"Tobin's q" and the Rate of Investment in General Equilibrium

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This paper is an exercise that was undertaken to practice some of the methods that were taught by R. E. Lucas, Jr., in his Economics 339 at the University of Chicago in the spring of 1977. The putty-clay version of the stochastic growth model used here was described by Lucas in his lectures. Propositions 1, 2, and 3 of this paper were proved by Lucas in class. Propositions 4, 5, and 6 are my own responsibility. Robert Litterman performed the calculations. Helpful comments on an earlier draft were made by Lars Hansen, Robert E. Lucas, Jr., and Jose Scheinkman. Thanks to the suggestions of Lars Hansen, the proof of Proposition 4 has been much simplified from the previous draft.

James Tobin's "q theory" is one of the most prominent current macroeconomic theories about firms' demand schedule for a flow of investment. According to that theory, there is at most times a discrepancy between the price of existing capital goods, say as reflected in the bond and equity markets, and the price of newly produced capital goods. Tobin calls the ratio of these two prices q. Tobin posits that q is an important argument of firms' demand schedule for investment. "The rate of investment--the speed at which investors wish to increase the capital stock--should be related, if to anything, to q, the value of capital relative to its replacement cost," [20, p. 21]. Such a theory must necessarily stem from a model in which "frictions" are present that prevent the price of existing capital from being driven equal at all times to the price of newly produced capital. For example, in "putty-putty" versions of one-sector growth models, q is always unity. Furthermore, in such models firms have no investment demand schedule, a point emphasized by Tobin [18, 19].

A simple model possessing the friction necessary to permit q to diverge from unity is the putty-clay version of the one-sector growth model. In this model, newly produced goods can either be consumed or used to augment the capital stock. But once in place, capital cannot be consumed. The irreversibility of investment is the friction that permits q to diverge from unity and which makes it possible for aggregate investment to be positively correlated with q. However, the population regression of aggregate investment on q is in no sense an "investment demand schedule," instead being a mongrel relation that reflects all of the parameters of the model. An econometrician studying such an economy would have no cause to fit such a regression if it is the economy's structure that he is after. Among other things, there is a massive "simultaneity problem." Not only does q, taken as a random process, influence investment decisions, but investment decisions influence q as a random process. But it is not merely a purely econometric simultaneity problem. There are serious theoretical questions about the precise sense even in which agents that can legitimately view q as exogenous exhibit investment behavior that can be described as a function mainly or solely of q. Indeed, the model in this paper exhibits a feature that probably characterizes virtually any model that possesses the friction necessary to make q diverge from unity: the very same source of friction that makes q diverge from unity also converts agents' decision problem into a nontrivial dynamic one, the solution of which will in general not assume a "myopic" form such as a simple contemporaneous demand schedule relating current investment to current q. Instead, investment decisions will necessarily be functions of agents' views about the future, the current state of which cannot in general be summarized by a single variable such as q.

This paper uses a putty-clay version of the stochastic one-sector growth model as a vehicle for making some observations about the q theory of investment. We are attracted to the stochastic one-sector growth model because it is perhaps the simplest coherent general equilibrium model available in which one can discuss the mutual determination of investment and q. The one-sector stochastic growth model has been well studied (see e.g., Mirman [15], Brock and Mirman [6] and Mirman and Zilcha [16]), so there is nothing analytically original here. However, because we are discussing a putty-clay version of the model, rather than the putty-putty (existing capital can be consumed) version that is extensively discussed in the literature, we have to spend some time discussing the nature of corner solutions in which the constraint that existing capital can't be consumed is binding. It is at best at this point that there is any analytical novelty.

2. A Market Interpretation of the Model

Production is governed by

$$y_t = f(k_t)\theta_t$$

where y is output per man, and k_t is capital per man at t; θ_t is a positive, independently and identically distributed random variable. We assume that $f(\cdot)$ is twice continuously differentiable and satisfies

$$f'(k) > 0,\ f''(k) < 0$$

$$f'(0) = \infty,\ f'(\infty) = 0.$$

All consumers are alike and have bounded one-period utility function $u(c_t, \varepsilon_t)$, which we assume is twice continuously differentiable. Here c_t is consumption per man and ε_t is a random shock to preferences. We assume that ε_t is independently and identically distributed. We assume

$$u(c, \varepsilon) < M \text{ for all } c, \varepsilon \quad \text{for some } M > 0$$

$$u_c(c,\varepsilon) > 0,\ u_{cc}(c,\varepsilon) < 0$$

$$u_{c\varepsilon}(c,\varepsilon) > 0$$

$$u_c(0,\varepsilon) = \infty,\ u_c(\infty,\varepsilon) = 0.$$

We assume that the random processes θ_t and ε_s are distributed independently of each other at all dates. We assume that the joint process $(\theta_t, \varepsilon_t)$ has a continuous joint probability density function with cumulative distribution function

$$F(\theta,\varepsilon) = \text{Prob}\{\theta_t \leq \theta, \varepsilon_t \leq \varepsilon\}$$

for all t. We assume that there exist numbers $\overline{\theta} > \underline{\theta} > 0$ and $\overline{\varepsilon} > \underline{\varepsilon} > 0$ such that $\text{Prob}\{\underline{\theta} \leq \theta \leq \overline{\theta},\ \underline{\varepsilon} \leq \varepsilon \leq \overline{\varepsilon}\} = 1$. We assume that $F(\theta, \varepsilon)$ has a strictly positive density on the rectangle $\{\underline{\theta} \leq \theta \leq \overline{\theta},\ \underline{\varepsilon} \leq \varepsilon < \overline{\varepsilon}\}$.

We assume that in a given period, all agents draw the same $(\theta_t, \varepsilon_t)$. Since all agents are assumed alike in the sense that they have the same utility functions and have access to the same technology and market opportunities, we shall assume that there is a single representative consumer. The consumer views himself as a perfect competitor and views economy-wide outcomes as independent of his own actions. This means that we must distinguish between the economy-wide state, which the consumer takes as given, and the consumer's own state variables, the evolution of some of which are a matter of choice to the consumer. In equilibrium, the economy-wide state variables equal the representative consumer's state variables, but the consumer is assumed to ignore this.[1/]

The state of the economy at time t can be characterized by the values of $(K_t, \theta_t, \varepsilon_t)$ where K_t is the economy-wide capital-labor ratio at the beginning of period t, θ_t is the random shock to productivity realized in period t and ε_t is the random shock to preferences realized in period t. The state of the individual consumer at time t is characterized by his stock of capital at the beginning of t, k_t, and also the same shocks ε_t and θ_t that affect all agents' preferences and opportunities. The consumer's supply of labor is identically one, so that k_t also equals his capital-labor ratio. At time t, the consumer can rent his capital to firms at a competitively determined rental r_t, measured in output per unit capital per unit time. Furthermore, during period t the consumer can buy or sell claims to existing capital to be carried into period (t+1) at a competitively determined relative price $p_K(t)$ measured in units of new output per unit of capital. According to one possible interpretation, the relative price $p_K(t)$ is precisely Tobin's q. During period t households also buy newly produced output, consuming an amount c_t and carrying an amount i_t into next period as capital. The relative price of newly produced capital goods in terms of consumption goods is unity. Finally, the consumer inelastically supplies one unit of labor and is paid a competitively determined real wage w_t measured in output per unit labor.

The consumer's problem is to maximize

$$(1) \qquad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, \varepsilon_t), \; 0 < \beta < 1,$$

where E_0 is the mathematical expectation operator conditional on information available at time 0, subject to the sequence of budget constraints for t=0, 1, 2, . . .

$$c_t + p_{Kt}k_t^d + i_t \leq w_t + r_t k_t + (1-\delta)p_{Kt}k_t$$

$$k_{t+1} = k_t^d + i_t,\ i_t \geq 0,\ c_t \geq 0,\ k_t^d \geq 0$$

where

δ = rate of depreciation of capital, $0 < \delta < 1$.

c_t = consumption per unit labor.

k_t^d = amount of old capital held at end of period t.

i_t = amount of newly produced goods to be used as capital.

k_t = amount of capital per unit of labor at beginning of period t.

The consumer seeks to maximize (1) with respect to the choice of stochastic processes for c_t, i_t, and k_t^d given the information he has at each period and given the constraints that he faces. To make the consumer's problem well posed, we suppose that the equilibrium relative prices in the system can be expressed as continuous functions of the economy-wide state variables, so that

$$r_t = r(K_t,\theta_t,\varepsilon_t)$$

(2) $$p_{Kt} = p_K(K_t,\theta_t,\varepsilon_t)$$

$$w_t = w(K_t,\theta_t,\varepsilon_t).$$

We assume that the representative agent in the economy knows the three functions listed in (2) and that at time t he knows the values of θ_t, ε_t, and the economy-wide capital stock K_t. We also suppose that K_t follows the law of motion

(3) $$K_{t+1} = h(K_t,\theta_t,\varepsilon_t)$$

where h is a continuous function. We assume that this aggregate law of motion is known to the representative agent and is perceived by the agent to be independent of his own decisions. Let us denote the four functions in (2) and (3) as f.

For a given selection of the four functions in (2) and (3), the household's problem is equivalent with finding an optimal value function $J(k,\theta,\varepsilon;K,f)$ which solves the functional equation

$$(4) \qquad J(k,\theta,\varepsilon;K,f) = \max_{i\geq 0,\ k^d \geq 0} \{u(w(\cdot)+r(\cdot)k+(1-\delta)p_K(\cdot)k-p_K(\cdot)k^d-i,\varepsilon) + \beta\int J(k^d+i,\theta',\varepsilon';h(K,\theta,\varepsilon),f)dF(\theta',\varepsilon')\}.$$

Here the functions $w(\cdot)$, $r(\cdot)$, and $p_K(\cdot)$ have as arguments (K,θ,ε). For a given selection of the functions in f, it is possible to prove that the functional equation has a unique, continuous bounded solution $J(k,\theta,\varepsilon;K,f)$.[2/] The right side of (4) can be shown to be uniquely attained by continuous functions[3/]

$$i = i(k,\theta,\varepsilon;K,f)$$

$$c = c(k,\theta,\varepsilon;K,f)$$

$$k^d = k^d(k,\theta,\varepsilon;K,f).$$

It can also be proved that $J(\cdot)$ is strictly concave in k and that J has a continuous and bounded partial derivative with respect to k.[4/]

The first-order necessary conditions for the maximization problem on the right side of (4) are[5/]

$$(5) \qquad k^d: \ -u_c(c,\varepsilon)p_K(K,\theta,\varepsilon) + \beta\int J_k(k^d+i,\theta',\varepsilon';h(K,\theta,\varepsilon),f)dF(\theta',\varepsilon') \leq 0, = 0 \text{ if } k^d > 0$$

$$(6) \qquad i: \; -u_c(c,\varepsilon) + \beta\int J_k(k^d+i,\theta',\varepsilon';h(K,\theta,\varepsilon),f)dF(\theta',\varepsilon') \leq 0,$$

$$= 0 \text{ if } i > 0.$$

The partial derivative of $J(\cdot)$ with respect to k can be calculated from (4) to be[6/]

$$(7) \qquad J_k(k,\theta,\varepsilon;K,f) = u_c(c(k,\theta,\varepsilon;K,f),\varepsilon)[r(K,\theta,\varepsilon)+(1-\delta)p_K(K,\theta,\varepsilon)]$$

Conditions (5) and (6) tell something about the sense in which there is a "q theory" of investment in the present model. Use (7) to write (5) and (6) as

$$(8) \qquad -u_c(c_t,\varepsilon_t)p_K(K_t,\theta_t,\varepsilon_t)$$

$$+ \beta\int u_c(c_{t+1},\varepsilon_{t+1})[r(K_{t+1},\theta_{t+1},\varepsilon_{t+1})+(1-\delta)p_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})]dF(\theta_{t+1},\varepsilon_{t+1}) \leq 0,$$

$$= 0 \text{ if } k_t^d > 0$$

$$(9) \qquad -u_c(c_t,\varepsilon_t)$$

$$+ \beta\int u_c(c_{t+1},\varepsilon_{t+1})[r(K_{t+1},\theta_{t+1},\varepsilon_{t+1})+(1-\delta)p_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})]dF(\theta_{t+1},\varepsilon_{t+1}) \leq 0,$$

$$= 0 \text{ if } i_t > 0.$$

Now in equilibrium, k_t^d must exceed zero, so that (8) will be satisfied with equality. It then follows that $p_K(K,\theta,\varepsilon) \leq 1$. The marginal condition (9) shows that i_t will be >0 only if $p_{Kt} = p_K(K_t,\theta_t,\varepsilon_t) = 1$. However, notice that the marginal conditions (8) and (9) necessarily involve the agent's perceptions of the distribution of one-period-ahead values of the rental $r_{t+1} = r(K_{t+1},\theta_{t+1},\varepsilon_{t+1})$ and the relative price $p_{Kt+1} = p_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})$. In general, the agent's choice of i_t depends on all of the current state variables that help determine the conditional distribution of future values of p_K and r. In a limited sense, the first-order conditions (8) and (9) do provide some foundation for the "q theory" of investment demand. But it is really the function $p_K(K,\theta,\varepsilon)$ or put differently, p_K as a stochastic process, and not only the current realized value of p_{Kt} that influences investment at time t.

The marginal conditions (8) and (9) make it clear that some carefully spelled out view about the stochastic processes (law of motion) of K_{t+1}, r_{t+1}, and p_{Kt+1} must be attributed to agents in order for the decision problem to be well specified. The restriction that we have imposed, that agents' perceptions of those laws of motion are accurate, is the hypothesis of rational expectations.

We can think of production as being determined by competitive firms which rent capital and hire labor to maximize profits

$$\pi = n^d f(k) - w(\cdot)n^d - r(\cdot)k \cdot n^d$$

where f(k) is output per man, k is the capital-labor ratio of the representative firm, and n^d is the employment level of the representative firm. The first-order necessary conditions for a maximum of profits are

$$f'(k) = r(K,\theta,\varepsilon)$$

$$f(k) - kf'(k) = w(K,\theta,\varepsilon).$$

We can now give a definition of equilibrium.

Definition: An equilibrium is a five-tuple of functions $r(K,\theta,\varepsilon)$, $p_K(K,\theta,\varepsilon)$, $w(K,\theta,\varepsilon)$, $h(K,\theta,\varepsilon)$, and $J(k,\theta,\varepsilon;K,f)$ such that--

i. The functional equation (4) is satisfied with the right-hand side being attained by the continuous policy functions $i(k,\theta,\varepsilon;K,f)$, $c(k,\theta,\varepsilon;K,f)$, and $k^d(k,\theta,\varepsilon;K,f)$.

ii. $(1-\delta)K = k^d(K, \theta, \varepsilon; K, f)$.

iii. $K_{t+1} = i(K, \theta, \varepsilon; K, f) + k^d(K, \theta, \varepsilon; K, f) \equiv h(K, \theta, \varepsilon)$.

iv. The marginal conditions for firms are satisfied with

$$f'(K) = r(K, \theta, \varepsilon)$$

$$f(K) - Kf'(K) = w(K, \theta, \varepsilon).$$

Condition (i) says that consumers are maximizing expected utility, given the random processes they are facing, which includes the Markov process (law of motion) for the economy-wide capital labor ratio K. Condition (ii) says that the market for existing capital clears at the end of each period, so that when the representative agent starts a period with a capital stock of K, he ends up demanding exactly $(1-\delta)K$ units of existing (old) capital to carry into the next period. Condition (iii) says that the consumer's perceptions of the law of motion for the aggregate K turn out to be correct; that is, those perceptions are implied by the representative agent's solution of the maximum problem on the right side of (4). Condition (iv) states that firms are on their demand schedules for factors and that the factor markets always clear.

We shall follow Lucas and Prescott [14] by studying the equilibrium of the model only indirectly by studying the planning problem that reproduces the competitive equilibrium. In the next section we study the version of the Cass-Koopmans planning model that is isomorphic with the market model of this section and which generates as a shadow prices for capital the correct function $p_K(K, \theta, \varepsilon)$.

3. The Planning Model

The planning problem is to choose a contingency plan for I_t which maximizes

(10) $$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, \varepsilon_t)$$

subject to

$$C_t + I_t \leq f(K_t)\theta_t$$

$$C_t \geq 0,\ I_t \geq 0$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

where

C_t = consumption per man.

I_t = gross investment per man.

K_t = capital per man.

Solving the planning problem is equivalent with solving the following functional equation in the optimum value function $v(K,\theta,\varepsilon)$

(11) $$v(K,\theta,\varepsilon) = \max_{I \geq 0}\{u(f(K)\theta - I,\varepsilon) + \beta\int v((1-\delta)K+I,\theta',\varepsilon')dF(\theta',\varepsilon')\}.$$

The solution $v(K,\theta,\varepsilon)$ gives the maximum value of (10) starting from state (K,θ,ε) at time 0. Associated with the functional equation (10) is the operator T defined by

(12) $$Tg(K,\theta,\varepsilon) = \max_{I \geq 0}\{u(f(K)\theta - I,\varepsilon) + \beta\int g((1-\delta)K+I,\theta',\varepsilon')dF(\theta',\varepsilon')\}.$$

Let L^{3+} be the space of bounded continuous functions mapping R^{3+} into the real line. Then it is readily verified that T maps bounded functions into bounded functions. Application of the "maximum theorem" of Berge [3, p. 215, 216] shows that T maps continuous functions g into continuous functions Tg. Therefore, T is an operator on the space of bounded continuous functions L^{3+}, mapping bounded continuous functions into bounded continuous functions.

As a norm on L^{3+}, take

$$||g_1-g_2|| = \sup_{K,\theta,\varepsilon} |g_1(K,\theta,\varepsilon)-g_2(K,\theta,\varepsilon)|$$

where $g_1 \varepsilon L^{3+}$, $g_2 \varepsilon L^{3+}$. With this norm, the space $(L^{3+}, ||\cdot||)$ is complete, so that the contraction mapping theorem is potentially applicable.[7/]

It can be verified that the operator T satisfies Blackwell's [5] pair of sufficient conditions for T to be a contraction operator:

i. T is monotone, i.e., if $g_1(K,\theta,\varepsilon) \geq g_2(K,\theta,\varepsilon)$ for all (K,θ,ε) R^{3+}, then $Tg_1(K,\theta,\varepsilon) \geq Tg_2(K,\theta,\varepsilon)$ for all $(K,\theta,\varepsilon)\varepsilon R^{3+}$.

ii. For all constants γ and all $g \varepsilon L^{3+}$, $T(g+\gamma) = Tg + \beta\gamma$.

By virtue of Blackwell's [5] theorem 5, satisfaction of (i) and (ii) implies that T is a contraction mapping. Therefore, application of the contraction mapping theorem proves:[8/]

Proposition 1: The functional equation $v(K,\theta,\varepsilon) = Tv(K,\theta,\varepsilon)$ has a unique continuous bounded solution $v(K,\theta,\varepsilon)$. Furthermore, given any $v_0 \varepsilon L^{3+}$, $\lim_{n\to\infty} T^n v_0 \to v$ where the convergence is in the sup norm. This implies that the convergence is uniform.

It is also possible to prove:

Proposition 2: The value function $v(K,\theta,\varepsilon)$ is strictly concave in K for each fixed pair (θ,ε).

This follows because T maps concave functions into strictly concave functions.

We also have:

Proposition 3: The value function $v(K,\theta,\varepsilon)$ is uniquely attained by the single-valued policy function $I = I(K,\theta,\varepsilon)$. The function $I(K,\theta,\varepsilon)$ is continuous.

Uniqueness of the maximizing value of I is implied by the strict concavity of $u(\cdot)$ and $v(\cdot)$. Continuity of the policy function $I(\cdot)$ is implied by the "maximum theorem" of Berge [5, p. 215-216].

Now choose $v^0(K,\theta,\varepsilon)$ to be nondecreasing in K, strictly concave in K, and continuously differentiable in K. Define $v^{j+1}(K,\theta,\varepsilon) = Tv^j(K,\theta,\varepsilon)$. We shall show that $v^{j+1}(K,\theta,\varepsilon)$ is continuously differentiable in K for each fixed (θ,ε), provided that $v^j(K,\theta,\varepsilon)$ is continuously differentiable in K for each fixed (θ,ε). Consider

$$(13) \qquad v^{j+1}(K,\theta,\varepsilon) = \max_{I^j \geq 0} \{u(f(K)\theta-I,\varepsilon)+\beta\int v^j((1-\delta)K+I^j,\theta',\varepsilon')dF(\theta',\varepsilon')\}$$

and assume that $v^j(K,\theta,\varepsilon)$ is nondecreasing in K, concave and continuously differentiable in K for each fixed (θ,ε). The first-order necessary condition for the maximum problem on the right-hand side is 9/

$$(14) \qquad -u_c(f(K)\theta-I^j,\varepsilon) + \beta\int v_K^j((1-\delta)K+I^j,\theta',\varepsilon')dF(\theta',\varepsilon') \leq 0,$$

$$= 0 \text{ if } I^j > 0.$$

Let $\tilde{I}^j = \tilde{g}^j(K,\theta,\varepsilon)$ be the solution of (14) with equality replacing the inequality, so that $\tilde{g}^j(K,\theta,\varepsilon)$ would be the optimal rate of investment given terminal reward function $v^j(\cdot)$ if the inequality constraint $I^j \geq 0$ were not present. Then the optimum rate of investment I^j implied by (14) is

$$I^j = I^j(K,\theta,\varepsilon) = \max(0,\tilde{g}^j(K,\theta,\varepsilon)).$$

That $I^j(K,\theta,\varepsilon)$ is continuous is implied by the maximum theorem of Berge. We consider three sets for (K,θ,ε):

i. the set of (K,θ,ε) such that $I^j > 0$.

ii. The set of (K,θ,ε) such that $I^j = 0$ and $\tilde{g}^j(K,\theta,\varepsilon) < 0$.

iii. Points (K,θ,ε) such that $\tilde{g}^j(K,\theta,\varepsilon) = 0$.

On the first set of points (K,θ,ε) such that $I^j > 0$, Benveniste and Scheinkman's [2] theorem implies that $v^{j+1}(K,\theta,\varepsilon)$ is differentiable in K with derivative given by

$$(15)\qquad v_K^{j+1}(K,\theta,\varepsilon) = u_c(f(K)\theta - I^j(K,\theta,\varepsilon),\varepsilon)[f'(K)\theta+(1-\delta)].$$

On the second set of points (K,θ,ε) such that $I^j(K,\theta,\varepsilon) = 0$ and $\tilde{g}^j(K,\theta,\varepsilon) < 0$, $I^j(K,\theta,\varepsilon)$ is differentiable in K with derivative zero. Then direct calculations on (13) show that $v^{j+1}(K,\theta,\varepsilon)$ is differentiable with respect to K and that

$$(16)\qquad v_K^{j+1}(K,\theta,\varepsilon) = u_c(f(K)\theta,\varepsilon)f'(K)\theta$$
$$+ \beta(1-\delta)\int v_K^j((1-\delta)K+I^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Now consider the third set of points (K,θ,ε) such that $I^j(K,\theta,\varepsilon) = 0 = \tilde{g}^j(K,\theta,\varepsilon)$. At such points $I^j(K,\theta,\varepsilon)$ has a right-hand derivative with respect to K. The argument used for set (ii) implies that $v^{j+1}(K,\theta,\varepsilon)$ is differentiable from the right at points in set (iii), with right-hand derivative given by formula (16). We now undertake to show that $v^{j+1}(K,\theta,\varepsilon)$ is also differentiable from the left in set (iii) and that the left-hand derivative is also given by (16).

First note that in region (i) since $I^j > 0$, the first order necessary condition (14) holds with equality. Substituting (14) with equality into (15) yields

$$v_K^{j+1}(K,\theta,\varepsilon) = u_c(f(K)\theta - I^j(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$

(17)

$$+ \beta(1-\delta)\int v_K^j((1-\delta)K+I^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Thus, (17) holds for sets (i) and (ii) and also gives the right-hand derivative on set iii. We wish to show that the left-hand derivative of $v^{j+1}(K,\theta,\varepsilon)$ exists at points in set (iii) and also equals (17). Let (K,θ,ε) be in set (iii), and let $\Delta > 0$. We know that $v^{j+1}(K,\theta,\varepsilon)$ is continuous on the closed interval $[K-\Delta,K]$ and is differentiable on the open interval $(K-\Delta,K)$, each point of which is in set (i). By the mean value theorem for derivatives, there exists a point ξ belonging to the open interval $(K-\Delta,K)$ for which

$$\frac{v^{j+1}(K,\theta,\varepsilon) - v^{j+1}(K-\Delta,\theta,\varepsilon)}{\Delta} = v_K^{j+1}(\xi,\theta,\varepsilon).$$

Taking the limit as Δ goes to zero proves that the left-hand derivative of $v^{j+1}(K,\theta,\varepsilon)$ at (K,θ,ε) exists and equals the limit of the derivatives $v_K^{j+1}(\xi,\theta,\varepsilon)$ as ξ approaches K from the left. From (15) or (17), we know that this latter limit exists since the right-hand side of (15) or (17) is continuous in K. Therefore, we have that the left-hand derivative of $v^{j+1}(K,\theta,\varepsilon)$ at K exists and equals the right side of (17), as does the right-hand derivative. In summary, it follows that for (K,θ,ε) in all three regions, the partial derivative of $v^{j+1}(K,\theta,\varepsilon)$ with respect to K exists and is given by (17).

We have now proved:

Proposition 4: Choose $v^0(K,\theta,\varepsilon)$ to be nondecreasing and concave in K with bounded and continuous partial derivative in K. Generate the sequence $v^j(K,\theta,\varepsilon) = T^j v^0(K,\theta,\varepsilon)$. For all $j \geq 0$, $v^{j+1}(K,\theta,\varepsilon)$ is continuously differentiable with respect to K with a partial derivative $v_K^{j+1}(K,\theta,\varepsilon)$ satisfying equation (17).

Now choose $v^0(K,\theta,\varepsilon)$ so that it has a continuous bounded partial derivative in K for each (θ,ε). Consider the mapping S associated with (17), namely,

$$(Sg)(K,\theta,\varepsilon) \equiv u_c(f(K)\theta - I^j(K,\theta,\varepsilon),\varepsilon)f'(K)\theta + \beta(1-\delta)\int g((1-\delta)K + I^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon'). \tag{18}$$

The mapping S is an operator that maps bounded continuous functions $g(K,\theta,\varepsilon)$ into bounded continuous functions $(Sg)(K,\theta,\varepsilon)$. Further, we have:

i. S is monotone, i.e., if for every (K,θ,ε), $g_1(K,\theta,\varepsilon) \geq g_2(K,\theta,\varepsilon)$ where $g_1 \varepsilon L^{3+}$, $g_2 \varepsilon L^{3+}$, then $(Sg_1)(K,\theta,\varepsilon) \geq (Sg_2)(K,\theta,\varepsilon)$ for every (K,θ,ε).

ii. For every constant γ and every $g \varepsilon L^{3+}$, $S(g+\gamma)(K,\theta,\varepsilon) = (Sg)(K,\theta,\varepsilon) + \beta(1-\delta)\gamma$.

Application of Blackwell's theorem 5 then shows that S is a contraction mapping. We therefore have proved

Proposition 5: Suppose $v_K^0(K,\theta,\varepsilon)$ is continuous and bounded. Then $v_K^j(K,\theta,\varepsilon)$ exists for all $j \geq 1$ and the sequence of functions $v_K^j(K,\theta,\varepsilon) = S^j v_K^0(K,\theta,\varepsilon)$ converges uniformly (i.e., in the sup norm) to a bounded, continuous function $\tilde{v}_K(K,\theta,\varepsilon)$.

From the uniform convergence of $v^j(K,\theta,\varepsilon)$ to $v(K,\theta,\varepsilon)$ and the uniform convergence of $v^j_K(K,\theta,\varepsilon)$ to $\tilde{v}_K(K,\theta,\varepsilon)$ we immediately have (see Apostol [1, p. 238-239]),

<u>Proposition 6</u>: The value function $v(K,\theta,\varepsilon)$ is continuously differentiable in K with $v_K(K,\theta,\varepsilon) = \tilde{v}_K(K,\theta,\varepsilon)$. The partial derivative obeys the equation

$$(19) \qquad v_K(K,\theta,\varepsilon) = u_c(f(K)\theta - I(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$

$$\beta(1-\delta)\int v_K((1-\delta)K+I(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Proposition 6 implies that the first-order necessary condition for the maximum problem on the right side of (11) is

$$(20) \qquad -u_c(f(K)\theta - I,\varepsilon) + \beta\int v_K((1-\delta)K+I,\theta',\varepsilon')dF(\theta',\varepsilon') \leq 0,$$

$$= 0 \text{ if } I > 0.$$

Proposition 6 implies that obvious candidates for the equilibrium price functions $r(K,\theta,\varepsilon)$, $p_K(K,\theta,\varepsilon)$, and $w(K,\theta,\varepsilon)$ are

$$r(K,\theta,\varepsilon) = f'(K)\theta$$

$$(21) \qquad w(K,\theta,\varepsilon) = f(K)\theta - Kf'(K)\theta$$

$$p_K(K,\theta,\varepsilon) = \{u_c(f(K)\theta - I(K,\theta,\varepsilon),\varepsilon)\}^{-1}.$$

$$\beta\int v_K((1-\delta)K+I(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

It can be verified that with these price functions and with $h(K,\theta,\varepsilon)$ taken to be given by

$$(22) \qquad h(K,\theta,\varepsilon) \equiv (1-\delta)K + I(K,\theta,\varepsilon)$$

the market model of Section 2 is in equilibrium with the representative consumer's choice of $i(K,\theta,\varepsilon;\ K,f)$ equaling $I(K,\theta,\varepsilon)$ the planner's investment plan, and with the representative consumer's choice of $c(K,\theta,\varepsilon;\ K,f)$ equaling $f(K)\theta - I(K,\theta,\varepsilon)$. This can be verified by noting first that with (21), firms' marginal conditions are satisfied. Second, note that with (21), (22), and the proposed choices of i( ) and c( ), the marginal conditions for the representative agent in the market problem exactly match the planner's marginal condition (20). For example, with the suggested substitutions condition (9) becomes

$$
\begin{aligned}
&-u_c(f(K_t)\theta_t - I(K_t,\theta_t,\varepsilon_t),\varepsilon_t) \\
&\quad + \beta\int\{u_c(f(K_{t+1})\theta_{t+1} - I(K_{t+1},\theta_{t+1},\varepsilon_{t+1}),\varepsilon_{t+1}) \\
&\cdot[f'(K_{t+1})\theta_{t+1} + (1-\delta)u_c(f(K_{t+1})\theta_{t+1} - I(K_{t+1},\theta_{t+1},\varepsilon_{t+1}),\varepsilon_{t+1})^{-1} \\
&\cdot\beta\int v_K((1-\delta)K_{t+1} + I(K_{t+1},\theta_{t+1},\varepsilon_{t+1}),\theta_{t+2},\varepsilon_{t+2})dF(\theta_{t+2},\varepsilon_{t+2})]\} \\
&dF(\theta_{t+1},\varepsilon_{t+1}) \le 0
\end{aligned}
$$

with equality if $I(K_t,\theta_t,\varepsilon_t) > 0$. But notice that from (19), the term in braces simply equals $v_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})$. Therefore the above inequality becomes

$$
\begin{aligned}
&-u_c(f(K_t)\theta_t - I(K_t,\theta_t,\varepsilon_t),\varepsilon_t) \\
&\quad + \beta\int v_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})dF(\theta_{t+1},\varepsilon_{t+1}) \le 0;\ = 0 \text{ if } I_t > 0.
\end{aligned}
$$

This is equivalent with (20), as claimed.

Martingale Properties

From (17) and the first-order necessary condition (20) we have

$$v_K(K_t,\theta_t,\varepsilon_t) = v_K((1-\delta)K_{t-1}+I(K_{t-1},\theta_{t-1},\varepsilon_{t-1}),\theta_t,\varepsilon_t)$$

$$\geq [f'(K_t)\theta_t+(1-\delta)]\beta\int v_K((1-\delta)K_t+I(K_t,\theta_t,\varepsilon_t),\varepsilon_{t+1},\theta_{t+1})dF(\theta_{t+1},\varepsilon_{t+1})$$

with equality for $I(K_t,\theta_t,\varepsilon_t) > 0$. Integrating both sides with respect to $dF(\theta_t,\varepsilon_t)$ gives

$$\beta\int v((1-\delta)K_{t-1}+I(K_{t-1},\theta_{t-1},\varepsilon_{t-1}),\theta_t,\varepsilon_t)dF(\theta_t,\varepsilon_t)$$

$$\geq \beta\int[f'(K_t)\theta_t+(1-\delta)]\beta\int v_K((1-\delta)K_t+I(K_t,\theta_t,\varepsilon_t),\varepsilon_{t+1},\theta_{t+1})dF(\theta_{t+1},\varepsilon_{t+1})dF(\theta_t,\varepsilon_t)$$

or

$$p_{Kt-1} \geq E_{t-1}\{[(1-\delta)+f'(K_t)\theta_t]\cdot\beta p_{Kt}\frac{u_c(c(K_t,\theta_t,\varepsilon_t),\varepsilon_t)}{u_c(c(K_{t-1},\theta_{t-1},\varepsilon_{t-1}),\varepsilon_{t-1})}\} \quad (23)$$

where $c(K,\theta,\varepsilon) \equiv f(K)\theta - I(K,\theta,\varepsilon)$. Expression (23) shows that even adjusted for "dividends" and time preference, the relative price of existing capital is not a martingale, for essentially the same reason that the martingale property fails to hold in the models of Lucas [11] and Danthine [7]: the presence of corners, making (23) an inequality, and the presence of risk aversion, which is reflected in the failure of $u_c(\quad)$ to be constant as a function of consumption. The same message emphasized by Lucas and Danthine is carried by the present model: failure of the relative price p_K to be a martingale does not reflect on whether or not markets are in equilibrium.

Restrictions on 'Slopes'

The evolution of the aggregate capital stock is governed by the stochastic difference equation

$$K_{t+1} = (1-\delta)K_t + I(K_t,\theta_t,\varepsilon_t)$$

$$\equiv b(K_t,\theta_t,\varepsilon_t).$$

In studying the "stability" of this difference equation, we will need information about the slopes of b with respect to K, θ, and ε. The following argument is taken from Lucas.[10/] Rewrite the functional equation (11) as

$$(24) \qquad v(K,\theta,\varepsilon) = \max_{y \geq (1-\delta)k} \{u[(1-\delta)K+f(K)\theta-y,\varepsilon] + \beta\int v(y,\theta',\varepsilon')dF(\theta',\varepsilon')\}$$

where the right-hand side is uniquely attained by

$$y = b(K,\theta,\varepsilon)$$

$$\equiv I(K,\theta,\varepsilon) + (1-\delta)K.$$

Let us choose $v^0(K,\theta,\varepsilon)$ to be continuous, bounded, strictly concave, and twice differentiable in K. Then it follows that for all $j \geq 1$, $v^j(K,\theta,\varepsilon) = T^j v^0(K,\theta,\varepsilon)$ is twice differentiable in K (almost everywhere). This property is useful in establishing restrictions on the "slopes" of $b(K,\theta,\varepsilon)$. To establish this property, assume the $v^j(K,\theta,\varepsilon)$ is almost everywhere twice differentiable in K. Let $b^j(K,\theta,\varepsilon)$ attain $v^{j+1}(K,\theta,\varepsilon)$. Then off corners, the first-order necessary conditions for the maximization of $\{u[(1-\delta)K+f(K)\theta-y,\varepsilon]+\beta\int v^j(y,\theta',\varepsilon')dF(\theta',\varepsilon')\}$ are satisfied with equality. Differentiating the first-order condition shows that off corners, $b^j(K,\theta,\varepsilon)$ is differentiable with

$$\frac{\partial b^j}{\partial K} = \frac{u_{cc}\cdot[(1-\delta)+f'(K)\theta]}{u_{cc}+\beta\int v^j_{KK}(y,\theta',\varepsilon')dF(\theta',\varepsilon')} > 0 \tag{25a}$$

$$\frac{\partial b^j}{\partial \varepsilon} = \frac{u_{c\varepsilon}}{u_{cc}+\beta\int v^j_{KK}(y,\theta',\varepsilon')dF(\theta',\varepsilon')} < 0 \tag{25b}$$

$$\frac{\partial b^j}{\partial \theta} = \frac{u_{cc}f(K)}{u_{cc}+\beta\int v^j_{KK}(y,\theta',\varepsilon')dF(\theta',\varepsilon')} > 0. \tag{25c}$$

The terms $\int v^j_{KK}(y,\theta',\varepsilon')dF(\theta',\varepsilon')$ are well defined by the assumed (almost everywhere) twice differentiability of v^j, and the assumption that $F(\theta,\varepsilon)$ has a continuous density function and so assigns zero probability to points where $v^j(K,\theta,\varepsilon)$ is not twice differentiable. Where $b^j(K,\theta,\varepsilon) = (1-\delta)K$ and $g^j(K,\theta,\varepsilon) < 0$ (i.e., in our region ii), $b^j(K,\theta,\varepsilon)$ is differentiable with $\partial b^j/\partial K = (1-\delta)$, $\partial b^j/\partial\varepsilon = \partial b^j/\partial\theta = 0$. In region (iii), which is a set of Lebesque measure zero, $b^j(K,\theta,\varepsilon)$ is not differentiable. Now write (17) as

$$v^{j+1}_K(K,\theta,\varepsilon) = u_c(f(K)\theta+(1-\delta)K-b^j(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$
$$+ \beta(1-\delta)\int v^j_K(b^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Differentiating with respect to K gives

$$v_{KK}^{j+1}(K,\theta,\varepsilon) = u_{cc}f'(K)\theta[f'(K)\theta+(1-\delta)-b_K^j(K,\theta,\varepsilon)]$$

$$+ \beta(1-\delta)\int v_{KK}^j(b^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon')\cdot b_K^j(K,\theta,\varepsilon) + u_c f''(K)\theta.$$

Since the right-hand side exists almost everywhere, so does the left. So we have established that if $v^j(K,\theta,\varepsilon)$ is twice differentiable (a.e.) in K, then so is $v^{j+1}(K,\theta,\varepsilon)$. It follows that iterating with T on a $v^0(K,\theta,\varepsilon)$ that is continuous, bounded, strictly concave and twice differentiable in K gives rise to a sequence $b^j(K,\theta,\varepsilon)$ of approximate policy functions each member of which satisfies (25) off corners.

Notice that where $v_{KK}^{j+1}(K,\theta,\varepsilon)$ is attained with $b^j(K,\theta,\varepsilon) > (1-\delta)K$ so that (25) applies, we have

$$v_{KK}^{j+1}(K,\theta,\varepsilon) = u_c f''(K)\theta +$$

$$u_{cc}f'(K)\theta[f'(K)\theta+(1-\delta)- \frac{u_{cc}}{u_{cc}+\beta\int v_{KK}^j dF(\theta',\varepsilon')}((1-\delta)+f'(K)\theta)]$$

$$+ \beta(1-\delta)\int v_{KK}^j dF(\theta',\varepsilon') \cdot \frac{u_{cc}}{u_{cc}+\beta\int v_{KK}^j dF(\theta',\varepsilon')}((1-\delta) + f'(K)\theta)$$

or

$$v_{KK}^{j+1}(K,\theta,\varepsilon) = \frac{\beta\int v_{KK}^j dF(\theta',\varepsilon')}{u_{cc}+\beta\int v_{KK}^j dF(\theta',\varepsilon')} \cdot u_{cc} \cdot [(1-\delta)+f'(K)\theta]^2 + u_c f''(K)\theta.$$

It follows that off corners

(26) $$v_{KK}^{j+1}(K,\theta,\varepsilon) \geq u_{cc} \cdot [(1-\delta)+f'(K)\theta]^2 + u_c f''(K)\theta$$

"On corners," i.e., when $v_{KK}^{j+1}(K,\theta,\varepsilon)$ is attained where $b^j(K,\theta,\varepsilon) = (1-\delta)K$, we have

$$(27)\qquad v_{KK}^{j+1}(K,\theta,\varepsilon) = u_{cc}\cdot[f'(K)\theta]^2 + u_c f''(K)\theta + \beta(1-\delta)^2\int v_{KK}^{j}((1-\delta)K,\theta',\varepsilon')dF(\theta',\varepsilon').$$

Evidently, (26) and (27) imply that $\int v_{KK}^{j}(b^{j}(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon')$ is uniformly (in j and K) bounded in absolute value on the compact interval $[K_e, K_u]$, where $K_u > K_e > 0$.

The boundedness of $\int v_{KK}^{j}dF(\theta',\varepsilon')$ together with (25a), (25b) and (25c) imply that off corners for K in the compact interval $[K_e,K_u]$, $K_u>K_e>0$, the derivatives $\partial b^j/\partial K$, $\partial b^j/\partial\theta$, $\partial b^j/\partial\varepsilon$ remain uniformly strictly bounded away from zero in the directions given by (25).

The differentiability of $b^j(K,\theta,\varepsilon)$ does not necessarily carry over to the pointwise limit function $b(K,\theta,\varepsilon)$. However, the restrictions that the derivatives in (25) impose on the finite differences of $b^j(K,\theta,\varepsilon)$ <u>do</u> carry over to $b(\cdot,\cdot,\cdot)$. In particular, we have that off corners

$$b(K_2,\theta,\varepsilon) - b(K_1,\theta,\varepsilon) \geq \alpha_1(K_2-K_1) \quad , \alpha_1 > 0 \quad K_1, K_2\varepsilon[K_e,K_u]$$

$$b(K,\theta_2,\varepsilon) - b(K,\theta_1,\varepsilon) \leq \alpha_2(\theta_2-\theta_1) \quad \alpha_2 < 0$$

$$b(K,\theta,\varepsilon_2) - b(K,\theta,\varepsilon_1) \leq \alpha_3(\varepsilon_2-\varepsilon_1), \qquad \alpha_3 > 0.$$

These restrictions are used in the appendix, where we discuss how to adapt Mirman or Lucas's proof of the stochastic stability of the difference equation $K_{t+1} = b(K_t,\theta_t,\varepsilon_t)$.

4. Sample Economies

The preceding section shows that aggregate investment I_t and the relative price of existing capital p_{Kt} can each be expressed as continuous functions of the aggregate state $(K_t, \theta_t, \varepsilon_t)$

$$p_{Kt} = p_K(K_t, \theta_t, \varepsilon_t)$$

$$I_t = I(K_t, \theta_t, \varepsilon_t).$$

It was shown that each of these functions reflects <u>all</u> of the parameters of the economy. In particular, the forms of <u>both</u> $p_K(\cdot)$ and $I(\cdot)$ depend on (i) the form of the utility function $u(c,\varepsilon)$, (ii) the form of the production function $f(K)\theta$, and (iii) the nature of the distribution of random shocks $F(\theta_t, \varepsilon_t)$. Thus, while the model can be seen to imply a pattern of covariation between I_t and P_{Kt}, the nature of that covariation reflects consumers' preferences, technology, and the probability distribution of the shocks θ and ε.

To make this point more formally, let

$$P(K'|K) = \text{prob}\{K_{t+1} \leq K' | K_t = K\}$$

$$= \int_{A(K',K)} dF(\theta,\varepsilon)$$

where

$$A(K',K) = \{(\theta,\varepsilon): (1-\delta)K + I(K,\theta,\varepsilon) \leq K'\}.$$

Here the stochastic kernel $P(K'|K)$ defines a first-order Markov process for capital per man. Let

$$\Psi_0(K) = \text{Prob}\{K_0 \leq K\}$$

be given. In the appendix, it is proved that the Markov process for K possesses a unique stationary distribution $\Psi(K)$ which is approached by iterations on

$$\Psi_{t+1}(K') = \int P(K'|K)d\Psi_t(K)$$

where $\Psi_{t+1}(K') = \text{Prob}\{K_{t+1} \leq K'\}$. The stationary distribution $\Psi(K)$ uniquely solves

$$\Psi(K') = \int P(K'|K)d\Psi(K).$$

Since (ε, θ) is a serially independent process, it follows that (K, θ, ε) are mutually independent contemporaneously. Therefore, the stationary moments of p_K and I can be calculated, for example, by

$$E(I \cdot p_K) = \iint p_K(K, \theta, \varepsilon) \cdot I(K, \theta, \varepsilon)dF(\theta, \varepsilon)d\psi(K)$$

$$E(p_K^2) = \iint p_K^2(K, \theta, \varepsilon)dF(\theta, \varepsilon)d\psi(K).$$

It is then clear that, for example, the regression coefficient of I on p_K, is in general a function of all of the parameters in the model. Further, the strong law of large numbers for Markov processes stated by Doob [8] tells us that sample moments such as

$$\frac{1}{T}\sum_{t=1}^{T} I_t p_{Kt}, \ \frac{1}{T}\sum_{t=1}^{T} p_{Kt}^2, \ \frac{1}{T}\sum_{t=1}^{T} p_{Kt}, \text{ etc.}$$

converge with probability one to the corresponding moments of the stationary distribution EIp_K, Ep_K^2, Ep_K, etc., respectively.

We carried out some calculations designed to illustrate how the regression of I on p_K depends on various parameters. We assumed that the distributions of ε_i and θ_i were concentrated on two points with

$$\text{Prob}\{\theta=\theta_1\} = p_1$$

$$\text{Prob}\{\theta=\theta_2\} = 1-p_1 \equiv p_2$$

$$\text{Prob}\{\varepsilon=\varepsilon_1\} = q_1$$

$$\text{Prob}\{\varepsilon=\varepsilon_2\} = 1-q_1 \equiv q_2$$

$$\text{Prob}\{\theta=\theta_i,\varepsilon=\varepsilon_j\} = p_i q_j, \; i=1,\ 2;\ j=1,\ 2.$$

We specified a grid of admissible points along the capital-labor axis, restricting the planner to choose among this finite set of feasible points, call it $\overline{K}$. The functional equation for the optimal value function is

$$(28) \qquad v(K_a, \theta_i, \varepsilon_j) = \max_{\substack{I \geq 0 \\ I+(1-\delta)K_a \varepsilon \overline{K}}} \{u(f(K_a)\theta_i - I, \varepsilon_j) + \beta \sum_s \sum_m v((1-\delta)K_a + I, \theta_s, \varepsilon_m) p_s q_m\}$$

where $K_a \varepsilon \overline{K}$. Notice that next period's capital stock $I+(1-\delta)K_a$ is required to belong to the set $\overline{K}$. The grid of feasible points $\overline{K}$ was chosen as follows. Where the grid contains n points and $\approx{K}$ was chosen as the highest capital-labor ratio in the grid, we chose

$$K_{n-j+1} = (1-\delta)^{j/m} \cdot \overset{\approx}{K} \qquad j=1, \ldots, n$$

where m is a positive integer. Notice that the grid is chosen so that the "corner points" $(1-\delta)K$ are included. In practice, $\overset{\approx}{K}$ and m were chosen so that the grid at least covered the set of ergodic states for the capital-labor ratio.

We solved the functional equation (28) by in effect iterating on the "T mapping" described in the discussion of Proposition 1. In practice we used an algorithm described by Bertsekas [4, pp. 237-241] to speed up the convergence. We are constrained to consider variations in the investment rate of Δ where Δ is the distance between adjacent points in $\overline{K}$. The necessary condition for the maximum problem on the right side of (28) is that for $\hat{I}$ optimal

$$u(f(K_a)\theta_i - \hat{I}, \varepsilon_j) + \beta \sum_s \sum_m v((1-\delta)K_a + \hat{I}, \theta_s, \varepsilon_m) p_s q_m$$

$$\geq u(f(K_a)\theta_i - (\hat{I}+\Delta), \varepsilon_j) + \beta \sum_s \sum_m v((1-\delta)K_a + (\hat{I}+\Delta), \theta_s, \varepsilon_m) p_s q_m$$

for all $\Delta > 0$ and for all $\hat{I} + \Delta \geq 0$ or $\Delta \geq -\hat{I}$, where $\hat{I} + (1-\delta)K_a \varepsilon \overline{K}$ and $\hat{I} + \Delta + (1-\delta)K_a \varepsilon \overline{K}$. The optimizing I thus satisfies the condition that it is the largest value of I for which

$$(29) \qquad \frac{u(f(K_a)\theta_i - I, \varepsilon_j) - u(f(K_a)\theta_i - (I+\Delta), \varepsilon_j)}{\Delta}$$

$$\geq \frac{\beta \sum_s \sum_m (v((1-\delta)K_a + I + \Delta, \theta_s, \varepsilon_m) - v((1-\delta)K_a + I, \theta_s, \varepsilon_m)) p_s q_m}{\Delta}$$

for all admissible $\Delta > 0$. For the smallest admissible Δ, we take the left side of (29) as our estimate of $u_c(c, \varepsilon_j)$, while we take the right side as our estimate of $p_K \cdot u_c(c,\varepsilon)$. We form our estimate of $p_K(K_a,\theta_i,\varepsilon_j)$ by dividing the latter by the former. The optimum policy function $I(K_a,\theta_i,\varepsilon_j)$ is obtained as a by-product of solving for the optimal value function.

We generated the stochastic matrix associated with the Markov process for K from

$$P_{ij} = \text{Prob}\{K_{t+1}=K_i | K_t=K_j\}$$

$$= \text{Prob}\{I(K_j,\theta,\varepsilon)+(1-\delta)K_j=K_i\}$$

$$= \sum_{s,m\varepsilon T} p_s q_m$$

where $T = \{(s,m): \; I(K_j,\theta_s,\varepsilon_m)+(1-\delta)K_j=K_i\}$. An (nxn) stochastic matrix P with elements P_{ij} was formed, with n being the number of points in the set of admissible capital stocks $\overline{K}$. Then the stationary distribution of K was determined by taking any column of $\lim_{t\to\infty} P^t$ (in the limit the columns of P^t are all the same, if P possesses a unique stationary distribution). For the stationary distribution of K_t we denote

$$\text{Prob}\{K_t=K_i\} = \pi_i, \; K_i \varepsilon \overline{K}, \; i=1, \; \ldots, \; n.$$

We calculated the population moments of I and P_K from, e.g.,

$$EI(K,\theta,\varepsilon) = \sum_{h=1}^{n} \sum_{i=1}^{2} \sum_{j=1}^{2} I(K_h,\theta_i,\varepsilon_j)\pi_h p_i q_j$$

$$EI(K,\theta,\varepsilon) \cdot p_K(K,\theta,\varepsilon) = \sum_{h=1}^{n} \sum_{i=1}^{2} \sum_{j=1}^{2} I(K_h,\theta_i,\varepsilon_j) p_K(K_h,\theta_i,\varepsilon_j)\pi_h p_i q_j.$$

Table 1

$u(c) = \varepsilon \cdot \ln c$ $\quad\quad$ $\delta = .05$

$f(K) = K^{(.25)}$ $\quad\quad$ $\beta = .95$

Economy 1 (64 points in $\overline{K}$)

$P\{\theta=.9\} = .5,\ P\{\theta=1.1\} = .5$

$P\{\theta=.9\} = .75,\ P\{\theta=1.1\} = .25$

$$\frac{\text{cov}(I, p_K)}{\text{var } p_K} = 1.5798$$

$$\frac{\text{cov}(I, p_K)}{\sqrt{\text{var } I \cdot \text{var } p_K}} = .4213$$

$$\frac{\text{cov}(I, p_K)}{\text{var } I} = .1123$$

Economy 2 (64 points in $\overline{K}$)

$P\{\theta=.9\} = .5$ $\quad\quad$ $P\{\theta=1.1\} = .5$

$P\{\varepsilon=.9\} = .25$ $\quad\quad$ $P\{\theta=1.1\} = .75$

$$\frac{\text{cov}(I, p_K)}{\text{var } p_K} = .8653$$

$$\frac{\text{cov}(I, p_K)}{\sqrt{\text{var } I \cdot \text{var } p_K}} = .0934$$

$$\frac{\text{cov}(I, p_K)}{\text{var } I} = .0101$$

Economy 3

$P\{\theta=.9\} = .5 \qquad P\{\theta=1.1\} = .5$

$P\{\varepsilon=.9\} = .5 \qquad P\{\varepsilon=1.1\} = .5$

	(32 states in $\overline{K}$)	(48 states in $\overline{K}$)	(64 states in $\overline{K}$)	(80 states in $\overline{K}$)
$\dfrac{\text{cov}(I,p_K)}{\text{var } p_K}$	= 1.31913	2.1889	2.5057	2.7491
$\dfrac{\text{cov}(I,p_K)}{\sqrt{\text{var } I\cdot\text{var } p_K}}$	= .3115	.4572	.4933	.5261
$\dfrac{\text{cov}(I,p_K)}{\text{var } I}$	= .0735	.0955	.0971	.1007

Table 1 gives examples for an economy in which $u(c) = \varepsilon \ln c$ and $f(K) = K^{.25}$, $\beta = .95$, and $\delta = .05$. The set $\overline{K}$ included sixty-four states, except where otherwise noted. For the parameters of economy 3, we have calculated the sample moments for alternative $\overline{K}$'s including 48, 64, and 80 states. These calculations for increasingly fine grids on K are interesting if one views these finite economies as approximations to the continuous-state economy analyzed in previous sections. From the behavior of these moments with increasingly fine grids, our grids are evidently not yet fine enough to approximate the corresponding continuous-state economies very well. An alternative way to view these calculations is not as giving approximations but exact evaluations of the population moments of the indicated finite-state economies. The three economies are identical except that they are characterized by different distributions of the shock to preferences $\{\varepsilon\}$. Notice the effects of alterations in the distribution on the population values of the regression coefficient of I on p_K, given by $\text{cov}(I,p_K)/\text{var } p_K$, and on the correlation coefficient between I and p_K, given by $\mathrm{cov}(I,P_K)/\sqrt{\mathrm{var}\ I \cdot \mathrm{var}\ p_K}$. The table illustrates how, in the jargon of macroeconomists, shifts in the distribution of the consumption function cannot be expected to leave the regression of I on p_K unaltered. Graph 1 depicts the population discrete density function giving the unique stationary distribution associated with economy 3 with 64 states.

These examples illustrate how in such an economy, the regression of investment on p_K does not recover the law governing the demand to accumulate capital. The problem is not a failure to correct for simultaneous equation bias, say by using instrumental variables, nor is it a failure to include enough lagged values of q. In these economies it would be impossible to recover a structural investment schedule by pursuing such modifications.

It is straightforward to describe econometric procedures that would permit recovery of the economy's structural parameters from time series data on y_t, K_t, and p_{Kt}. It would be necessary to specify functional forms for $u(c,\varepsilon)$ and $f(K)\theta$, as well as a form for the distribution $F(\theta,\varepsilon)$. Then for each point in the space of parameters determining β, δ, $u(\cdot,\cdot)$, $f(\cdot)$, and $F(\cdot,\cdot)$, there is a unique pair of functions $I(K,\theta,\varepsilon)$ and $p_K(K,\theta,\varepsilon)$. The likelihood function of a vector of time series on (y_t, K_t, p_{Kt}) can then be characterized as a function of the free parameters of $\{\beta,\delta, u(\cdot,\cdot), f(\cdot)$, and $F(\cdot,\cdot)\}$. The method of maximum likelihood could then be used to estimate the structural parameters of the economy. As of now, such procedures would be very expensive even for the very simple economy that we have described. They would be prohibitively expensive for any "realistic" model.

Of course, in our sample economies the least squares regression of I on p_K is predicted to remain the same so long as the distributions of all shocks remain unaltered. It is possible to construct examples, as we have in Table 1, in which p_K explains a large part of the variation in investment. But one wants a structural model of investment in order to be able to analyze interventions in the forms of alterations in certain random processes, in particular, in processes describing various aspects of fiscal policy. It is for analyzing such policy changes that our analysis suggests that it will be inadequate to rely on the maintenance of historical patterns between I and p_K.

5. Concluding Remarks

The following two features of our model deserve brief discussion: first, whenever p_K is less than unity, the aggregate rate of investment is zero; and second, it is impossible for p_K ever to be above unity. It is easy to conceive of variations on the present model in which aggregate investment is positive even when an aggregate index corresponding to p_K is less than unity. For example, consider a model with two goods, x and y, both of which are consumed while good y can also be used to augment the capital stock of industries x and y. Assume that new output of y can be costlessly allocated across consumption, investment in industry x, or investment in industry y. But once in place, capital in industries x and y cannot be consumed. This setup will give rise to two distinct prices of existing capital in industries x and y, say p_{Kx} and p_{Ky}, respectively, relative to newly produced capital. Investment in industry x will be positive only if p_{Kx} is unity and investment in industry y will be positive only if p_{Ky} is unity. But aggregate investment can be positive when an aggregate index of the price of existing capital relative to newly produced capital is less than unity. Conceptually, analysis of such a model is no more complicated than the one-sector model studied in this paper; it is only much more cumbersome notationally.

The second peculiarity of our model, the inability of p_K to rise above unity, stems from the asymmetry in the "friction" that we have posited. That is, the technological rigidity that we have posited impedes rapid decreases in the capital stock, but not increases. It seems clear that general equilibrium versions of the cost-of-change models of Lucas [12], Gould [10], and Treadway [21] which posit more or less symmetrical costs of adjustment, could be constructed in which measures of p_K would rise above unity.

There is no reason to believe that modifications along either of these lines would alter the basic message of this paper: that the same "frictions" or "adjustment costs" that make it possible for p_K or q to diverge from unity also establish a presumption that agents' investment decisions are not expressible in any simple way as a function of p_K.

Footnotes

1/Lucas uses a single representative consumer in exactly this way [11].

2/The proof of this proposition exactly parallels Lucas's [11] anlaogous proposition and will be omitted.

3/The proof parallels Lucas's [11]. Actually, only the sum $i + k^d$ is determined as a continuous function of the state variables k, θ, ε, K. This is because when $p_K(K,\theta,\varepsilon) = 1$, the agent is indifferent as to the breakdown of $i + k^d$ between i and k^d. Suppose we adopt the convention that when $p_K(K,\theta,\varepsilon) = 1$,

$$k^d = (1-\delta)k \qquad \text{if } k^d + i \geq (1-\delta)k$$

$$k^d = k^d + i \qquad \text{if } k^d + i < (1-\delta)k.$$

This convention resolves the indeterminancy when $p_K = 1$ and makes the resulting demand functions for i and k^d continuous.

4/The concavity of $J(\cdot)$ in k can be proved as in Lucas [11]. The differentiability of $J(\cdot)$ can be proved by following an argument analogous to the one used below in Section 3 to prove differentiability of v( ) with respect to K.

5/The condition that $u_c(0,\varepsilon) = \infty$ rules out the possibility of corner solutions with c = 0.

6/Calculated using the methods in Section 3 below.

7/See Naylor and Sell [17].

8/Propositions 1, 2, and 3 and their proofs mimic analogous propositions in Lucas [11] and Lucas and Prescott [13, 14]. For this reason, we only sketch the proofs.

9/Again, corner solutions with c = 0 are ruled out by the assumed form of the utility function.

10/From lectures in his Economics 337 class.

11/From lectures in Economics 337.

Appendix

Proving "Stochastic Stability"

We describe how to prove that the growth model possesses a unique stationary distribution over K to which the system converges starting from any arbitrary initial distribution over K. We will simply indicate how the proof in Mirman [15] or Lucas[11/] must be modified to account for the "corner" that is present in our problem. Let R be the real numbers, and $\mathcal{A}$ the Borel sets. Then a Markov process is defined by the stochastic kernel $P(x,A):D\times\mathcal{A}\to[0,1]$, where $D\subset R$. Here $P(x,A) = \text{Prob}\{x_{t+1}\varepsilon A|x_t=x\}$. For fixed x, P(x,A) is a probability distribution in A, while for any interval A, P(x,A) is a Baire function in x.

We need the following two definitions:

Definition: An interval $I \subset R$ is called an ergodic set if

i) $x\varepsilon I$ implies $P(x,I) = 1$.

ii) There is no $I' \subset I$ with $\lambda(I') < \lambda(I)$ such that $x\varepsilon I'$ implies $P(x,I') = 1$. (Here $\lambda(\cdot)$ is Lebesque measure.)

Definition: An ergodic set I is called noncyclic if for all $I' \subset I$ with $\lambda(I') > 0$, $x\varepsilon I'$ implies $P(x,I') > 0$.

Lucas and Mirman both use a version of Doob's

Condition D: There exists a probability measure φ on $(R,\mathcal{A})$, an integer n, and numbers $\varepsilon,\varepsilon' > 0$ such that $\varphi(A) \leq \varepsilon'$ implies that $P^n(x,A) \leq 1-\varepsilon$. Here $P^n(x,A) = \text{Prob}\{x_{t+n}\varepsilon A|x_t=x\}$.

Lucas's proof proceeds by verifying that for the stochastic growth model there obtain the hypotheses of the following theorem of Doob:

Theorem: Suppose the process defined by P(x,A) has a single, noncyclic ergodic set I and satisfies condition D. Then

i) There is a unique "invariant" probability distribution satisfying

$$q(A) = \int_R P(x,A)dq(x).$$

ii) $q(I) = 1$.

iii) For any q_0, $q_n(\cdot)$ converges in distribution to $q(\cdot)$ where $\{q_n\}$ is generated by iterations on

$$q_n(A) = \int_R P(x,A)dq_{n-1}(x).$$

We have assumed that there exist numbers $\bar{\theta} > \underline{\theta} > 0$ and $\bar{\varepsilon} > \underline{\varepsilon} > 0$ such that $\text{Prob}\{\underline{\theta}\leq\theta\leq\bar{\theta},\ \underline{\varepsilon}\leq\varepsilon\leq\bar{\varepsilon}\}= 1$. We have also assumed that $F(\theta,\varepsilon)$ has a continuous and strictly positive density on the rectangle $\{\underline{\theta}\leq\theta\leq\bar{\theta},\ \underline{\varepsilon}\leq\varepsilon\leq\bar{\varepsilon}\}$.

Figure (1) plots $b(K,\bar{\theta},\underline{\varepsilon})$ and $b(K,\underline{\theta},\bar{\varepsilon})$. We define $\underline{\underline{K}}$ and $\bar{\bar{K}}$ by $\underline{\underline{K}} = h(\underline{\underline{K}},\underline{\theta},\bar{\varepsilon})$ and $\bar{\bar{K}} = h(\bar{\bar{K}},\bar{\theta},\underline{\varepsilon})$ respectively. The following argument, adapted from Lucas, proves that the solution to e.g., $\bar{\bar{K}} = h(\bar{\bar{K}},\bar{\theta},\underline{\varepsilon})$ is unique, so that figure 1 is drawn correctly. At the steady-state values $\bar{\bar{K}},\underline{\theta},\bar{\varepsilon}$ the functional equation is satisfied with

$$v(\bar{\bar{K}},\bar{\theta},\underline{\varepsilon}) = u((1-\delta)\bar{\bar{K}} + f(\bar{\bar{K}})\bar{\theta}-\bar{\bar{K}},\underline{\varepsilon})$$
$$+ \beta\int v(\bar{\bar{K}},\theta',\varepsilon')dF(\theta',\varepsilon').$$

The definition of $v(K,\theta,\varepsilon)$ as the maximum attainable value starting from K implies that

$$v(K,\theta,\varepsilon) \geq u(f(K)\theta-\delta K,\varepsilon) + \beta\int v(K,\theta',\varepsilon')dF(\theta',\varepsilon').$$

Integrating both sides with respect to $dF(\theta,\varepsilon)$ and rearranging gives

$$\text{(A1)} \qquad \int v(K,\theta,\varepsilon)dF(\theta,\varepsilon) \geq \frac{1}{1-\beta}\int u(f(K)\theta-\delta K,\varepsilon)dF(\theta,\varepsilon).$$

Now since leaving the stationary point $\bar{\bar{K}} = h(\bar{\bar{K}},\bar{\theta},\underline{\varepsilon})$ must lower discounted expected utility, we have for $K \neq \bar{\bar{K}}$

(A2) $$v(\overline{\overline{K}},\overline{\theta},\underline{\varepsilon}) \geq u[(1-\delta)\overline{\overline{K}}+f(\overline{\overline{K}})\overline{\theta}-K,\underline{\varepsilon}] + \beta\int v(K,\theta',\varepsilon')dF(\theta',\varepsilon').$$

Combining (A2) with (A1) gives for all K

$$v(\overline{\overline{K}},\overline{\theta},\underline{\varepsilon}) \geq u[(1-\delta)\overline{\overline{K}}+f(\overline{\overline{K}})\overline{\theta}-K,\underline{\varepsilon}] + \frac{\beta}{1-\beta}\int u(f(K)\theta'-\delta K,\varepsilon')dF(\theta',\varepsilon').$$

Therefore $\overline{\overline{K}}$ solves the problem

$$\underset{K}{\text{maximize}}\{u[(1-\delta)\overline{\overline{K}}+f(\overline{\overline{K}})\overline{\theta}-K,\varepsilon] + \frac{\beta}{1-\beta}\int u(f(K)\theta'-\delta K,\varepsilon')dF(\theta',\varepsilon').$$

The first-order necessary condition for this problem is

$$0 = H = -u_c[(1-\delta)\overline{\overline{K}}+f(\overline{\overline{K}})\overline{\theta}-K,\varepsilon] + \frac{\beta}{1-\beta}\int u_c(f(K)\theta'-\delta K,\varepsilon')(f'(K)\theta'-\delta)dF(\theta',\varepsilon').$$

We calculate

$$\frac{dH}{dK} = u_{cc} + \frac{\beta}{1-\beta}\int u_{cc}(f(K)\theta'-\delta K,\varepsilon')(f'(K)\theta'-\delta)^2 dF(\theta',\varepsilon') + \frac{\beta}{1-\beta}\int u_c(f(K)\theta'-\delta K,\varepsilon')f''(K)\theta' dF(\theta',\varepsilon') < 0.$$

That $dH/dK < 0$ implies that there is a unique K that solves $H = 0$. This proves that there is a unique $\overline{\overline{K}}$ solving $\overline{\overline{K}} = h(\overline{\overline{K}},\overline{\theta},\underline{\varepsilon})$. Obviously, the same argument establishes the uniqueness of the solution $\underline{\underline{K}}$ to $\underline{\underline{K}} = h(\underline{\underline{K}},\underline{\theta},\overline{\varepsilon})$.

Following the argument of Mirman [15], it is possible to show that $I = [\underline{\underline{K}}, \overline{\overline{K}}]$ is the ergodic set. Further, I is noncyclic since for any interval E of positive length containing $K\varepsilon(\underline{\underline{K}},\overline{\overline{K}})$, $\text{Prob}\{K_{t+1}\varepsilon E|K_t=K\} > 0$.

As in Mirman [15], it can be shown that from any initial (distribution over) K, K will eventually remain in $(\underline{\underline{K}},\bar{\bar{K}})$ with probability one. Therefore, we will restrict our attention to the interval $(\underline{\underline{K}},\bar{\bar{K}})$.

For the purpose of indicating how to adapt Lucas's argument, nothing essential will be lost by assuming that $\underline{\varepsilon}=\bar{\varepsilon}$, so that ε is nonrandom. Suppressing the argument ε, we write the stochastic difference equation for K as $K_{t+1} = b(K_t,\theta_t)$. We rely heavily on Figures 2 and 3, which depict the function $K_{t+1} = b(K_t,\theta_t)$ in the (K_{t+1},K_t) and (K_{t+1},θ_t) planes, respectively. For $K_t \leq K_0$ corner solutions do not occur, while for $K_t > K_0$ they do occur with some probability, but for $K\varepsilon[\underline{\underline{K}},\bar{\bar{K}}]$ we shall show that this probability is uniformly (in K) bounded away from unity. This is enough to permit Lucas's proof to work.

Evidently, for each value of $K_t = K$, there is a value of $\theta = \tilde{\theta}(K)$ such that $K_{t+1} = b(K_t,\theta_t) = (1-\delta)K_t$ for $\theta_t < \tilde{\theta}(K_t)$ and $K_{t+1} = b(K_t,\theta_t) > (1-\delta)K_t$ for $\theta_t > \tilde{\theta}(K_t)$. There exists a value K_0, $\underline{\underline{K}} < K_0 < \bar{\bar{K}}$ such that for $K_t \leq K_0$, $\tilde{\theta}(K_t) = \underline{\theta}$; $\tilde{\theta}(K)$ is a continuous function of K and is increasing in K on $(K_0,\bar{\bar{K}})$. For $\delta > 0$, $\tilde{\theta}(\bar{\bar{K}}) < \bar{\theta}$. This follows because off corners, $b(K,\theta)$ is strictly increasing in θ, and $b(\bar{\bar{K}},\theta) > (1-\delta)\bar{\bar{K}}$. It follows that for all $K\varepsilon[\underline{\underline{K}},\bar{\bar{K}}]$

$$\text{Prob}\{K_{t+1}=(1-\delta)K|K_t=K\} \leq \text{Prob}\{\theta\leq\tilde{\theta}(\bar{\bar{K}})\}.$$

On our assumption that $F(\theta)$ has a strictly positive and continuous density on $[\underline{\theta},\bar{\theta}]$, it follows that the right-hand side of the above is strictly less than unity. Therefore for all $K\varepsilon[\underline{\underline{K}},\bar{\bar{K}}]$, there is a scalar $\rho = \text{Prob}\{\theta>\tilde{\theta}(\bar{\bar{K}})\} > 0$ such that

$$\text{(A3)} \qquad \text{Prob}\{K_{t+1}=(1-\delta)K|K_t=K\} \leq 1-\rho.$$

Notice that ρ is independent of K.

It is now straightforward to combine Lucas's results with (A3) to show that Condition D is satisfied. Following Lucas, let J=(a,d) be any interval in the bounded set $[\underline{\underline{K}},\bar{\bar{K}}]$. Choose measure $\varphi(J) = C_1(d-a)$, C_1 a normalizing constant. Now from section 3 we have that off corners, i.e., for $\theta_2 > \theta_1 > \tilde{\theta}(K)$, and for all $K \varepsilon [\underline{\underline{K}},\bar{\bar{K}}]$

$$b(K,\theta_2) - b(K,\theta_1) \geq \alpha(\theta_2-\theta_1)$$

where $\alpha > 0$. Notice that α is independent of K. Assume first that we are given a K and an interval I = [a,d] in which the system is off corners, the situation in figure (4).

Define $\theta_a(K)$ and $\theta_d(K)$ by $d = b(K,\theta_d(K))$, $a = b(K,\theta_a(K))$. Then we have

$$\theta_d(K) - \theta_a(K) \leq \frac{1}{\alpha}(b(K,\theta_d(K)), b(K,\theta_a(K))$$

$$\theta_d(K) - \theta_a(K) \leq \frac{1}{\alpha}(d-a),$$

uniformly in K. Now choose $\varepsilon' > 0$ such that $(d-a) \leq \varepsilon'$ implies

$$\text{Prob}\{\theta_a(K) \leq \theta \leq \theta_d(K)\} \leq \frac{\rho}{2}.$$

This is possible by the assumption that θ has a continuous density.

We now have to consider the situation in which the interval (a,d) includes values of K_{t+1} such that $K_{t+1} = (1-\delta)K_t$ for some values of $K_t = K$.

It will suffice to consider the "worst" possible placement of the interval [a,d] and choice of K depicted in figure 5, worst from the point of view of satisfying condition D. Here $d > a = (1-\delta)\overline{\overline{K}}$. This is the worst situation because evidently over all choices of $K_t = K$ and all positioning of intervals of length $C_1(d-a)$, this choice maximizes $\text{Prob}\{K_{t+1}=(1-\delta)K|K_t=K\}$. By the previous argument, we have that for $(d-a) \leq \varepsilon'$

$$\text{Prob}\{K_{t+1}\varepsilon I|K_t=\overline{\overline{K}}\} =$$

$$\text{Prob}\{K_{t+1}\varepsilon[a,d)|K_t=\overline{\overline{K}}\}+\text{Prob}\{K_{t+1}=(1-\delta)\overline{\overline{K}}|K_t=\overline{\overline{K}}\}.$$

$$\leq \frac{\rho}{2} + (1-\rho) = 1- \frac{\rho}{2}.$$

With the above choices of φ and ε', and with n=1 and $\varepsilon = \frac{\rho}{2} > 0$, Doob's condition D is satisfied.

So Doob's condition D is verified, as are the uniqueness and noncyclic nature of the ergodic set. Then application of the above quoted theory from Doob establishes that the stochastic growth model has a unique stationary distribution over K which assigns unit probability to $[\underline{\underline{K}},\overline{\overline{K}}]$, and to which the model converges in distribution starting from any arbitrary initial probability distribution over K.

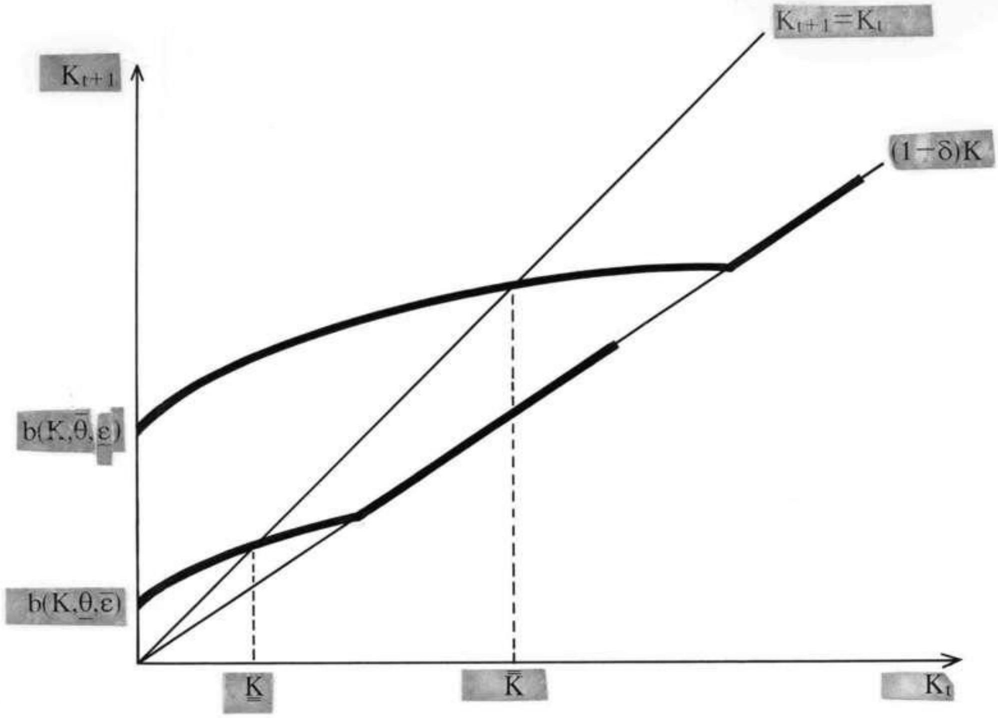

Figure 1

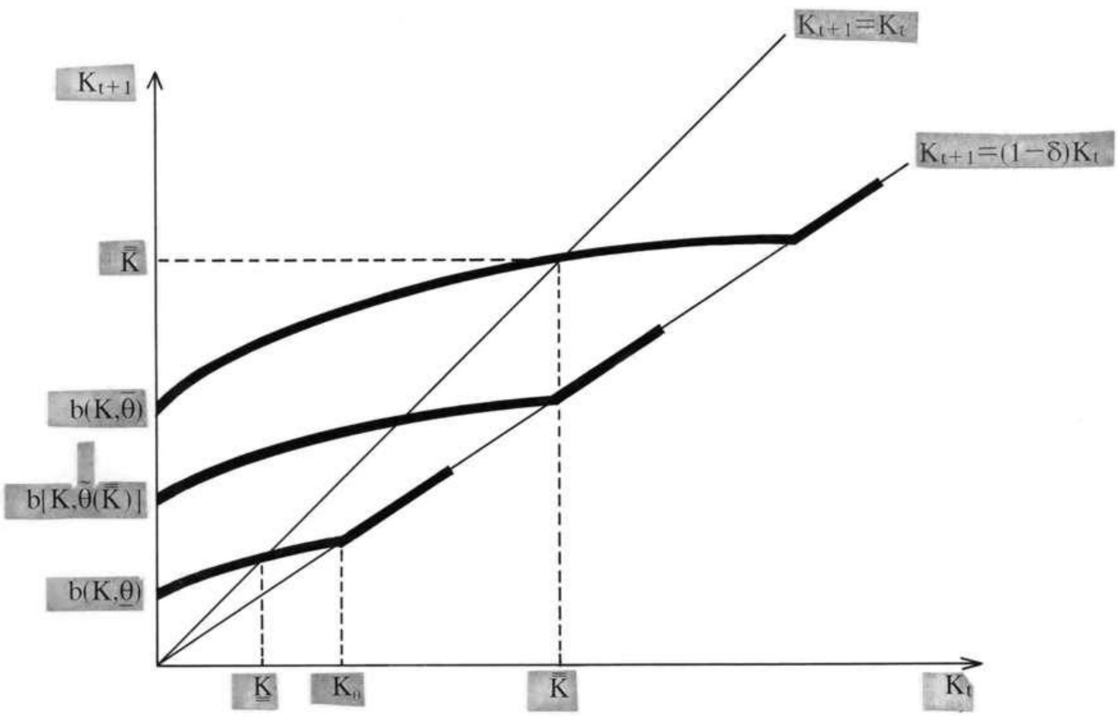

Figure 2

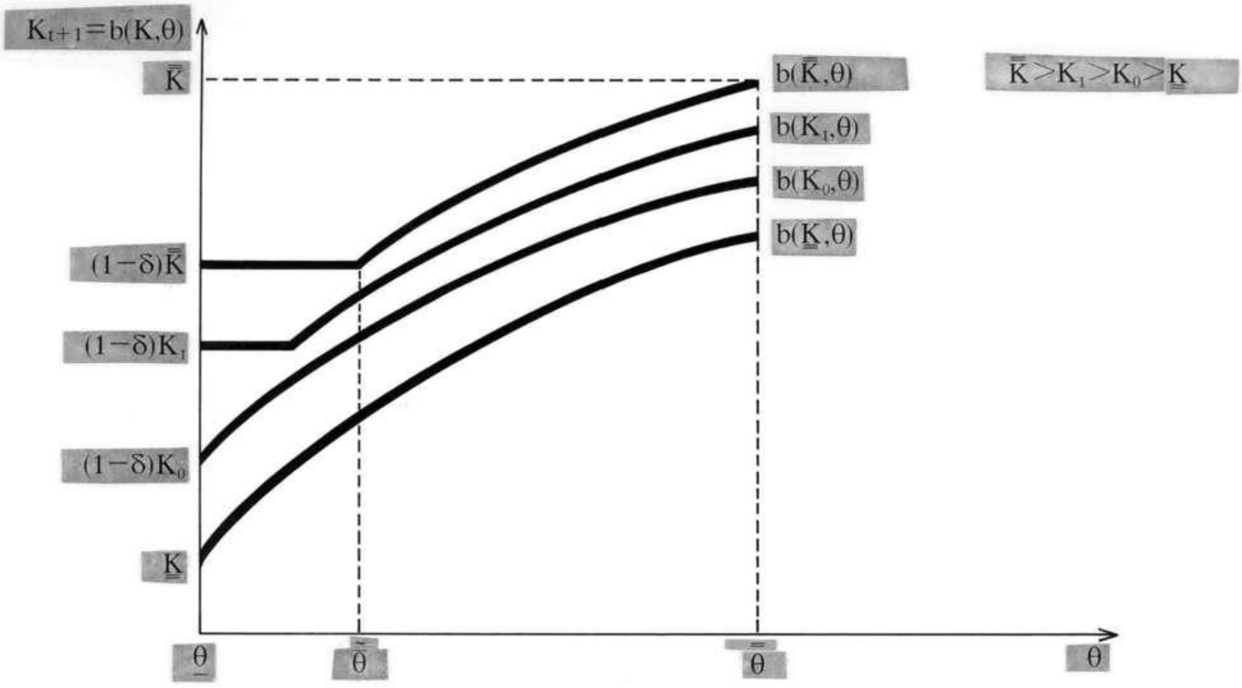

Figure 3

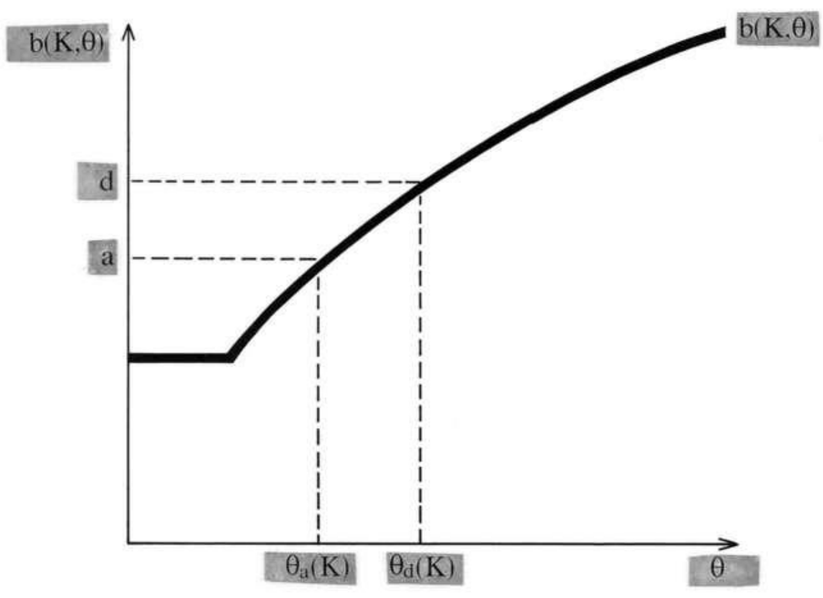

Figure 4

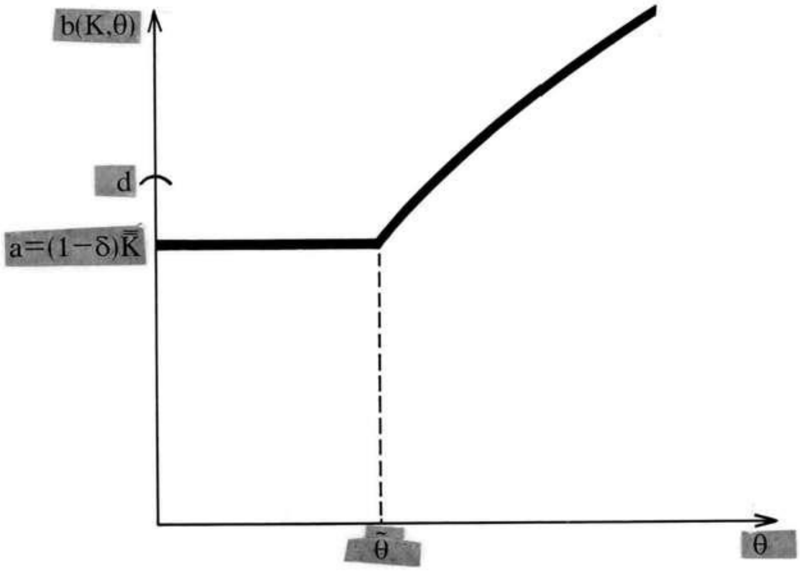

Figure 5

References

1. Apostol, Tom. Mathematical Analysis, second edition, Addison-Wesley, Reading Massachusetts, 1974.

2. Benveniste, L. M., and J. A. Scheinkman. "Differentiable Value Functions in Concave Dynamic Optimization Problems," manuscript, November 1975.

3. Berge, Claude. Topological Spaces (New York: Macmillan), 1963.

4. Bertsekas, Dimitri P. Dynamic Programming and Stochastic Control, Academic Press, New York, 1976.

5. Blackwell, David. "Discounted Dynamic Programming," Annals of Mathematical Statistics 36 (1965) 226-35.

6. Brock, William A., and L. Mirman. "Optimal Economic Growth and Uncertainty: The Discounted Case," Journal of Economic Theory, Vol. 4, 1972, 479-513.

7. Danthine, Jean-Pierre. "Martingale, Market Efficiency, and Commodity Prices," European Economic Review 10 (1977) pp. 1-17.

8. Doob, J. Stochastic Processes, Wiley, New York, 1953.

9. Feller, W. An Introduction to Probability Theory and Its Applications, Vol. II, Wiley, New York, 1966.

10. Gould, J. P. "Adjustment Costs in the Theory of Investment of the Firm," The Review of Economic Studies, January 1968, Volume XXXV(1), Number 101, pp. 47-56.

11. Lucas, Robert E., Jr. "Asset Prices in An Exchange Economy," Econometrica, forthcoming.

12. _______________. "Adjustment Costs and the Theory of Supply," The Journal of Political Economy, August 1967, Part 1, Volume 75, Number 4, pp. 321-334.

13. Lucas, Robert E., Jr., and Edward Prescott. "Equilibrium Search and Unemployment," Journal of Economic Theory, Vol. 7, Number 2, February 1974, 188-209.

14. Lucas, Robert E., Jr., and Edward C. Prescott. "Investment Under Uncertainty," Econometrica 39, (1971), 659-81.

15. Mirman, Leonard J. "The Steady State Behavior of a Class of One Sector Growth Models with Uncertain Technology," Journal of Economic Theory, Vol. 6, Number 3, June 1973, 219-242.

16. Mirman, L. J., and I. Zilcha. "On Optimal Growth Under Uncertainty," Journal of Economic Theory, 11, December 1975, 329-339.

17. Naylor, Arch W. and George Sell. Linear Operator Theory in Engineering and Science, Holt, Rinehart, and Winston, New York, 1971.

18. Tobin, James. Essays in Economics, Vol. 1: Macroeconomics, Chicago, Markham, 1971.

19. Tobin, James. "A Dynamic Aggregative Model," The Journal of Political Economy, April 1955, Volume LXIII, Number 2, pp. 103-115.

20. ______________. "A General Equilibrium Approach to Monetary Theory," Journal of Money, Credit, and Banking, February 1969, Volume 1, Number 1, pp. 15-29.

21. Treadway, A. B. "On Rational Entrepreneurial Behavior and the Demand for Investment," The Review of Economic Studies, April 1969, Volume XXXVI(2), Number 106, pp. 227-240.