning about the δ's is readily available in the "Kalman filter". The Kalman filter can be used to model how a rational agent would use observations on (ε_t, u_t, D_t, W_t) to revise his prior beliefs about the δ's.[31] However, with the δ's uncertain, it is no longer possible to give closed form formulas

for the optimal decision rule in terms of what are now the posterior
probability distributions over the δ's. The reason that no one
has yet obtained or is likely ever to obtain such closed formulas is
as follows. In deriving the closed form of the restrictions (21)
for the case in which the δ's are assumed known with certainty, the
Wiener-Kolmogorov prediction formula

$$E_t W_{t+i} = [\frac{\delta_w(L)^{-1}}{L^i}]_+ \; \delta(L) W_t$$

was used extensively. Here $[\sum_{j=-\infty}^{\infty} \alpha_j L^j]_+ = \sum_{j=0}^{\infty} \alpha_j L^j$, so that $[\;]_+$
means "ignore negative powers of L". The Wiener-Kolmogorov formula
is equivalent with the "chain rule" of forecasting (see Shiller [55]
or Sargent [54] for expositions). These equivalent forecasting rules
are known to be correct for the case in which the δ's are known with
certainty. However, as Chow [6] has pointed out, where there is
a nontrivial posterior density over the δ's, there is in general no
known closed form formula such as the above one for the i-step ahead
forecast. For example, it is not true that where δ is uncertain, the
correct expression for $E_t W_{t+i}$ is given by replacing the δ_{kj}'s with their
posterior means in the above formula. The fact that there is no closed
form prediction formula for sufficiently general cases implies that it is
impossible to derive closed form versions of decision rules (and hence
equilibria) that correspond to (21). As we shall see, for the kind of
empirical work we are advocating, it is important to have a closed form
for the mapping from the parameters of the objective functions (17) and
the dynamic constraints to the decision rule (20). From this viewpoint,
the suggestion that one ought to build a learning mechanism into rational

expectations models is not useful in suggesting practical *econometric* alternatives to the procedures recommended here.[32/]

Another drawback with incorporating learning is that, even if one could derive the decision rules in the face of uncertain δ's, the issue would arise of how to determine the prior used to initiate the learning model for the δ's. Would it be imposed a priori or estimated? If the initial prior were to be estimated, this would substantially complicate the estimation problem and add to the number of parameters.

Finally, in many settings the Bayesian learning model implies that the posterior distributions collapse about the true δ's as time passes without limit. In such settings, even if the researcher erroneously assumes that the δ's are known with certainty when in reality agents are learning about them in an optimal way, the researcher continues to obtain consistent estimators of the underlying parameters $\{A_0, A_1, A_2, f, d, \delta_W, \delta_D, \delta_\varepsilon, \delta_u\}$ using the methods described here and in Hansen [20] and Hansen and Sargent [23]. It does seem likely that by erroneously ignoring the phenomenon of learning about the δ's, the researcher is incorrectly calculating the asymptotic covariance matrix of his estimators. However, at present nothing is known about the nature of this error. Further, since we simply don't know how to compute optimum decision rules under the assumption that agents know the δ's with uncertainty, no consistent estimators of the underlying parameters have been proposed that incorporate agents' learning about the δ's in the optimal way, to say nothing of expressions for the associated asymptotic covariance matrices.

From the preceding considerations, I draw the conclusion that incorporating optimal Bayesian learning about the δ's on the part of agents is not a research avenue that soon promises appreciable dividends for the economist interested in applying dynamic competitive models of the sort described here.

A Model of the "Error Term"

We now derive a "dynamic supply curve" for the industry by using the industry-wide production function $Y_t = fK_t + \epsilon_t$ to eliminate K from (20) in favor of Y. Multiplying both sides of (20) by f then adding ϵ_{t+1} to both sides gives

$$Y_{t+1} = H_0 f + fH_w(L)W_t + fH_D(L)D_t$$

$$+ fH_\epsilon(L)\epsilon_t + fH_u(L)u_t$$

$$+ H_1 Y_t + \epsilon_{t+1} - H_1 \epsilon_t$$

Eliminating u_t by using $u_t = P_t - A_0 + A_1 Y_t - A_2 D_{1t}$ gives

$$(1.22) \quad Y_{t+1} = [H_0 f - fH_u(1)A_0] + fH_u(L)P_t + fH_w(L)W_t$$

$$+ [fH_D(L) - fH_u(L)A_2 \phi_D]D_t$$

$$+ [H_1 + fH_u(L)A_1]Y_t$$

$$+ [1 + fH_\epsilon(L)L - H_1 L]\epsilon_{t+1}$$

This can be written as

$$(1.23) \quad Y_{t+1} = S_0 + S_p(L)P_t + S_w(L)W_t$$

$$+ S_D(L)D_t + S_Y(L)Y_t + S_\epsilon(L)\epsilon_{t+1}$$

where

$$S_0 = H_0 f - fH_u(1)A_0$$

$$(1.24) \quad S_p(L) = fH_u(L)$$

$$S_w(L) = fH_w(L)$$

$$S_D(L) = [fH_D(L) \cdot - fH_u(L)A_2 \phi_D]$$

$$S_Y(L) = [H_1 + fH_u(L)A_1]$$

$$S_\epsilon(L) = [1 + fH_\epsilon(L)L - H_1L]$$

Recall that the demand curve is

$$(1.3) \qquad P_t = A_0 - A_1Y_t + A_2D_{1t} + u_t$$

Using $\delta_\epsilon(L)\epsilon_t = V_t^\epsilon$ and $\delta_u(L)u_t = V_t^u$, we can write the supply and demand curves as

$$(1.25) \qquad Y_{t+1} = S_0 + S_p(L)P_t + S_w(L)W_t + S_D(L)D_t$$

$$+ S_Y(L)Y_t + S_\epsilon(L)\delta_\epsilon(L)^{-1}V_{t+1}^\epsilon$$

$$(1.26) \qquad P_t = A_0 - A_1Y_t + A_2D_{1t} + \delta_u(L)^{-1}V_t^u$$

To discuss identification and estimation of the model, we need a theory about what is unknown to the econometrician. In constructing the model, we have taken the view that all of the variables on the right hand side of the supply and demand curves (23) and (3) (or equivalently (25) and (26)) are known to the representative firm. Thus, from the viewpoint of private agents, (25) and (26) describe exact linear functions of the right side variables in which there are no "random errors".[33/] The only tractable ways that have so far been discovered of introducing "random errors" into (25) and (26) have been to assume that the econometrician has less information than do the private agents. The smaller information set of the econometrician leads to what from his point of view are random terms

in relationships to be derived from (23) and (3) or (25) and (26).
The idea is to restrict the econometrician's information set relative
to that of private agents in a way both that is plausible and that
leads to a tractable statistical model of the error term. I shall
describe two models of the error term that can be constructed in
this way.

One model results from assuming that the econometrician has time
series on $\{P_t, W_t, D_t, Y_t\}$ but never observes the random processes
ε_t and u_t. On this interpretation, ε_t and u_t become random
terms in (23) and (3) from the econometrician's viewpoint.[34] In
constructing the model, we have already·imposed that V_{t+1}^{ε} is
orthogonal to all of the variables on the right side of (25), and
will assume that V_t^u is orthogonal to Y_t. We can also impose that
V_t^u is orthogonal to D_{1t}, if we wish,[35] although we might get by
with a weaker assumption.

The second model of the error term results from assuming that the
econometrician sees less of D_t and W_t than do private agents. It
is convenient to postpone a detailed discussion of this second model
of the error, and instead first to discuss identification and estimation
under the first model of the error term.

Identification and Estimation

With this model of the error terms, we can proceed to discuss identification and estimation. First, notice that every variable that appears on the right side of the demand schedule (26) also appears on the right side of the supply schedule (25). The dynamic economic theory leading to (25) makes the reason for this clear, since any variables that help predict future prices P will appear in the supply schedule of the representative firm. This immediately implies that any variables that help predict the demand shifters D_{1t} will appear in the supply schedule.[36/] The fact that no variables on the right side of the demand curve (26) are excluded from the supply schedule (25) means that if the supply schedule is to be identified, the source of identification must be found in restrictions of a kind different from the usual exclusion restrictions treated extensively in econometrics textbooks.[37/] According to the standard "order condition" for identification, equation (25) is hopelessly underidentified.[38/] So if the parameters of the model are to be identified, sources of prior information not of the exclusion variety must be available. The main source of these restrictions in the present model is the extensive body of cross equation restrictions embodied in equations (21) and (24). Equations (21) and (24) give the parameters of the supply schedule (25) as nonlinear functions of the parameters $\{A_0, A_1, A_2,$ f, d, β, n, $\delta_w(L)$, $\delta_D(L)$, $\delta_u(L)$, and $\delta_\varepsilon(L)\}$. In general, provided that the parameters r_D and r_w, which determine the order of the autoregressive processes for D and W and the parameters p and q, the number of elements in the vectors D and W, respectively, are large enough, these cross equation restrictions identify or over-

identify the parameters of the model. The strength of overidentification generally increases with increases in the orders r_D and r_w and the dimensions p and q. $\underline{39/}$

At this point it is useful to collect together the equations comprising the model as

(1.25) $Y_{t+1} = S_0 + S_p(L)P_t + S_w(L)W_t + S_D(L)D_t$

$$+ S_Y(L)Y_t + S_\epsilon(L) \delta_\epsilon(L)^{-1}v_{t+1}^\epsilon$$

(1.26) $P_t = A_0 - A_1Y_t + A_2D_{1t} + \delta_u(L)^{-1}v_t^u$

(1.8) $\delta_w(L)W_t = v_t^w$

(1.5) $\delta_D(L)D_t = v_t^D$

where

$$S_0 = +f^2 \lambda n \beta A_0 d^{-1} - fH_u(1)A_0$$

$$S_p(L) = +f^2 n \lambda \beta d^{-1} [\frac{L^{-1}(I - \delta_u(\lambda\beta)^{-1}\delta_u(L))}{1-\lambda\beta L^{-1}}]$$

$$S_w(L) = \frac{-n\lambda\beta f}{d} \phi_w [\frac{L^{-1}(I - \delta_w(\lambda\beta)^{-1}\delta_w(L))}{1-\lambda\beta L^{-1}}]$$

(27)

$$S_D(L) = + f^2 n \lambda \beta d^{-1}A_2\phi_D[\frac{L^{-1}(I - \delta_D(\lambda\beta)^{-1}\delta_D(L))}{1-\lambda\beta L^{-1}}]$$

$$- f^2 d^{-1}n\lambda\beta[\frac{L^{-1}(I - \delta_u(\lambda\beta)^{-1}\delta_u(L))}{1-\lambda\beta L^{-1}}] A_2\phi_D$$

$$S_Y(L) = \quad + f^2 d^{-1} n \lambda \beta A_1 \left[\frac{L^{-1}(I - \delta_u(\lambda\beta)^{-1}\delta_u(L))}{1 - \lambda\beta L^{-1}} \right] + \lambda$$

$$S_\varepsilon(L) = 1 - \frac{A_1 f^2 n \lambda \beta}{d} \left[\frac{I - \delta_\varepsilon(\lambda\beta)^{-1}\delta_\varepsilon(L)}{1 - \lambda\beta L^{-1}} \right] - \lambda L \cdot$$

$$(1 - (\lambda\beta)^{-1}L)(1 - \lambda L) = [1 - (1 + \beta^{-1} + A_1 f^2 n d^{-1})L + \beta^{-1}L^2].$$

Equations (25), (26), (8), and (5) form a statistical model for the joint process (P_t, Y_t, W_t, D_t). The model is linear in the variables, but is characterized by the extensive set of cross-equation restrictions described by (27). With the model of the error terms currently under discussion, the statistical model of the (P_t, Y_t, W_t, D_t) process has been spelled out sufficiently completely that we could write down the likelihood function for a sample (P_t, Y_t, W_t, D_t), $t = 1, \ldots, T$ assuming a normal probability density for $(V_t^W, V_t^u, V_t^D, V_t^\varepsilon)$.[40] Maximum likelihood estimates of the free parameters of the model $\{A_0, A_1, A_2, f, d, \delta_D(L), \delta_W(L), \delta_u(L), \delta_\varepsilon(L)\}$ could then be obtained. Computational details of such procedures are described by Sargent [51, 52] and Hansen and Sargent [23]. From the point of view of computing the estimates, it is a great practical advantage that (27) gives a set of closed-form formulas for the cross-equation restrictions imposed by the dynamic economic theory.

Application of Bayesian Methods

The fact that for the present model of the error terms it is possible to write down a normal likelihood function means that in principle Bayesian methods are applicable. Letting θ be the list of parameters of the model, and Z be the data, we have

$$f_{post}\{\theta \mid Z\} = \frac{\ell\{Z \mid \theta\} \; f_{prior}\{\theta\}}{f(Z)}$$

or

(1.28) $\quad f_{post}\{\theta \mid Z\} = \ell(Z \mid \theta) f(\theta) / \int \ell(z \mid \theta) \; f_{prior}(\theta) d\theta$

where $f_{post}\{\theta \mid Z\}$ denotes the posterior probability density, $f(Z)$ the probability density of Z, $f_{prior}\{\theta\}$ the prior density on θ, and $\ell\{Z \mid \theta\}$ the likelihood function. Measures of the location and dispersion of the posterior distribution of θ can be calculated, for example, by integrating $\theta^k \cdot f_{post}\{\theta \mid Z\}$ over θ for appropriate values of k. In the Bayesian view, the role of data analysis is to trace out in as revealing a way as possible the mapping defined by (28) from the prior to the posterior distribution. For such an analysis to be practical, it substantially eases matters if the mapping (28) can be characterized analytically, so that for example, posterior moments such as $\int \theta f_{post}(\theta \mid Z) d\theta$ can be calculated without the need to resort to numerical integration. Leamer [32] and Zellner [61] describe forms of prior densities $f_{prior}(\theta)$ that have the property that the mapping (28) is one that can be written as an analytic closed form when $\ell(Z \mid \theta)$ is the normal likelihood function.

In the context of dynamic economic models of the class represented by (25), (26), (8) and (5), the question of whether the mapping (28) can be characterized analytically hinges on which parameters one regards

as being in the list θ about which the researcher has formulated prior information. One possibility is that θ consists of the S's of (25), the A's of (26), and δ_w, δ_u and δ_D of (26), (8) and (5). With this interpretation of θ, then since (25), (26), (8), and (5) are linear in the S's, A's, δ_w and δ_D, it is possible to get analytic characterizations of the mapping from $f_{prior}(\theta)$ to $f_{post}(\theta|Z)$. For example, Shiller [55] and Leamer [31] have shown how priors of various forms on the S's in (25) can tractably be mapped into posteriors, in contexts where (25) is appropriately viewed as a regression equation. In effect, Leamer [31] and Shiller [55] provided formal Bayesian methods for imposing restrictions on lag distributions of a general kind, examples of which had long been imposed by applied econometricians. These restrictions usually corresponded to restrictions directly on our S_j's. Pre-dating the work of Shiller and Leamer were the restrictions on distributed lags proposed by Koyck [ ], Friedman [ ] and Cagan [ ], Jorgenson [ ], and Almon [ ]. There was also the frequently used identifying restriction that various distributed lag weights sum to unity.[41/] All of these approaches view the S's themselves as among the "free parameters" of the model about which the researcher can reasonably be imagined to have formed views summarized by a prior distribution.

Unfortunately, the tractability of the Leamer-Shiller approach is purchased at the cost of ignoring the essential aspects of the dynamic economic theory leading to (25). According to that theory, the S's are not free parameters, but are complicated functions of the parameters $\{A_0, A_1, A_2, f, d, \beta, n, \delta_w(L), \delta_D(L), \delta_u(L), \delta_\epsilon(L)\}$. It is _this_ list of parameters about which it seems most appropriate to expect an economist to have prior beliefs. The parameters $\{A_0, A_1, A_2, f, d\}$ are the parameters describing preferences and the technology,

about which the economic theorist may have some prior beliefs. The
economists' "prior beliefs" about the parameters $\{\delta_w, \delta_D, \delta_u, \delta_\varepsilon\}$
are presumably on a different theoretical footing from his beliefs
about $\{A_0, A_1, A_2, f, d\}$, since the former list simply characterizes
the serial correlation properties of the "shift variables" about which
economic theory itself suggests little, although casual general
observations may suggest a presumption in favor of high serial
correlation, at least in some types of variables. In any event,
it is the "deep" parameters $\{A_0, A_1, A_2, f, d, \beta, n, \delta_w, \delta_D, \delta_u, \delta_\varepsilon\}$
that must be estimated, if one is to build a model that
potentially overcomes Lucas's critique of econometric policy evaluation
procedures. $\underline{42/}$

When this list of "deep" parameters contains the objects of interest,
Bayesian analysis using (28) becomes much less tractable. This is
because the likelihood function $\ell(Z|\theta)$ becomes a very complicated
function of the free parameters in θ, by virtue of the complicated
nature of the cross-equation restrictions illustrated in (27). Although
Bayesian analysis is still possible, the researcher will be forced to
use numerical methods to characterize the mapping from the prior to the
posterior given in (28). For example, for a given prior, numerical
integration will have to be used to calculate the moments of the posterior
distribution. My own judgement is that given current computer technology,
formal Bayesian estimation procedures seem prohibitively expensive for
most members of the class of dynamic models considered here. This is
obviously not an objection to Bayesian methods in principle. However,
I believe that the high cost attached to applying Bayesian methods
correctly helps to explain why they have not yet been applied extensively
to estimating rational expectations models.

A Second Model of the "Error Term"

More serious limitations on the domain of Bayesian techniques emerge if the researcher embraces a second model of the error term, which we now discuss. In the second model of the error term, it is assumed that the econometrician possesses only observations on subsets $\bar{W}_t \subset W_t$ and $\tilde{D}_t \subset D_t$ of the information variables that private agents use to forecast future w_t's and D_{1t}'s.[43/] It is assumed that these subsets of information variables follow autoregressive processes

$$\tilde{\delta}_D(L)\tilde{D}_t = \tilde{V}_t^D$$

$$\tilde{\delta}_W(L)\tilde{W}_t = \tilde{V}_t^W$$

where $\bar{\delta}_D(L)$ and $\tilde{\delta}_W(L)$ are polynomials in the lag operator of of order $\bar{r}_D$ and $\bar{r}_W$, respectively. Then it turns that the equilibrium law of motion for capital (20) can be written in a form identical to (20), except that W, D, δ_D, δ_W, H_D, and H_W are to be replaced by the corresponding objects with tildes above them, and that there appears a random disturbance $\dot{\eta}_t$ on the right side of (20). The cross-equation restrictions (21) continue to characterize the objects with tildes over them.[44/] The random variable η_t can be shown to be orthogonal to all of the current and lagged values of $\tilde{W}$ and $\tilde{D}$.[45/] However, it turns out that η_t is in general serially correlated, with serial correlation properties that depend on the joint covariance properties of those variables in D_t and W_t that the econometrician does not have observations on. In the context of this setup, it is not even possible to write down the likelihood function without specifying details of the moments of information variables in D

and W that are unobservable to the econometrician. It would seem

attractive to adopt an estimation procedure that avoids the implicit

theorizing about the stochastic properties of the unobserved D's

and W's that an estimator using the likelihood function requires.

One such estimation strategy that exploits the orthogonality of

η to $\bar{D}$ and $\bar{W}$, without requiring all of the added details

required to write down a likelihood function, has been developed

by Lars Hansen [20]. The "generalized method of moments" estimators

of Hansen have the advantage of delivering estimators of the free

parameters whose desirable statistical properties don't depend on

any arbitrary assumptions about the serial correlation properties

of the η_t's.[46/] These generalized method of moments estimators

were invented precisely to handle situations in which the researcher

is substantially more confident of the orthogonality conditions

delivered by his theorizing than he is about the serial correlation

properties of the error. These methods construct statistically

consistent estimators, while avoiding the need to form the likelihood

function. However, in acknowledging that he does not have enough

information about the distrubances to construct the likelihood

function, the researcher looses the ability to employ Bayesian methods,

since knowledge of the likelihood function is essential for using

Bayes' law as in (28).

2. A Model of the Corn-Hog Cycle

This section illustrates in a general way how a model of an important phenomenon, the "corn-hog cycle", might be constructed in a fashion consistent with the principles recommended above. Technically, the corn-hog cycle is defined as the occurrence of distinctive patterns of serial correlation and cross-serial correlation between various measures of output flows, stocks, and prices of corn and hogs. A model of the behavior of such series might be useful either for the positive analysis of various tax and subsidy schemes designed to ameliorate the "cycle", or for the welfare of analysis of alternative policies. The simple model I shall describe is mainly directed at the first use, but might also be used for the second purpose.

The corn-hog example also illustrates an important methodological point. The Lucas-Prescott example described in section 1 has the feature that all the state variables W_t that help predict future values of the input price w_t appear in the equilibrium laws of motion for capital and output. This suggests that since the "hog industry" purchases corn, the state variables which characterize the corn market will appear in the equilibrium laws of motion for the state variables that characterize the hog market. Similar reasoning suggests that the corn market will inherit all of the state variables characterizing the hog market. To take this into account properly will require, for example, that for the hog industry the stochastic process for one of its inputs, which corresponds to $\delta_w(L)$ in (1.8), can no longer be regarded as determined from outside the model. Instead, the markets become intimately interrelated, and their equilibria must be defined jointly.

The purpose of this section is to illustrate the nontrivial modifications of the setup of section 1 that would be required for various "realistic" applications. The model of the corn-hog cycle described here still does not realistically describe the dynamics of the actual corn and hog industries. More realism could be attained at the cost of proliferating state variables, but at no real cost in terms of analytical difficulty. However, the example described here will convey the sense in which the internal logic of models of this class tends to push the analyst toward a general equilibrium approach.

We now define the following variables:

c_t = output of the representative corn farmer

m = number of corn farmers

C_t = total output of corn   ($C_t = mc_t$)

P_{ct} = price of corn

D_{1t}^c = a $(p_1^c \times 1)$ vector of random variables appearing in the demand schedule for final consumption of corn

D_{2t}^c = a $(p^c - p_1^c) \times 1$ vector of random variables that help predict the future values of the collection of random variables D_{1t}^c, $p^c \geq p_1^c$.

D_t^c = $\begin{bmatrix} D_{1t}^c \\ D_{2t}^c \end{bmatrix}$

k_{ct} = capital stock of representative corn farmer

K_{ct} = mk_{ct} = total capital stock in the corn industry.

w_t^c = real rental rate on capital in the corn industry.

W_t^c = a $(q^c \times 1)$ vector whose first element is w_t^c; the remaining variables of W_t^c are variables that help predict future values of w^c.

h_t = flow output of hogs of the representative hog farmer.

n = number of hog farmers.

H_t = total output of hogs $(H_t = nh_t)$.

c_{ht} = consumption of corn by hogs at the representative hog farm.

k_{ht} = number of hogs at the representative hog farm.

K_{ht} = total number of hogs at hog farms $K_{ht} = nk_{ht}$.

$w_t^h k_{ht}$ = miscellaneous expenses needed to maintain k_{ht} hogs at the representative hog farm.

W_t^h = a $(q^h \times 1)$ vector whose first element is w_t^h; the remaining variables of W_t^h are variables that help predict future values of w^h.

P_{ht} = flow price of hogs

D_{1t}^h = a $(p_1^h \times 1)$ vector of random variables appearing in the demand schedule for final consumption of hogs.

D_{2t}^h = a $(p^h - p_1^h)$ vector of random variables that help predict the future values of the collection of variables D_1^h, $p^h \geq p_1^h$.

$$D_t^h = \begin{pmatrix} D_{1t}^h \\ D_{2t}^h \end{pmatrix}$$

ε_{ct} = a random shock in the production function for corn.

ε_{ht} = a random shock in the production technology for hog farming.

u_{ct} = a random shock to the demand for final consumption of corn.

u_{ht} = a random shock to the flow demand for consumption of hogs.

We further define the following polynomials in the lag operator

L:

$$\delta_u^c(L) = 1 - \sum_{j=1}^{r_u^c} \delta_{uj}^c L^j, \qquad \text{where } \delta_{uj}^c \text{ is a scalar}$$

$$\delta_u^h(L) = 1 - \sum_{j=1}^{r_u^h} \delta_{uj}^h L^j, \qquad \text{where } \delta_{uj}^h \text{ is a scalar}$$

$$\delta_D^c(L) = I_{p^c} - \sum_{j=1}^{r_D^c} \delta_{Dj}^c L^j, \qquad \text{where } \delta_{Dj}^c \text{ is a } p^c \times p^c \text{ matrix, and}$$
$$I_{p^c} \text{ is a } (p^c \times p^c) \text{ identity matrix}$$

$$\delta_D^h(L) = I_{p^h} - \sum_{j=1}^{r_D^h} \delta_{Dj}^h L^j, \qquad \text{where } \delta_{Dj}^h \text{ is a } p^h \times p^h \text{ matrix and}$$
$$I_{p^h} \text{ is the } (p^h \times p^h) \text{ identity matrix}$$

$$\delta_\varepsilon^c(L) = 1 - \sum_{j=1}^{r_\varepsilon^c} \delta_{\varepsilon j}^c L^j, \qquad \text{where } \delta_{\varepsilon j}^c \text{ is a scalar}$$

$$\delta_\varepsilon^h(L) = 1 - \sum_{j=1}^{r_\varepsilon^h} \delta_{\varepsilon j}^h L^j, \qquad \text{where } \delta_{\varepsilon j}^h \text{ is a scalar.}$$

$$\delta_w^c(L) = I_{q^c} - \sum_{j=1}^{r_w^c} \delta_{wj}^c L^j, \qquad \text{where } \delta_{wj}^c \text{ is a } (q_c \times q_c) \text{ matrix.}$$

$$\delta_w^h(L) = I_{q^h} - \sum_{j=1}^{r_w^h} \delta_{wj}^h L^j, \qquad \text{where } \delta_{wj}^h \text{ is a } (q_h \times q_h) \text{ matrix.}$$

I shall assume that for each polynomial δ the zeroes of $\det(z)$ exceed $\sqrt{\beta}$ in modulus, where β is the discount factor introduced below.

There are m identical corn farmers, each of whom maximizes

$$(2.1) \quad E_0 \sum_{t=0}^{\infty} \beta^t \{ P_{ct} c_t - w_t^c k_{ct} - \frac{d}{2}(k_{ct+1} - k_{ct})^2 \}, \qquad d > 0$$
$$0 < \beta < 1$$

where P_{ct} is the price of corn, c_t is output of corn, k_{ct} is

the capital stock of the corn producer, and w_t^c is the rental rate
on capital in the corn-growing industry. Output of corn obeys

$$(2.2) \qquad c_t = fk_{ct} + m^{-1}\varepsilon_{ct}, \qquad\qquad f > 0$$

where ε_{ct} is a random process. The productivity shock ε_{ct} is
assumed to follow the r_ε^c th order Markov process

$$(2.3) \qquad \delta_\varepsilon^c(L)\varepsilon_{ct} = V_{\varepsilon t}^c$$

where $V_{\varepsilon t}^c$ is a fundamental white noise for ε_{ct}. The corn producer
faces the stochastic processes for P_{ct} and w_t^c as a price taker.
The rental w_t is assumed to be the first element of a $q^c \times 1$ vector
random process that obeys the r_w^c th order vector Markov process

$$(2.4) \qquad \delta_w^c(L)W_t^c = V_{wt}^c$$

where V_{wt}^c is a $(q \times 1)$ vector white noise that is fundamental
for W_t^c. The assumptions about the stochastic process for P_{ct} and
the representative firm's perceptions of it will be filled in later.

The hog industry consists of n identical producers each of whom
maximizes

$$(2.5) \qquad E_0 \sum_{t=0}^{\infty} \beta^t \{P_{ht}h_t - w_t^h k_{ht} - e/2 \ (k_{ht+1} - k_{ht})^2 - P_{ct}c_{ht}\}, \ e > 0,$$

where P_{ht} is the price of hogs, c_{ht} is the consumption of corn by
k_{ht} hogs, k_{ht} is the number of hogs, h_t is sales of hogs, and $w_t^h k_{ht}$
is miscellaneous expenses required to maintain k_{ht} hogs. The technology
is assumed to be

$$c_{ht} = \gamma k_{ht}, \qquad\qquad \gamma > 0$$

(2.6)
$$h_t = (1 + \phi)k_{ht} - k_{ht+1} + n^{-1}\varepsilon_{ht}, \qquad \phi > 0$$

where ϕ is determined by the reproduction rate of pigs, which is assumed exogenous here, but would be a decision variable in a more realistic analysis. In (5), the term $e/2 \, (k_{ht+1} - k_{ht})^2$ represents costs of adjusting the number of pigs rapidly. The productivity shock ε_{ht} follows the r_ε^h th order process

(2.7) $\qquad \delta_\varepsilon^h(L)\varepsilon_{ht} = V_{\varepsilon t}^h$

where $V_{\varepsilon t}^h$ is a fundamental white noise for ε_{ht}. The price w_t^h is the first element of a $(q^h \times 1)$ vector W_t^h that follows the r_w^h th order vector Markov process

(2.8) $\qquad \delta_w^h(L)W_t^h = V_{wt}^h$

where V_{wt}^h is a fundamental white noise for W_t^h.

The flow demand for hogs is given by

(2.9) $\qquad P_{ht} = A_0 - A_1 H_t + A_2 D_{1t}^h + u_{ht}; \qquad A_0 > 0, \; A_1 > 0$

where A_0, A_1 are scalars, A_2 is a $(1 \times p_1^h)$ vector, $H_t = nh_t$, and where u_{ht} is a stochastic shock to demand that obeys the r_u^h th order Markov process

(2.10) $\qquad \delta_u^h(L)u_{ht} = V_{ut}^h$

where V_{ut}^h is fundamental for u_{ht}. The vector of demand shifters D_{1t}^h is $(p_1^h \times 1)$ and consists of the first p_1^h elements of the

$(p^h \times 1)$ vector D_t^h. It is assumed that D_t^h evolves according to the r_D^h th order vector Markov process

$$(2.11) \qquad \delta_D^h(L)D_t^h = V_{Dt}^h$$

where V_{Dt}^h is a $(p^h \times 1)$ vector white noise that is fundamental for D_t^h.

The demand for corn is the sum of the demand derived from hog production, $C_{ht} = nc_{ht}$, and the demand for final consumption, C_{ct}. The demand for final consumption obeys

$$(2.12) \qquad C_{ct} = B_0 - B_1 P_{ct} + B_2 D_{1t}^c + u_{ct}, \qquad\qquad B_0, B_1 > 0$$

where B_0 and B_1 are scalars, where B_2 is a $(1 \times p_1^c)$ vector, where u_{ct} is a demand shock that obeys the r_u^c th order Markov process

$$(2.13) \qquad \delta_u^c(L)u_{ct} = V_{ut}^c \quad ,$$

where V_{ut}^c is a fundamental white noise for u_{ct}. The $(p_1^c \times 1)$ vector of corn-demand shifters D_{1t}^c consists of the first p_1^c elements of a $(p^c \times 1)$ vector D_t^c that follows the r_D^c th order Markov process

$$(2.14) \qquad \delta_D^c(L)D_t^c = V_{Dt}^c$$

where V_{ct}^D is a $(p^c \times 1)$ vector white noise that is fundamental for D_t^c. The hog-derived demand for corn, from (6), is

$$C_{ht} = \gamma K_{ht}$$

where $K_{ht} = nk_{ht}$ and $K_{ct} = mk_{ct}$. The equilibrium condition in the market for corn is

$$C_{ct} + C_{ht} = fK_{ct} + \varepsilon_{ct}$$

or

$$B_0 - B_1 P_{ct} + B_2 D_{1t}^c + u_{ct} + \gamma K_{ht} = f K_{ct} + \varepsilon_{ct}$$

which implies that

$$(2.15) \qquad P_{ct} = B_1^{-1} [\gamma K_{ht} + u_{ct} + B_0 + B_2 D_{1t}^c - f K_{ct} - \varepsilon_{ct}]$$

The farmers in each market need to form views about future prices of corn and hogs in order to solve the maximum problems (5) and (1). Since future corn and hog prices will depend on future state variables in each market, including the capital stocks in each market, farmers in each market need to form a view about the laws of motion of the market-wide stocks of capital in both markets. We assume that farmers' views about these laws of motion are correct. It will turn out that the laws of motion for the market-wide capital stocks in the two industries will have the forms

$$
\begin{aligned}
K_{ct+1} = {}& H_0 + H_w^c(L) W_t^c + H_w^h(L) W_t^h + H_D^c(L) D_t^c \\
& + H_D^h(L) D_t^h + H_\varepsilon^c(L) \varepsilon_{ct} + H_\varepsilon^h(L) \varepsilon_{ht} \\
& + H_u^c(L) u_{ct} + H_u^h(L) u_{ht} + H_1^c K_{ct} + H_1^h K_{ht}
\end{aligned}
$$

(2.16)

$$
\begin{aligned}
K_{ht+1} = {}& G_0 + G_w^c(L) W_t^c + G_w^h(L) W_t^h + G_D^c(L) D_t^c \\
& + G_D^h(L) D_t^h + G_\varepsilon^c(L) \varepsilon_{ct} + G_\varepsilon^h(L) \varepsilon_{ht} \\
& + G_u^c(L) u_{ct} + G_u^h(L) u_{ht} + G_1^c K_{ct} + G_1^h K_{ht}
\end{aligned}
$$

(2.17)

where H_0, H_1^c, H_1^h, G_0, G_1^c, G_1^h are scalars and where

$$H_w^c(L) \quad \text{and} \quad G_w^c(L) \quad \text{are} \quad r_w^c - 1 \quad \text{order polynomials in} \quad L.$$

$$H_w^h(L) \quad \text{and} \quad G_w^h(L) \quad \text{are} \quad r_w^h - 1 \quad \text{order polynomials in} \quad L.$$

$$H_D^c(L) \quad \text{and} \quad G_D^c(L) \quad \text{are} \quad r_D^c - 1 \quad \text{order polynomials in} \quad L.$$

$$(2.18) \quad H_D^h(L) \quad \text{and} \quad G_D^h(L) \quad \text{are} \quad r_D^h - 1 \quad \text{order polynomials in} \quad L.$$

$$H_\varepsilon^c(L) \quad \text{and} \quad G_\varepsilon^c(L) \quad \text{are} \quad r_\varepsilon^c - 1 \quad \text{order polynomials in} \quad L.$$

$$H_\varepsilon^h(L) \quad \text{and} \quad G_\varepsilon^h(L) \quad \text{are} \quad r_\varepsilon^h - 1 \quad \text{order polynomials in} \quad L.$$

$$H_u^c(L) \quad \text{and} \quad G_u^c(L) \quad \text{are} \quad r_u^c - 1 \quad \text{order polynomials in} \quad L.$$

$$H_u^h(L) \quad \text{and} \quad G_u^h(L) \quad \text{are} \quad r_u^h - 1 \quad \text{order polynomials in} \quad L.$$

Each of the $H(L)$ and $G(L)$ polynomials is conformable in dimension with the process it premultiplies. Equations (16) and (17) describe farmers' perceptions of the laws of motion for K_c and K_h.

The important thing to notice about (16) and (17) is that any variable that appears on the right hand side of one appears on the right side of the other. Thus, the <u>state</u> variables are asserted to be identical for the two industries' aggregate capital stocks. That (16) and (17) embody the correct choices of state variables will now become evident, as we express the optimum problems of the representative firm in each industry as completely stated dynamic optimization problems. That is, (16) and (17) will turn out to have the correct right hand side variables in the sense that if firms' <u>perceptions</u> of the laws of motion for K_c and K_h are (16) and (17), respectively, then firms' optimizing behavior will imply <u>actual</u> laws of motion for K_c and K_h, respectively, of the same functional forms as assumed for those perceptions.

From (6) we have that $H_t = (1+\phi)K_{ht} - K_{ht+1} + \varepsilon_{ht}$. Substituting this into the demand schedule for hogs gives,

$$(2.19) \quad P_{ht} = A_0 - A_1((1 + \phi)K_{ht} - K_{ht+1}) + A_2 D_{1t}^h + u_{ht} - A_1 \varepsilon_{ht}$$

Substituting (19) and (15) into (1) and (5) gives the following criterion functions for representative hog farmers and corn farmers, respectively:

$$E_o \sum_{t=0}^{\infty} \beta^t \{ [A_0 - A_1((1 + \phi)K_{ht} - K_{ht+1}) + A_2 D_{1t}^h + u_{ht} - A_1 \varepsilon_{ht}]$$

$$(2.20) \qquad [(1+\phi)k_{ht} - k_{ht+1} + \varepsilon_{ht}] - w_t^h k_{ht} - \frac{e}{2}(k_{ht+1} - k_{ht})^2$$

$$- \gamma k_{ht}[B_1^{-1}(\gamma K_{ht} + u_{ct} + B_0 + B_2 D_{1t}^c - fK_{ct} - \varepsilon_{ct})] \}$$

$$(2.21) \quad E_o \sum_{t=0}^{\infty} \beta^t \{ B_1^{-1}(\gamma K_{ht} + u_{ct} + B_0 + B_2 D_{1t}^c - fK_{ct} - \varepsilon_{ct})fk_{ct}$$

$$- w_t^c k_{ct} - \frac{d}{2}(k_{ct+1} - k_{ct})^2 \}$$

The maximization of (20) and (21) each takes place subject to the known laws of motion

$$(2.4) \qquad \delta_w^c(L)W_t^c = V_{wt}^c$$

$$(2.8) \qquad \delta_w^h(L)W_t^h = V_{wt}^h$$

$$(2.10) \qquad \delta_u^h(L)u_{nt} = V_{ut}^h$$

$$(2.11) \qquad \delta_D^h(L)D_t^h = V_{Dt}^h$$

$$(2.13) \qquad \delta_u^c(L)u_{ct} = V_{ut}^c$$

$$(2.14) \qquad \delta_D^c(L)D_t^c = V_{Dt}^c$$

$$(2.3) \qquad \delta_\varepsilon^c(L)\varepsilon_{ct} = V_{\varepsilon t}^c$$

$$(2.7) \qquad \delta_\varepsilon^h(L)\varepsilon_{ht} = V_{\varepsilon t}^h$$

$$K_{ct+1} = H_0 + H_w^c(L)W_t^c + H_w^h(L)W_t^h + H_D^c(L)D_t^c$$

(2.16)
$$+ H_D^h(L)D_t^h + H_\epsilon^c(L)\epsilon_{ct} + H_\epsilon^h(L)\epsilon_{ht}$$

$$+ H_u^c(L)u_{ct} + H_u^h(L)u_{ht} + H_1^c K_{ct} + H_1^h K_{ht}$$

$$K_{nt+1} = G_0 + G_w^c(L)W_t^c + G_w^h(L)W_t^h + G_D^c(L)D_t^c$$

(2.17)
$$+ G_D^h(L)D_t^h + G_\epsilon^c(L)\epsilon_{ct} + G_\epsilon^h(L)\epsilon_{ht}$$

$$+ G_u^c(L)u_{ct} + G_u^h(L)u_{ht} + G_1^c K_{ct} + G_1^h K_{ht}$$

The optimization for the representative firm in each industry is
carried out subject to the information set at time t,[47]/

$$\{K_{ht}, K_{ct}, W_t^h, W_{t-1}^h, \ldots, W_{t-r_w^h+1}^h, D_t^h, \ldots, D_{t-r_D^h+1}^h,$$

$$\epsilon_t^h, \ldots, \epsilon_{t-r_\epsilon^h+1}^h, u_{ht}, \ldots, u_{ht-r_u^h+1}, W_t^c, \ldots, W_{t-r_w^c+1}^c,$$

$$D_t^c, \ldots, D_{t-r_D^c+1}^c, \epsilon_{ct}, \ldots, \epsilon_{ct-r_\epsilon^c+1}, u_{ct}, \ldots, u_{ct-r_u^c+1}\}$$

In addition, at time t the firm knows its own current stock of capital
k_{ct} or k_{ht}, as it chooses k_{ct+1} or k_{ht+1}.

The solutions to the problems of the constrained maximization of
(20) and (21) are a pair of linear contingency plans of the form

$$k_{ht+1} = g_0 + g_w^c(L)W_t^c + g_w^h(L)W_t^h + g_D^c(L)D_t^c$$

(2.22)
$$+ g_D^h(L)D_t^h + g_\epsilon^c(L)\epsilon_{ct} + g_\epsilon^h(L)\epsilon_{ht}$$

$$+ g_u^c(L)u_{ct} + g_u^h(L)u_{ht} + g_1^c K_{ct} + g_1^h K_{ht}$$

$$+ g_2^h k_{ht}$$

$$k_{ct+1} = h_0 + h_w^c(L)W_t^c + h_w^h(L)W_t^h + h_D^c(L)D_t^c$$

$$+ h_D^h(L)D_t^h + h_\epsilon^c(L)\epsilon_{ct} + h_\epsilon^h(L)\epsilon_{ht}$$

$$+ h_u^c(L)u_{ct} + h_u^h(L)u_{ht} + h_1^c K_{ct} + h_1^h K_{ht}$$

$$+ h_2^c k_{ct}$$

where h_2^c and g_2^h are scalars and the remaining constants and polynomials in L are of the same dimensions and orders as the corresponding capital lettered quantities in (16) and (17), as defined in (18).

Multiplying (22) by n and (23) by m gives

$$K_{ht+1} = ng_0 + ng_w^c(t)W_t^c + ng_w^h(L)W_t^h + ng_D^c(L)D_t^c$$

$$+ ng_D^h(L)D_t^h + ng_\epsilon^c(L)\epsilon_{ct} + ng_\epsilon^h(L)\epsilon_{ht}$$

(2.24)

$$+ ng_u^c(L)u_{ct} + ng_u^h(L)u_{ht}$$

$$ng_1^c K_{ct} + (ng_1^h + g_2^h)K_{ht}$$

$$K_{ct+1} = mh_0 + mh_w^c(L)W_t^c + mh_w^h(L)W_t^h + mh_D^c(L)D_t^c$$

(2.25)

$$+ mh_D^h(L)D_t^h + mh_\epsilon^c(L)\epsilon_{ct} + mh_\epsilon^h(L)\epsilon_{ht}$$

$$+ mh_u^c(L)u_{ct} + mh_u^h(L)u_{ht} + mh_1^h K_{ht}$$

$$+ (mh_1^c + h_2^c)K_{ct}$$

Equations (24) and (25) are the <u>actual</u> laws of motion for K_{ht+1}
and K_{ct+1}, respectively, that correspond to the <u>perceived</u> laws of
motion (17) and (16), respectively. A <u>rational expectations equilibrium</u>
is defined as a pair of schedules (16) and (25) which are identically
equal, and a pair of schedules (17) and (24) which are identically
equal. This amounts to equating the perceived and actual laws of
motion.

Inspection of the criterion functions (20) and (21) and the
constraints (4), (8), (10), (11), (13), (14), (3), and (7) will
convince the reader that the correct state variables have been included
in our "guesses" about the forms of the laws of motion (16) and (17)
for the aggregate capital stocks K_h and K_c. The correct principle
for the selection of these state variables is this: any state variables
that appear either in the objective functions (20) and (21) or in the
constraints (4), (8), (10), (11), (13), (14), (3), and (7) at time
t + 1, belong in the laws of motion for the aggregate capital stocks
of both industries.

Having defined a rational expectations equilibrium for the model,
I now turn to describe how the equilibrium can be calculated via the
method corresponding to that used by Lucas and Prescott [37]. As in
section 1, the idea is to "discover" a fictitious social planning problem
that the equilibrium implicitly solves. First, notice that since the
flow demand curve for hogs is $P_{ht} = A_0 - A_1 H_t + A_2 D^h_{1t} + u_{ht}$, the area under
the demand curve for hogs is

$$\int_0^{H_t} (A_0 - A_1 x + A_2 D^h_{1t} + u_{ht}) dx$$

$$= A_0 H_t - \frac{A_1}{2} H_t^2 + H_t A_2 D^h_{1t} + H_t u_{ht}$$

$$= [(1 + \phi)K_{ht} - K_{ht+1} + \varepsilon_{ht}][A_0 + A_2 D_{1t}^h + u_{ht} - \frac{A_1}{2}((1 + \phi)K_{ht} -$$

$$- K_{ht+1} + \varepsilon_{ht})]$$

Since the demand curve for final consumption of corn is $P_{ct} = B_1^{-1}[B_0 + B_2 D_{1t}^c + u_{ct} - C_{ct}]$, the area under the demand curve for final consumption of corn is

$$(2.27) \qquad \int_0^{C_{ct}} B_1^{-1}[B_0 + B_2 D_{1t}^c + u_{ct} - x]dx =$$

$$B_0 B_1^{-1} C_{ct} + B_1^{-1} C_{ct} B_2 D_{1t}^c + B_1^{-1} u_{ct} C_{ct} - (2B_1)^{-1} C_{ct}^2$$

Substituting for C_{ct} from the equilibrium condition $C_{ct} = fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}$ gives the following formula for the area under the demand curve for the final consumption of corn:

$$(2.28) \qquad B_0 B_1^{-1}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}] + B_1^{-1} u_{ct}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}]$$

$$-(2B_1)^{-1}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}]^2 + B_1^{-1}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}]B_2 D_{1t}^c$$

Now consider the following fictitious social planning criterion:

$$(2.29) \qquad E_0 \sum_{t=0}^{\infty} \beta^t \{\{[(1 + \phi)K_{ht} - K_{ht+1} + \varepsilon_{ht}][A_0 + A_2 D_{1t}^h + u_{ht}$$

$$- \frac{A_1}{2}((1 + \phi)K_{ht} - K_{ht+1} + \varepsilon_{ht})]\}$$

$$+ \{B_0 B_1^{-1}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}] + B_1^{-1} u_{ct}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}]$$

$$- (2B_1)^{-1}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}]^2 + B_1^{-1}[fK_{ct} + \varepsilon_{ct} - \gamma K_{ht}]B_2 D_{1t}^c\}$$

$$- w_t^c K_{ct} - w_t^h K_{ht} - \frac{d}{2} m^{-1} (K_{ct+1} - K_{ct})^2$$

$$- \frac{e}{2} n^{-1} (K_{ht+1} - K_{ht})^2\} .$$

Consider the maximization of (29), assumed to take place subject to the laws of motion 4, 8, 10, 11, 13, 14, 3, and 7. The information set is at time t consists of observations on K_h, K_c, ε_c, ε_h, u_c, u_h, W^c, W^h, D^c, D^h dated t and earlier. At time t, the fictitious planner is assumed to set K_{ht+1} and K_{ct+1} as linear functions of the variables in the information set at t. The reader can verify that these linear contingency plans will be of precisely the same linear forms as the laws of motion (16) and (17) that are the laws of motion for K_c and K_h in the rational expectations equilibrium. Furthermore, by a straightforward argument, it can be established directly that the laws of motion for K_c and K_h that maximize the social planning criterion (29) are precisely the unique laws of motion for the rational expectations equilibrium.[48/]

The problem of maximizing the fictitious social planning criterion (29) subject to the constraints stated above is known in general to have a solution in which all the relevant histories of the state variables $\{K_h, K_c, \varepsilon_c, \varepsilon_h, u_c, u_h, W^c, W^h, D^c, D^h\}$ appear in the optimal law of motion for <u>both</u> K_c and K_h. Computationally efficient methods for obtaining solutions for a general class of "interrelated factor demand" problems, of which (29) is a special case, have been extensively described by Hansen and Sargent [22].

By a series of manipulations which parallel those used in section 2, it is possible to move from the laws of motion (16) and (17) for the aggregate capital stocks to dynamic flow supply curves for the corn and hog industries. These dynamic supply curves take the form

$$C_{t+1} = S_0 + S_w^c(L)W_t^c + S_w^h(L)W_t^h + S_D^c(L)D_t^c$$

$$(2.30) \qquad + S_D^h(L)D_t^h + S_\varepsilon^c(L)\varepsilon_{ct} + S_\varepsilon^h(L)\varepsilon_{ht}$$

$$+ S_p^c(L)P_{ct} + S_p^h(L)P_{ht} + S_1^c K_{ct} + S_1^h K_{ht}$$

and

$$H_t = \sigma_0 + \sigma_w^c(L)W_t^c + \sigma_w^h(L)W_t^h + \sigma_D^c(L)D_t^c$$

$$(2.31) \qquad + \sigma_D^h(L)D_t^h + \sigma_\varepsilon^c(L)\varepsilon_{ct} + \sigma_\varepsilon^h(L)\varepsilon_{ht}$$

$$+ \sigma_p^c(L)P_{ct} + \sigma_p^h(L)P_{ht} + \sigma_1^c K_{ct} + \sigma_1^h K_{ht}$$

In (30) and (31) all polynomials in the lag operator L are comparable in dimension and order with the corresponding polynomials that appear in (16) and (17), respectively.   The point to be emphasized is that each of the dynamic supply curves (30) and (31) inherits as arguments all of the random variables that appear as demand shifters in the flow demand curves for both commodities.

As in the development in section 1, equations (30) and (31) are relationships that are exact linear functions of the variables in the information sets of private agents. They can be converted to equations that contain what are random errors from the viewpoint of the econometrician by one of the devices described in section 1. For example, we might assume that the econometrician has data on all of the variables on the right sides of (30) and (31), except for current and past values of the shocks in the production relations, ε_{ct} and ε_{ht}. Under this interpretation, the distributed lags in ε_{ct} and ε_{ht} become the error terms from the econometrician's perspective.

It is evident from (30) and (31) that the dynamic economic theory that we are using subverts the usual identification conditions of the exclusion variety. Because of the appearance of all of the state variables in the dynamic supply curves (30) and (31), the supply schedules for hogs and corn are underidentified according to the usual order condition. However, as in section 1, the dynamic theory from which these curves are derived furnishes an extensive set of restrictions between the parameters of (30) and (31) and the parameters of (4), (8), (10), (11), (13), (14), (3) and (7). These latter parameters are the parameters of the Markov laws for the driving variables.

Although the model of this section admittedly embodies oversimplified technologies, it could readily be extended to incorporate much more "realistic" technologies for the corn and hog industries, even while staying within the linear-quadratic framework.[49/] Nerlove, Grether and Caravalho [49] provide useful indications about how this might be done. Such generalizations can be accomplished at the cost of increases in the number of "state" variables required, but involve no genuine analytical complications beyond those encountered in our corn-hog model. Increasing the number of factors of production in each industry can also be accomplished in a conceptually straightforward manner. Hansen and Sargent [22] have devoted substantial effort to the development of fast methods of computing linear rational expectations equilibria for a wide class of models. The use of these methods will make it practical to construct models of interrelated industries with state vectors of at least moderate dimension.

The Lucas-Prescott example of section 1 and the corn-hog example illustrate a key principle: laws of motion for the endogenous state variables of one industry will generally be functions of all of the state variables characterizing the industries with which it interacts.

The analyst will often face a hard practical decision about which dynamics he takes as given from outside the model. To illustrate this further, suppose that one of the shifters D_{1t}^c in the demand schedule for corn is the price of wheat, and that the price of corn is also an argument in the demand schedule for wheat. It is then easy to imagine a rational expectations equilibrium model of the corn, hog, and wheat markets. The general point is that a rational expectations equilibrium of industries that use each others outputs as inputs is technically equivalent with the solution of an interrelated factor demand problem in which the laws of motion for states characterizing each one of the industries dynamically interact in interesting ways with the states of all the other industries. Thus, the internal logic of these models tends to propel the analyst toward an even more extended general equilibrium formulation. In any given application, the researcher will have to choose what laws of motion he takes as given from outside the model, for the purposes of the analysis at hand. As our examples indicate, the choice made will influence substantially the definition of equilibrium to be employed.

Superficially, the corn-hog model of this section may seem very much more complicated and difficult to estimate than the Lucas-Prescott example of section 1. However, to some extent this impression is illusory, as the number of free parameters of the model of this section is roughly of the order of magnitude of twice the number for the Lucas-Prescott example. The lag distributions in (2.24) and (2.25) turn out to be highly restricted by the theory. Hansen and Sargent [ ] have described formulas for describing these restrictions in a way that is convenient both for analysis and estimation. As a result of such research, it seems feasible to estimate models with sizable state-spaces and complicated interrelationships across industries.

The theoretical and econometric tractability of dynamic systems with large state spaces is the chief virtue of the linear-quadratic setups adopted in this paper. These linear-quadratic setups admittedly sacrifice generality vis-a-vis the methods used by Lucas-Prescott [ ] or Sargent [ ] in terms of admissible functional forms for objectives and constraints, but for practical applications they permit greater generality vis-a-vis those same alternatives in the sense of easily accommodating high-order serial dependence in the "forcing functions". A major goal of recent research by Hansen and Sargent [ ] and Taylor [ ] is to extend the limits of the dimension of models that can be analyzed and estimated by the above methods.

3. Conclusions

Re-making dynamic econometric practice so that it is consistent with the principle that agents' constraints influence their behavior is a task that is far from finished. Further, properly allowing for the implications of the principle will surely require abandoning many presently received ways of interpreting data. A variety of setups can be imagined that are consistent with the principle. For example, a variety of variations of the setups of this paper can be imagined in which agents optimize, but have smaller information sets than have been attributed to them here. Also, information discrepancies across classes of agents can be assumed. In many such cases, endogenous variables such as prices will play an important role in conveying information to agents. In models with dynamics as complicated as those of our examples, these variations introduce substantial analytical difficulties. To date there is very little work which investigates the econometric implications of such complications to setups like ours. With or without these complications, building a dynamic econometrics that is consistent with our simple principle from economic theory is a challenging task. It is sure to require substantial changes in the ways that applied economists interpret economic time series.

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FOOTNOTES

1. Examples of work in the general line are Taylor [59, 60], Meese [41], Kennan [27], Crawford [9], Blanco [5], Huntzinger [25], Nerlove, Grether and Carvalho [49], Craine [8], Geweke [18], Hansen and Sargent [21, 22, 23], and Sargent [51, 52].

2. The papers by Paul Anderson [ ] and Frederick Mishkin [ ] seem at least partly motivated by this interpretation.

3. The set of ideas I discuss in this paper has perhaps received most notoriety in the context of macroeconomic examples. In particular, substantial attention has been devoted to the sample economies of Lucas [ ] and Sargent and Wallace [ ] in which those systematic nonneutralities that come from imputing persistently suboptimal expectations to agents were shown to disappear when the hypothesis of rational expectations was imposed on agents. Crudely put, certain classes of systematic monetary policies, in particular those which operate solely via deception, were rendered impotent in the Lucas and Sargent and Wallace examples. Since the publication of these papers, many papers have been published that have described setups in which the choice of systematic policy matters, even when rational expectations prevail. These papers usually invoke a source of nonneutrality not based on deception, of which there are many in standard macroeconomic theory. Papers of this class have often been interpreted as providing a defense of "pre-rational expectations" activist policies along lines that were produced by calculating optimal controls for Keynesian econometric models of the style of the late 1960's. In fact, no such defense is implied partly because the methods by which optimal controls for government policy variables are calculated is very different in all rational expectations models from the procedures that were applied to pre-rational expectations models, but also because the

ways in which econometric estimates are to be constructed for rational

expectations models, with or without neutralities, differ substantially

from the methods applied to the Keynesian models of the 1960's. The

main point of the Lucas [ ] and Sargent and Wallace [ ] examples is

that substituting the assumption of rational expectations for "adaptive"

expectations makes a critical difference for the methods both by which

we should evaluate and optimally choose government policies. That

same message is present in the papers of Phelps and Taylor [ ], Fischer

[ ], and Hall [ ], even if superficially the differences in some

qualitative features of the optimal policies under the two assumptions

on expectations may have seemed less dramatic than in Sargent and

Wallace's example or Lucas's.

4. Using the methods of discounted dynamic programming (e.g. Blackwell [4]),

theoretical results establishing existence and uniqueness of equilibria

and various qualitative features of the equilibria can often be obtained for

"weak" or "general" assumptions such as that utility is concave, constraint

sets are convex and monotone in shift variables, and so on. Lucas and

Prescott [37] and Lucas [36] give interesting illustrations of these methods.

These techniques were also used by Sargent [53] to make some general

observations on interpreting time series correlations between "Tobin's q"

variable and the aggregate rate of investment. However, for applied work,

it is necessary to be able to calculate equilibria as a function of the free

parameters of preferences and constraints, and it is highly desirable if the

equilibria can be calculated easily. While for general functional forms,

it is in principle possible to calculate equilibria of recursive competitive

models using a contraction mapping, in practice such methods are presently

too expensive to use in empirical work. For this reason, for empirical work

it is presently necessary to choose functional forms for which equilibria

can be calculated either analytically or "very quickly". Linear-quadratic

specifications are one of the few such choices of convenient functional forms available. (Various versions of logarithmic specification are also sometimes tractable, e.g. Merton [42]). A valuable treatment of recursive competitive equilibrium models with general specifications of functional forms is Prescott and Mehra [48].

5. I shall impose the condition that the zeroes of $\delta_\epsilon(z)$, $\delta_u(z)$, $\det \delta_D(z)$ and $\det \delta_w(z)$ each exceed unity in modulus. Actually, a weaker condition would suffice, namely that the zeroes of these polynomials each exceed $\sqrt{\beta}$ in modulus, where β is the discount factor introduced below. These conditions on the zeroes are regularity conditions that assure that the infinite series calculated on page   below converge.

6. An $(n \times 1)$ vector white noise v_t^x is said to be fundamental for an $(n \times 1)$ vector process x_t if the vector of one-step-ahead linear least squares errors in predicting x_t from past x's can be written as a linear combination of the n components of v_t^x.

7. In the models of both sections 1 and 2, simple static demand functions are posited. As a result, all of the interesting dynamics of the models comes from their supply sides. Specifying a demand schedule with interesting dynamics would complicate the presentation, but not alter the basic messages of our examples. Graves and Telser [17] analyze dynamic optimization problems in which much of the interesting dynamics comes from a demand curve that is specified. Sargent [54, Ch. 16] analyzes a model of the labor market in which the dynamics are influenced by nontrivial dynamic optimization problems solved by both suppliers and demanders.

8. It would be straightforward to modify this setup to assume that the {W, u, ε, D} processes are each finite order mixed moving average, autoregressive processes. For the details, see Hansen and Sargent [21].

9. These variables completely characterize the "state" vector for the firm's problem. We have in mind that the firm actually has observations on values of K, k, W, D, ϵ, and u for <u>all</u> dates t and earlier. It turns out that the firm's decisions are optimally a function only of the information set listed in the text.

10. It is assumed that each of v_t^W, v_t^u, v_t^D and v_t^ϵ is orthogonal to the information set $\{W_{t-s},\ u_{t-s},\ D_{t-s},\ \epsilon_{t-s},\ s \geq 1\}$.

11. This is because we want the stochastic difference equations describing the behavior of the system to be linear, and thereby to be readily susceptible to econometric analysis.

12. That the solution to the problem is of this form follows from linear optimal control theory. See Kushner [24, Chapter 9], Bertsekas [3, Chapter 3], Sargent [54, Chapter XIV], or Kwakernaak and Sivan [29].

13. For a discussion of the use of lag operators in the present context, see Sargent [54, chapters IX and XIV].

14. In effect, the transversality condition compels us to solve the unstable root forward in this manner.

15. As noted above, we shall want the relevant conditional expectations to be linear. So we shall regard the $E(\cdot \mid \Omega_t)$ that appears in (15) and elsewhere as wide-sense conditional expectations, that is, linear least squares predictors. This amounts to restricting the firm to linear decision rules, as desired.

16. The parameters β and n also belong in this list of underlying parameters of the model. I shall usually delete these two parameters from subsequent listings of the model's underlying parameters, though they should be understood. In some applications, the analyst may want to specify counterparts of β and n completely <u>a priori</u>, in which case they would not be included among the "free parameters" of the model over which the likelihood function or other measure of "fit"

17. The following argument in the text provides a way of "discovering"

Lucas and Prescott's [37] method of calculating the rational expectations

equilibrium by formulating a fictitious social planning problem that

reproduces the equilibrium. It is worth remarking that Kydland and

Prescott [30] describe a recursive method of calculating a linear

rational expectations equilibrium that is applicable to our

problem and is distinct from the Lucas-Prescott method upon which

the discussion in the text is based. Kydland and Prescott's method

successfully computes the equilibrium even in instances in which

the Lucas-Prescott method breaks down. These instances occur, for

example, in which there is feedback from the industry-wide aggregate

capital stock K to W or D, as would occur if lagged K's appeared

as states in the Markov law for W or D. In such instances, Lucas

and Prescott's social planning problem fails to reproduce the rational

expectations equilibrium essentially because the fictitious planner

takes into account the "externality" that the feedback from K to

W or D constitutes.

18. This was emphasized in a more general context by Lucas and Prescott
    [30].

19. It can also be proved that the transversality condition for (17)

    imposes the same condition on the solution as does the transversality

    condition of the representative firm.

20. This follows directly from the observation that if z_0 is a
    zero of $[1 - (1 + \beta^{-1} + A_1 f^2 nd^{-1})z + \beta^{-1}z^2]$, then so is βz_0^{-1}.

21. Note that the equation (1.19) satisfies the first-order necessary

    conditions for the optimization problem but gives the "planner" too

    much information (it is "anticipative" or "nonrealizable"). The

    correct solution to the problem taking the information set available

    to the "planner" into account is the solution of the first-order

    necessary conditions that expresses K_{t+1} as a function only of

information that the planner possesses at time t. Such a solution is said to be "realizable" or "nonanticipative".

22. By expanding the polynomial in L, it is possible to show that

(*) $$\frac{L^{-1}(I-\delta(\lambda\beta)^{-1}\delta(L))}{1-\lambda\beta L^{-1}} = \delta(\lambda\beta)^{-1} [\sum_{j=0}^{r-1} (\sum_{k=j+1}^{r} (\lambda\beta)^{k-j-1}\delta_k)L^j]$$

where $\delta(L) = I - \sum_{j=1}^{r} \delta_j L^j$. Notice that the polynomial on the left

side of (*) is one sided in nonnegative powers of L, despite the appearance of L^{-1}, and that it is a polynomial of order (r-1), as asserted in the text. The formula (*) can be derived by mimicking the procedures used in Hansen and Sargent [23]. The same mathematical techniques used by Hansen and Sargent [23] to derive expressions like (21) or (*) were independently utilized by Futia [15] to compute linear rational expectations equilibria. Also, without knowing of Hansen and Sargent's work, John Kennan independently derived formulas similar to (*) in a personal letter to me.

23. From this point of view, it is irrelevant whether Granger causality is consistent with one's notion of what "true" causality is. Christopher Sims [57] has described the relationship of the concept of Granger causality to that of strict econometric exogeneity. That a random process y fail to Granger-cause x is a necessary condition for x to be strictly econometrically exogenous with respect to y. For this reason, the concept of Granger causality is also useful in designing specification tests. For a discussion of the relationship between Granger causality and econometric exogeneity in the context of linear rational expectations models, see Hansen and Sargent [23].

24. Thus Franklin Fisher wrote: "In practice, except for such covariance restrictions [across disturbances in distinct structural equations], restrictions which relate the parameters of one equation to those of one or more others are extremely rare. There is no reason in principle why such cases cannot occur, however, and it may be worthwhile devoting a very short discussion to them." [10, p. 176].

25. By now, this is a routine and uncontroversial definition of an "intervention". Applications of the techniques of optimal control theory to the calculation of macroeconometric and microeconometric policy response functions employ precisely this concept of intervention (see e.g. Chow [6], Taylor [60], Kareken, Muench, and Wallace [26], and Arzac and Wilkinson [2].)

26. For example, if it becomes a cartel when before it had been competitive or noncooperative in some way.

27. A technical qualification needs to be added at this point. In order to have a model capable of predicting effects of interventions acting on the δ's, one can sometimes get by without having uniquely identified the parameters $\{A, f, d\}$. What the researcher must identify are the parameters of the characteristic polynomial of the Euler equation (16), namely the parameters ϕ_0, ϕ_1 in

$$[\beta d - (d(1+\beta) + A_1 f^2 n\beta)L + dL^2] = [\beta\phi_1 + \phi_0 L + \phi_1 L^2]$$

where $\phi_1 = d$, and $-\phi_0 = (d(1+\beta) + A_1 f^2 n\beta)$. The theory assumes that the ϕ_j's will be invariant with respect to interventions on the δ's. If the researcher can uniquely identify the ϕ_j's, he can proceed with econometric policy evaluation, even if he cannot uniquely identify all of (A_1, f, d). In some setups, the parameters of the characteristic polynomials of the Euler equations are identified even though only an equivalence class of the counterparts of (A_1, f, d)

is identified. This is enough for econometric policy evaluation to proceed. This problem is discussed by Hansen and Sargent [22]. It is technically related to the "inverse optimal control" problem (see Mosca and Zappa [44]).

28. This message is at least implicit in the work by Lucas and Prescott [37]. Gordon and Hynes [19] made the argument in an informal way. Lucas [34] forcefully brought the message to the attention of macroeconomists.

29. However, some economists continue to argue that existing macro-econometric models can be used to predict the effects of such arbitrary sequences. (See Friedman [13]).

30. See Benjamin Friedman [14] and Franco Modigliani [43].

31. See Anderson and Moore [1]. The Kalman filter provides a model of Bayesian learning about the δ's where the initial prior and the posteriors are multivariate normal. However, as Lars Hansen points out to me, normal posteriors for the δ's are inadmissible for dynamic models of the class described here. This is because the dynamic optimization problems we consider may be ill-posed for points in the parameter space of δ's for which the zeroes of det $\delta(z)$ are less than $\sqrt{\beta}$ in modulus. Only priors and posteriors that assign zero probability to this region of the parameter space are in general admissible for our problems. This rules out multivariate normal distributions. Taking account of this admissibility constraint severely complicates the task of building a model of optimal learning about the δ's.

32. Further, notice that if the decision rules could be calculated in closed form under uncertainty about the δ's, the resulting time series models would have time-varying coefficients and so be non-stationary. Even if calculating the decision rules were a tractable task under uncertainty

about the δ's, the loss of stationarity that it would imply might well be a price that the applied economist would not be prepared to pay even in exchange for the "greater realism" of the learning assumption.

33. This is a consequence of the fact that the representative firm views itself as playing a dynamic "game against nature", and so finds it optimal to use a nonrandom strategy, that is, a strategy that can be expressed as an exact function of its information variables and other state variables.

34. This is a version of the model of the error term analyzed by Hansen and Sargent [23] and Sargent [52].

35. We have assumed that V_t^ε and V_t^u are the "innovations" or one-step-ahead errors in predicting ε_t and u_t on the basis of observations on all variables in agents' information set at time $t-1$ (see footnote 6). This implies that V_{t+1}^ε is orthogonal to all variables on the right side of (25). If we assume that V_t^u is orthogonal to V_t^ε, it also implies that V_t^u is orthogonal to Y_t. Imposing that V_t^u is orthogonal to D_{1t} amounts to assuming that D_{1t} is strictly exogenous in (26), which is stronger than the Granger causality assumptions already imposed on D_{1t}, namely, that except for lagged D's, no other variables in the model Granger-cause D_{1t}.

36. There is a singular class of exceptions to this statement. In the special case that

$$(+) \quad E\, D_{1t+j} | \{D_{t-s}\}_{s=0}^\infty = E\, D_{1t+j} | \{D_{2t-s}\}_{s=0}^\infty$$

for all $j \geq 1$, D_{1t}'s will appear in the demand schedule but not in the supply schedule. The condition (+) is usually thought to be exceedingly unlikely for <u>any</u> economic time series $\{D_{1t}\}$.

37. These remarks about identification should be compared to Milton Friedman's discussion [11] of the conditions needed for "supply" and "demand" to provide a useful categorization of the factors

impinging on price and output. Friedman argued that the categorization was useful to the extent that it effectively sorted forces acting on price and output into mutually exclusive categories.

38. The fact that the demand curve excl _s some variables that appear in the supply schedule is due to the static specification for the demand curve. This feature of the model would not survive a variety of alterations that might plausibly be used to introduce dynamics into the demand curve. For example, if the demand schedule involved expected future prices as arguments, all variables that help to predict future prices would appear in the econometrically operational expression for current p_t that would correspond to (26).

39. This characteristic of identification in rational expectations models has been noted in various contexts by several authors including Lucas [35] and Sims [58].

40. See Hansen and Sargent [23] for a discussion of the details.

41. This restriction was criticized by Lucas [ ] and Sargent [ ] for essentially the same reasons given here.

42. A point related to that raised in footnote 29 is relevant here. Priors and posteriors that assign positive probability to points in regions for which zeroes of det $\delta(z)$ are less than $\sqrt{\beta}$ in modulus are inadmissible. This is because under such distributions, for some regions in the parameter space with positive probability, the dynamic optimum problems are not well posed. Taking this into account would substantially complicate the analysis since it would involve using mathematically less tractable distributions.

43. This model of the error term was originally proposed by Shiller [55] in a related but somewhat different context. The model was applied in the present context by Hansen and Sargent [23]. Nerlove, Grether and Carvalho [49] also recommend Shiller's model of the error term.

44. See Hansen and Sargent [23].

45. Again, see Hansen and Sargent [23].

46. Under regularity conditions provided by Hansen [20], the estimators of the underlying parameters are shown to be consistent, and most efficient within a restricted class of estimators. Hansen's discussion of the conditions for consistency, which also has implications for the conditions for consistency of maximum likelihood estimators, is at this date the key reference on issues of statistical consistency in linear rational expectations models.

47. I assume that each of the innovations

$$\{v^c_{wt}, v^h_{wt}, v^h_{ut}, v^h_{Dt}, v^c_{ut}, v^c_{Dt}, v^c_{\varepsilon t}, v^h_{\varepsilon t}\}$$

is orthogonal to the information set

$$\{W^h_{t-s}, W^c_{t-s}, D^h_{t-s}, D^c_{t-s}, \varepsilon_{ht-s}, \varepsilon_{ct-s}, u_{ht-s}, u_{ct-s}; s \geq 1\}.$$

48. This can be established as follows. First, obtain the Euler equations for the social planning problem of maximizing (29). Next, obtain the Euler equations for the optimum problems (20) and (21) of the representative hog and corn farmers, respectively. Upon using $K_{ht} = nk_{ht}$, and $K_{ct} = mk_{ct}$ to eliminate k_{ht} and k_{ct} from the representative firms' Euler equations in favor of K_{ht} and K_{ct}, it will be noticed that the resulting equations are the Euler equations of the social planning problem. This argument parallels that used in section 1. Furthermore, it can be demonstrated that the transversality conditions of the planning problem and the representative firms' problems also match up.

49. As I have interpreted the models described in this paper, agents are regarded as using a long list of state variables to forecast future prices. A criticism of this interpretation of the models, which is called to mind by the naming of commodities as "corn" and "hogs", might be that it ignores the existence of futures markets which confront agents with prices at which they can purchase or sell for future delivery. So maybe agents are being modeled as having to forecast too much. In the presence of a sufficiently large set of markets in state contingent futures claims, agents would have to do no forecasting. It would be a more or less standard application of the Arrow-Debreu apparatus to reinterpret the current models as ones in which there is a complete set of Arrow-Debreu contingent claims markets. Lucas and Prescott [ ] and Sargent [, pp. 376-7] provide examples of such reinterpretations in related contexts. So there is a limited sense in which the models described here can accomodate futures markets, limited because the models seem to explain "too many" such futures markets.