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ANOTHER NOTE ON DEADWEIGHT LOSS

Michael J. Stutzer

Federal Reserve Bank of Minneapolis

ABSTRACT

In a recent article, J. A. Kay has proposed a useful measure of the deadweight loss arising from a commodity tax system. The measure answers the question, How much more would the taxed consumer be willing to pay in a lump sum rather than as a commodity tax? Kay's computation of the marginal deadweight loss does not yield the change in this measure for small changes in commodity tax rates, however. This note clarifies Kay's otherwise excellent contribution, derives the measure for Cobb-Douglas utilities, and examines a useful property of it.

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Another Note on Deadweight Loss

In a recently published note, J. A. Kay (1980) proposed a useful measure of the deadweight loss resulting from a commodity tax system. Kay's measure of deadweight loss is

$$\overline{L} = EV - R \tag{1}$$

where R is the revenue raised by the commodity tax and EV is the equivalent variation in income which would leave the taxpayer as bad off as the tax does. As Kay points out, $\overline{L}$ answers the question, How much more would the taxed consumer be willing to pay in a lump sum rather than as commodity taxes? However, Kay's computation of the marginal deadweight loss resulting from a small change in a tax rate is not the derivative of $\overline{L}$ with respect to that tax rate. This derivative is of value in computing optimal tax systems, as well as in computing the welfare loss resulting from small changes in existing tax systems. The purpose of this note is to correct and clarify Kay's otherwise excellent contribution.

With the initial vector of commodity prices denoted by $\underline{p} = (p_1,\ldots,p_n)$, the after tax vector of commodity prices denoted by $\underline{q} = \underline{p} + \underline{t} = (p_1+t_1,\ldots,p_n+t_n)$, and with U_1 denoting the utility attained by the consumer in the commodity tax equilibrium, Kay's measure takes the form of his formula 4,

$$\overline{L} = [E(\underline{q},U_1) - E(\underline{p},U_1)] - \sum_{i=1}^{n} (q_i-p_i)x_i \tag{2}$$

where x_i is the ordinary demand function for the i^{th} good and E is the expenditure function defined by

$$E(\underline{r},U_1) = \min_{x_1,\ldots,x_n} \sum_{i=1}^{n} r_i x_i \tag{3}$$

$$\text{s.t. } U(x_1,\ldots,x_n) = U_1$$

and whose solution yields the compensated demand functions

$$x_i^c\ (\underline{r},U_1),\ i=1,\ldots,N. \tag{4}$$

U_1 is the utility attained by the consumer with income M in the commodity tax equilibrium and is defined by the indirect utility function V as follows:

$$U_1 = V(\underline{q},M) = \max_{x_1,\ldots,x_n} U(x_1,\ldots,x_n) \tag{5}$$

$$\text{s.t. } \sum_{i=1}^{n} q_i x_i = M.$$

Diagrammatic Exposition

A simple example of this measure is presented in figure 1 for the case of n=2 goods, where units are chosen so that the initial prices of x_1 and x_2 are $p_1 = p_2 = 1$. A tax of t_1 per unit is placed on x_1, which shifts the budget line inward to $M - M/q_1$, where $q_1 = p_1 + t_1 = 1 + t_1$. The tax shifts the consumer equilibrium from I to T. The equivalent lump sum tax would take away EV dollars of income and would result in the consumer purchasing bundle LS. The deadweight loss $\overline{L}$ is the difference between the two vertical distances EV and R.

When the tax rate in the example is increased to $t_1 + dt_1$, in figure 2, the commodity tax equilibrium, the lump sum tax equilibrium, and the utility attained by both changes to T′, LS′, and U_1', respectively. Both EV and R change to EV′ and R′, which results in $\overline{L}$ changing to $\overline{L}'$. The marginal deadweight loss $\partial L/\partial q_1$ is approximately equal to $\overline{L}' - \overline{L} = d\overline{L}$.

Kay's computation of the marginal deadweight loss $\partial L/\partial q_i$, used in his equations 8 and 9 as well as in his equations 11 through 16, ignores the change in attainable utility from U_1 to U_1'. In our example, his computation is illustrated in figure 3. There, Kay's measure is

$$\frac{\partial \overline{L}}{\partial q_1} \approx d\overline{L}^k = [[E(\underline{q} + dq_1, U_1) - E(\underline{p}, U_1)] - EV] - [R' - R] \qquad (6)$$
$$= EV^k - dR.$$

Kay shows in his formula 9′ that it is possible to derive an optimal tax rule which minimizes deadweight loss subject to a revenue constraint and which does take account of the change in attainable utility. But he does not take this into account in his equations 11 to 16, where it is clearly appropriate to do so. Thus, computations dependent on Kay's marginal deadweight loss are not compatible with the definition he states for deadweight loss.

In the next section, the differences between Kay's computation and the derivative of (2) are investigated algebraically.

Algebraic Exposition

The deadweight loss formula (2) is differentiated with respect to q_i to produce the marginal deadweight loss with respect to the tax on x_i. To do so, the derivatives of both the expenditure and the indirect utility functions are needed. This requires the repeated use of the envelope theorem [Silberberg (1971)]. It states that the derivative of the maximand or the minimand with respect to a parameter in a constrained optimization problem equals the derivative of its Lagrangian with respect to that parameter.

Kay's procedure, which holds U_1 constant, goes as follows. From (2), compute

$$\frac{\partial \overline{L}}{\partial q_i} = \frac{\partial E}{\partial q_i} (q,U_1) - \left(x_i + q_i \frac{\partial x_i}{\partial q_i} + \sum_{j \neq i} q_j \frac{\partial x_j}{\partial q_i} - \sum_j p_j \frac{\partial x_j}{\partial q_i}\right) \tag{7}$$

where the second term is $\partial R/\partial q_i$ and is correct. The envelope theorem applied to (3) yields the first term, which gives

$$\frac{\partial \overline{L}}{\partial q_i} = x_i^c (\underline{q},U_1) - x_i - q_i \frac{\partial x_i}{\partial q_i} - \sum_{j \neq i} q_j \frac{\partial x_j}{\partial q_i} + \sum_j p_j \frac{\partial x_j}{\partial q_i}. \tag{8}$$

Because the compensated demand x_i^c equals the ordinary demand x_i at $(\underline{q},U_1)$, the following expression used by Kay results:

$$\frac{\partial \overline{L}}{\partial q_i} = - \sum_j (q_j - p_j) \frac{\partial x_j}{\partial q_i} = - \sum_j t_j \frac{\partial x_j}{\partial q_i} = - \frac{\partial R}{\partial q_i} + x_i. \tag{9}$$

A true measure of the marginal deadweight loss lets the utility level U_1 vary, as it must, when q_i changes (see figure 2). To do so, one must substitute the indirect utility function (5) into (2) before differentiating. Thus, start with

$$\overline{L} = E\big(\underline{q},V(\underline{q},M)\big) - E\big(\underline{p},V(\underline{q},M)\big) - \sum_{i=1}^{N} (q_i - p_i)x_i. \tag{10}$$

Then the marginal deadweight loss is

$$\frac{\partial \overline{L}}{\partial q_i} = \frac{\partial E}{\partial q_i} \big(\underline{q},V(\underline{q},M)\big) - \frac{\partial E}{\partial q_i} \big(\underline{p},V(\underline{q},M)\big) - \frac{\partial R}{\partial q_i}. \tag{11}$$

The last term is the same as in (7). The first term is equal to zero, as $E\big(\underline{q},V(\underline{q},M)\big) \equiv M$. If this fact is not evident from figure 2, the reader can verify it by direct differentiation. Applying the composite function rule to the second term, find

$$\frac{\partial E}{\partial q_i}\left(\underline{p},V(\underline{q},M)\right) = \frac{\partial E}{\partial V}\left(\underline{p},V(\underline{q},M)\right)\frac{\partial V}{\partial q_i}(\underline{q},M). \tag{12}$$

The envelope theorem applied to (3) yields

$$\frac{\partial E}{\partial V}\left(\underline{p},V(\underline{q},M)\right) = \lambda^* = p_i/\left[\frac{\partial U}{\partial x_i}\left(x_1(\underline{p},U_1),\ldots,x_n(\underline{p},U_1)\right)\right] \tag{13}$$

where λ^* is the Lagrange multiplier for (3) and where the latter equality follows from its first order conditions. The marginal utility of x_i is evaluated at the equivalent variation solution $x_j(\underline{p},U_1)$, $j=1,\ldots,n$. The envelope theorem applied to (5), when coupled with its first order conditions, yields

$$\frac{\partial V}{\partial q_i}(\underline{q},M) = -\lambda x_i = -\frac{\partial U}{\partial x_i}\left(x_1(\underline{q},M),\ldots,x_n(\underline{q},M)\right)\cdot x_i(\underline{q},M)/q_i \tag{14}$$

where λ is the Lagrange multiplier for (5) and where $x_j(\underline{q},M)$, $j=1,\ldots,n$ is the commodity tax equilibrium. Finally, substituting (13) and (14) into (12), and the latter into (11), find

$$\frac{\partial \overline{L}}{\partial q_i}\left(\underline{q},V(\underline{q},M)\right) = \frac{p_i\,\partial U/\partial x_i\left(x_1(\underline{q},M),\ldots,x_n(\underline{q},M)\right)x_i(\underline{q},M)}{q_i\,\partial U/\partial x_i\left(x_1(\underline{p},U_1),\ldots,x_n(\underline{p},U_1)\right)} - x_i(\underline{q},M) - \sum_{j=1}^{n} t_j\frac{\partial x_j}{\partial q_i}. \tag{15}$$

Denoting the ratio of marginal utilities in (15) by d_i, we have

$$\frac{\partial \overline{L}}{\partial q_i}\left(\underline{q},V(\underline{q},M)\right) = \left[\frac{p_i d_i}{q_i} - 1\right]x_i(\underline{q},M) - \sum_{j=1}^{n} t_j\frac{\partial x_j}{\partial q_i} \tag{16}$$

which differs from Kay's expression by the presence of the first term.

<u>Example</u>

For the homogeneous Cobb-Douglas utility

$$U(x_1,\ldots,x_n) = \sum a_i \ln x_i, \quad \sum a_i = 1 \tag{17}$$

it is well-known [Varian (1978, Chapter 3)] that

$$V(\underline{q},M) = \sum a_i \ln(\frac{a_i}{q_i}) + \ln M \tag{18}$$

$$E(\underline{p},U) = \exp[U - \sum a_i \ln(\frac{a_i}{p_i})] \tag{19}$$

$$x_i(\underline{q},M) = a_i \frac{M}{q_i} . \tag{20}$$

Substituting (18)-(20) into (10) and simplifying, the deadweight loss is

$$\overline{L} = M \, [\sum \frac{p_i a_i}{q_i} - \prod_i (\frac{p_i}{q_i})^{a_i}] . \tag{21}$$

Differentiating (21) with respect to the i^{th} good's after tax price, find

$$\frac{\partial \overline{L}}{\partial q_i} = \frac{Mp_i a_i}{q_i^{\,2}} \left[1 - (\frac{q_i}{p_i})^{1-a_i} \prod_{j \neq i} (\frac{p_j}{q_j})^{a_j} \right]. \tag{22}$$

Choose units so that $p_i = p_j = 1$, and tax only the i^{th} good, so that $q_i = 1+t_i > 1$. Then (22) simplifies to

$$\frac{\partial \overline{L}}{\partial q_i} = - Ma_i \left[\frac{1}{q_i^{\,2}} - \frac{1}{q_i^{\,(1+a_i)}} \right]. \tag{23}$$

Kay's marginal measure for this case is easily computed from (9) to be

$$\frac{\partial \overline{L}}{\partial q_i} = - Ma_i \left[\frac{1}{q_i^{\,2}} - \frac{1}{q_i} \right].$$

which, because $q_i = 1+t_i > 1$, is clearly greater than (23). Whether or not Kay's marginal measure always exceeds (16) for arbitrary utilities is still an open question.

In closing, note that for Cobb-Douglas utilities, a glance at (21) shows that deadweight loss as a fraction of income, $\overline{L}/M$, is independent of income. This is a useful property for applied work and will hold for all homothetic utilities, because for them [Varian (1978, Chapter 3, Exercises)]

$$V(\underline{q},M) = Mg(\underline{q}) \tag{24}$$

$$E(\underline{p},U) = U/g(\underline{p}) \tag{25}$$

and, via Roy's identity,

$$x_i(\underline{q},M) = \frac{M\dfrac{\partial g}{\partial q_i}}{g(\underline{q})} \tag{26}$$

where g is a convex function homogeneous of degree -1. Substituting (24)-(26) into (10) yields

$$\frac{\overline{L}}{M} = \left[1 - \frac{g(\underline{q})}{g(\underline{p})} + \frac{\sum t_i \dfrac{\partial g}{\partial q_i}}{g(\underline{q})}\right] \tag{27}$$

which is independent of M.

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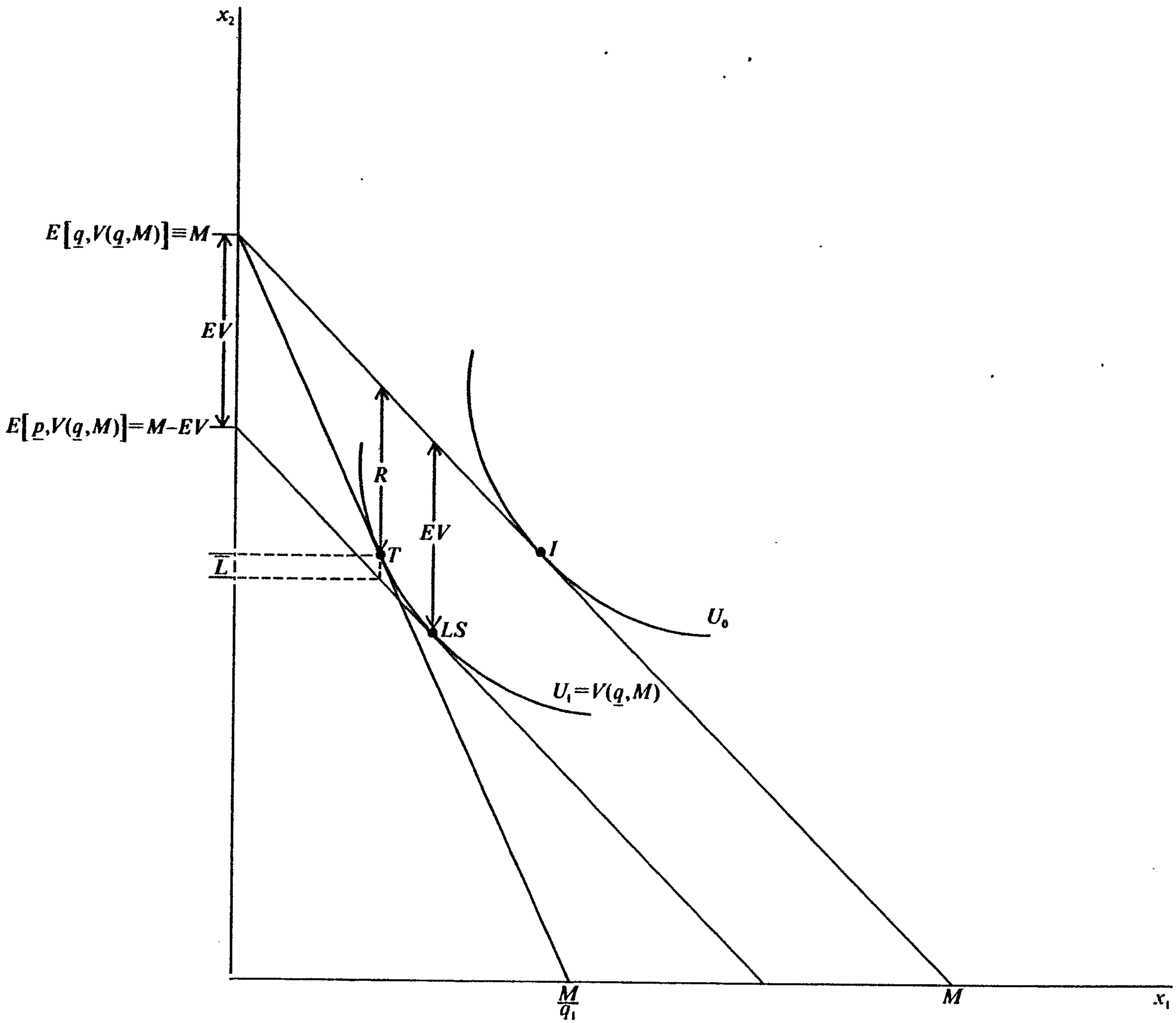

Figure 1: Deadweight loss measure $\overline{L}$

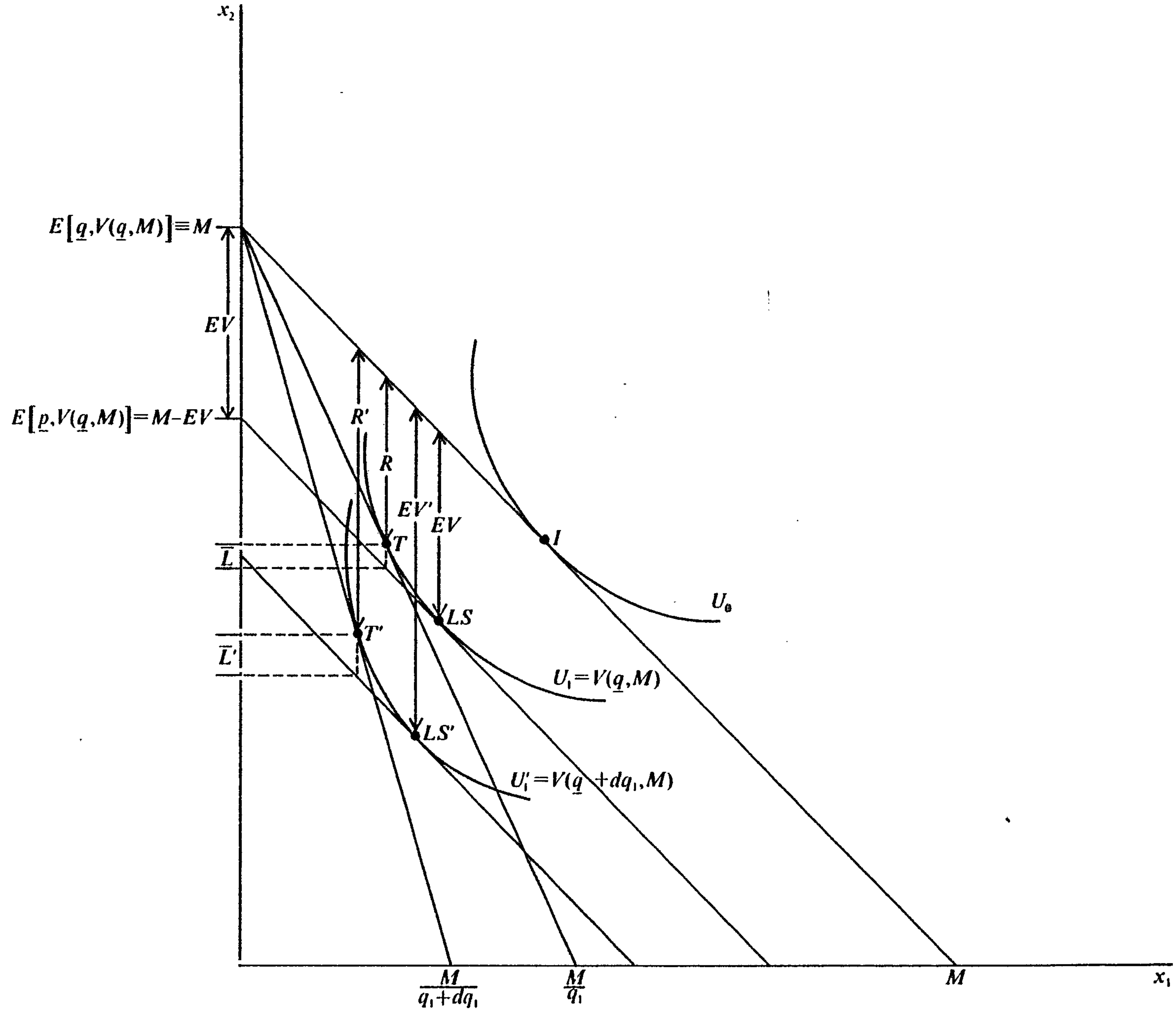

Figure 2: Marginal deadweight loss $\overline{L}' - \overline{L}$

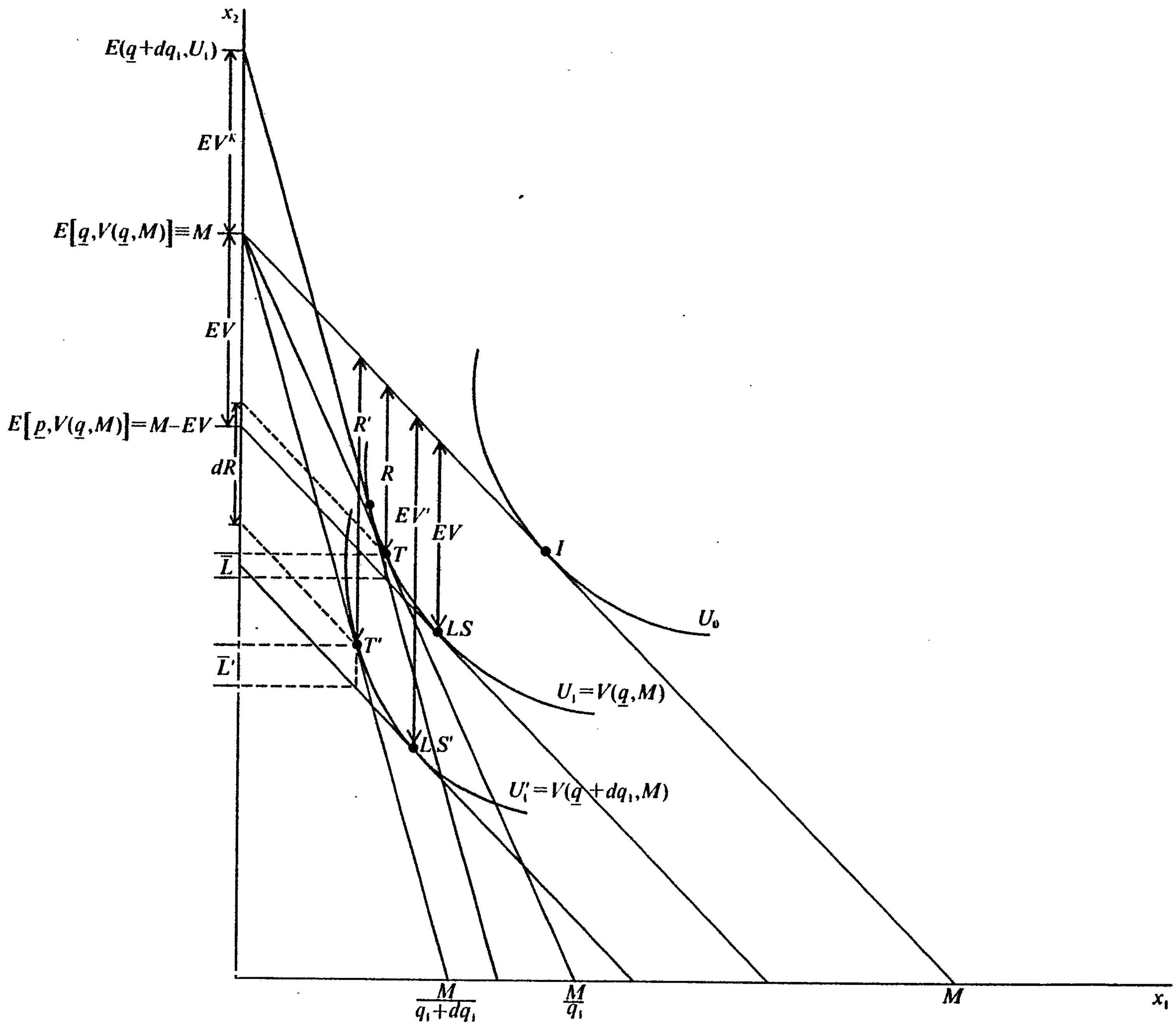

Figure 3: Kay's marginal deadweight loss EV^K-dR