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Chaotic Dynamics and Bifurcation in a Macro Model

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The qualitative dynamics of a discrete time version of a deterministic, continuous time, nonlinear macro model formulated by Haavelmo are fully characterized. The methods of symbolic dynamics and ergodic theory are shown to provide a simple, effective means of analyzing the behavior of the resulting one-parameter family of first-order, deterministic, nonlinear difference equations. A complex periodic and random "aperiodic" orbit structure dependent on a key structural parameter is present, which contrasts with the total absence of such complexity in Haavelmo's continuous time version. Several implications for dynamic economic modelling are discussed.

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Chaotic Dynamics and Bifurcation in a Macro Model

Consider the following simple macroeconomic growth model analyzed by Haavelmo (1954):

(1) $$\frac{\dot{N}}{N} = \alpha - \beta N/Y = \alpha - \frac{\beta}{\frac{Y}{N}};\ \alpha,\ \beta > 0$$

(2) $$Y = AN^{a},\ A > 0,\ 0 < a < 1$$

where N and Y are functions of time and the dot denotes a time derivative. Equation (2) has real output Y produced with a constant output elasticity, a, under decreasing returns by the labor force, N. In equation (1), N is seen to grow autonomously at the proportional rate α, minus a rate that depends inversely on per capita output. Therefore, the growth rate increases with per capita output and is bounded above by α. Substituting (2) into (1), find:

(3) $$\dot{N} = N[\alpha-\beta N^{1-a}/A].$$

By dividing both sides of (3) by N, we see that the growth law is a generalization of the familiar logistic (S-shaped) form used in the industrial growth studies of Kuznets (1933), technological diffusion study of Griliches (1957), and the simulation study of Kaldor's trade cycle model by Klein and Preston (1969). This has been referred to generically in the econometric literature as the "logit" model [Maddala (1977)]. The logistic law also plays an important role in studying the growth of biological populations.

Haavelmo solves (3), obtaining the solution

$$N(t) = [1/A((x(0)/N(0)-\beta/\alpha)e^{\alpha(a-1)t}+\beta/\alpha)]^{1/a-1}$$

$$Y(t) = AN(t)^{a}$$

where $x(0) = AN(0)^{a-1}$.

The dynamics are quite simple. If the initial condition $x(0)/N(0) \overset{>}{(<)} \beta/\alpha$, then both N and Y will increase (decrease) monotonically, approaching their unique steady-state values $(A\alpha/\beta)^{1/1-a}$ and $A(A\alpha/\beta)^{a/1-a}$, respectively. The solution is graphed in Figure 1 below.

For econometric estimation with period data, or for ease of analysis by those untrained in the solution of nonlinear differential equations, discrete time logit models might be considered. A common practice is to replace time derivatives by first differences. Doing so in (1) and (2), one obtains:

$$(1') \qquad \frac{N_{t+1}-N_t}{N_t} = \alpha - \beta N_t/Y_t$$

$$(2') \qquad Y_t = AN_t^a.$$

Combining (1') and (2'), find:

$$(3') \qquad N_{t+1} = N_t[(1+\alpha)-\beta N_t^{1-a}/A].$$

To simplify the notation, perform a positive linear change of variable in (3') by letting $N_t = (A(1+\alpha)/\beta)^{1/1-a}x_t$, which yields:

$$(4) \qquad x_{t+1} = (1+\alpha)x_t[1-x_t^{1-a}] = F(x_t;\alpha,a).$$

As the change of variables is just a change of the scale measuring N, one can analyze the dynamics of (4) without loss of generality. In doing so, we will see that the discrete time version (3') has vastly different qualitative properties from that of the continuous time model (3). No longer does the system always monotonically approach a steady state. As the autonomous growth rate α exceeds a certain value, the steady state ceases being approached monotonically, and an oscillatory approach occurs. If α is increased further, the steady state becomes unstable and repels nearby points. As α is increased even further, one can find a value of α, call it α_k, in which the system would possess a cycle of period k for any arbitrary k. Also, there exists an uncountable number of initial conditions from which emanate trajectories which fluctuate in a bounded and aperiodic fashion and which are indistinguishable from a realization of some stochastic process! Such behavior has been dubbed "chaotic" by Li and Yorke (1975), whose work is heavily relied upon in this paper.

The following conclusions can be drawn from the detailed analysis to follow:

A. The discrete time analog of a continuous time system cannot reliably be assumed to be found by replacing derivatives with first differences. This corroborates evidence arrived at independently in different contexts by Sims (1971), Sargent and Hanson (1979), and Graves and Telser (1968). In this model even the qualitative properties of the continuous and discrete time systems are as different as night and day. Alternatively, if one is not sure which representation is the "real" system, these results stress the fundamental importance that choice of time domain and "unit time" length can have on the qualitative properties of models.

B. Simple, nonlinear, first-order, deterministic difference equations may exhibit chaotic, seemingly random fluctuations which might mistakenly be attributed to the influence of excluded variables or the influence of included, but assumed random, variables. Such phenomena are absent in deterministic low-order linear difference equations. Mathematical modelling of fluctuations typically leads to additive random terms (e.g., white noise) whose fluctuations are propagated through time by the rest of the system. Our results show that there must be a better basis for introducing such stochastic effects than merely the desire to reproduce "realistic" trajectories.[1/] In the context of linear difference equation models of macroeconomic phenomena, the introduction of plausible, theoretically justifiable nonlinearities into the structural equations might explain observed fluctuations as well, or better than, the addition of random variables. This possibility is more likely in models where linearity is usually an ad hoc assumption justified solely as an analytical convenience. The practice of explaining fluctuation through the use of linear models with higher-order lags and leads is also subject to this criticism.

C. The bounded, aperiodic, "chaotic" fluctuations referred to earlier might best be described statistically, e.g., by computing over a long span of time the fraction of time the trajectory spends in given intervals. This is sometimes analytically possible, but requires rapidly developing techniques from ergodic theory and topological dynamics not familiar to most economists. Thus, while economists spend much time learning techniques to find order in the chaos of the real world, it may be necessary to learn techniques to find chaos resulting from the order of the real world.

D. Relatively "small" changes in structural parameters can lead to large, qualitative changes in system behavior. While (3') is technically structurally stable in the sense of Smale (1967) for most values of α, as a practical matter the range of perturbation of α, which will preserve the qualitative properties of the solution, can be quite small.

E. The evolution of nonlinear low-order systems can also be drastically affected by the initial conditions of the system. In the construction of models, this dependence is often viewed as something to avoid. After all, how can one know what conditions prevailed when the system started--whenever that was? In time series modelling, such reasoning is often the basis for justifying stationarity assumptions. This is not sufficient cause to ignore the very real possibility of qualitative dependence on initial conditions. Rather, a more detailed theory predicting plausible values for the initial conditions is essential. Haavelmo (1954, pages 56-63), in a spirited defense of this position, argues that the distinction between structural parameters and initial conditions depends on the level of detail in the questions posed of a model. For example, he argues that if one merely wants to know how long it will take for an investment growing at a constant rate to double, then knowledge of the growth rate (a "structural" parameter) is sufficient. However, if one also wanted to know how much money one would have after this period, then knowledge of the initial amount invested (an "initial condition") is essential as well.

F. The latter two effects (i.e., qualitative changes caused by small changes in structural parameters and initial conditions) coupled with the possibility of measurement error in these variables, casts doubt on the ability to predict and control such nonlinear systems. Thus, even if the model specification is exact, prediction and control may be impossible in practice, due to unavoidable measurement error.

Definitions

Consider a first-order difference equation, $x_{t+1} = F(x_t)$, where $F: J \to J$ is continuous, and J is a closed and bounded interval of the real line. Denote the n-fold composition of F with itself by $F^n(x)$, with $F^0(x) \equiv x$ denoting the identity map.

A point $p \varepsilon J$ is termed a nondegenerate (degenerate) periodic point with period n, or an n-period point, if and only if $F^n(p) = p$ and $p \underset{(=)}{\neq} F^k(p)$, for all (some) $1 \leq k < n$. A point $p \epsilon J$ is termed <u>periodic</u> if it is an n-period point for some $n \geq 1$. A 1-period point is termed a <u>steady-state</u>, an <u>equilibrium</u>, or <u>fixed</u> point of F.

If p is an n-period point, then each point in the collection of points $\{p, F(p), \ldots, F^{n-1}(p)\}$ is also an n-period point, and the collection is termed the <u>periodic orbit</u>, or <u>cycle</u>, of p. If p is nondegenerate, then each point in the periodic orbit is distinct, and the orbit is said to have <u>length</u>, or <u>period</u>, n.

A point $q \epsilon J$ is <u>asymptotically periodic</u> if there is a periodic point $p \neq q$ for which:

$$\lim_{n \to \infty} [F^n(q) - F^n(p)] = 0.$$

A k-period point p, and its corresponding periodic orbit, are said to be <u>locally asymptotically stable</u> if, for some open interval I about p,

$$|F^k(p) - x| < |p - x|, \text{ for all } x \epsilon I.$$

For $k = 1$, i.e., when p is an equilibrium point, the definition is the usual definition of a locally stable equilibrium point. For $k > 1$, the definition guarantees that all points $x \epsilon I$ are asymptotically periodic to p.

The term <u>chaotic dynamics</u> refers to the dynamic behavior of certain equations F which possess: (a) a nondegenerate n-period point for each $n \geq 1$, and (b) an uncountable set $S \subset J$, containing no periodic points and no asymptotically periodic points. The trajectories of such points wander around in J "randomly" and, for practical purposes, may be indistinguishable from a realization of a stochastic process.

<u>Analysis of Discrete Time System (4)</u>

From inspection of (4), note that $F(0;\alpha,a) = F(1;\alpha,a) = 0$, for all α and a. Differentiating (4), find:

(5) $F'(x;\alpha,a) = (1+\alpha)(1-(2-a)x^{1-a}) \gtreqless 0 \text{ as } \frac{1}{2-a} \gtreqless x^{1-a}$,

or as

$$(\frac{1}{2-a})^{1/1-a} \gtreqless x.$$

Noting that $F''(x;\alpha,a) = -(1+\alpha)(2-a)(1-a)x^{-a} < 0$, one sees that the geometry of F when, say, a = 1/2, is depicted in Figure 2.

As x_t increases, note that x_{t+1} at first increases for low values of x_t, then peaks, and then decreases for all x_t higher than this. This <u>density dependence</u> is a consequence of the autonomous growth of the labor force coupled with limited production capacity to sustain it. In population biology, density dependence arises from autonomously growing populations competing for fixed resources with no production capacity. Figure 2 also shows that the autonomous growth rate α "tunes" the nonlinearity of F, the severity of the nonlinearity increasing with α.

While it is not essential in obtaining the results to follow, the admissible α will be restricted for each $0 < a < 1$ so that F maps J = [0,1] into itself. Hence, assume:

(6) $F(x;\alpha,a) \leq 1$,

or computing F from (4) at its maximum,

(7) $$\alpha \leq \frac{1}{(\frac{1}{2-a})^{1/1-a}(1-\frac{1}{2-a})} - 1 = \alpha^*$$

α^* is tabled for various values of a in Table 1 below:

a	α^*
.1	3.307619
.25	3.920707
.5	5.750000
.75	11.207031
.9	27.531167

<u>Table 1</u>: Largest α such that F: [0,1] → [0,1]

The qualitative properties of (4) can be found by examining it for any particular value of a, say $a = 1/2$. None of the qualitative properties are affected by the particular choice of $0 < a < 1$. This will become clear as the analysis proceeds.

Thus, for the rest of the paper, consider the single-parameter family of first-order difference equations (graphed in Figure 2).

$$(8) \qquad x_{t+1} = (1+\alpha)x_t(1-\sqrt{x_t}) = F(x_t;\alpha);\ 0 \leq \alpha \leq 5.75;\ x_0 \varepsilon [0,1].$$

For each value of α, equilibrium points are found by locating the intersection of the graph of $F(x_t;\alpha)$ with the 45-degree line in Figure 2. For each value of α, note that there are two equilibrium points: $\bar{x} = 0$ and $\bar{x} = (\frac{\alpha}{1+\alpha})^2 = \bar{x}(\alpha)$. The latter is found by solving analytically for the intersection point. The point $\bar{x} = 0$ is clearly unstable and repels nearby points. The local stability of the other can be determined by linearizing about $\bar{x}(\alpha) = (\frac{\alpha}{1+\alpha})^2$ in (8), obtaining:

$$(9) \qquad F'(\bar{x};\alpha) = 1 - 1/2\alpha = \lambda(\alpha).$$

As is well known, the eigenvalue $\lambda(\alpha)$ determines the local stability of $\bar{x}$. When $0 < \lambda(\alpha) < 1$, $\bar{x}$ attracts nearby points in an exponential, monotonic fashion. When $0 > \lambda(\alpha) > -1$, $\bar{x}$ attracts nearby points in a (damped) oscillatory manner. When $\lambda(\alpha) = -1$, $\bar{x}$ is neither stable nor unstable, neither attracting nor repelling nearby points. Finally, when $|\lambda(\alpha)| > 1$, $\bar{x}$ is unstable and repels neighboring points. Examination of (9) shows that these behaviors occur when: $0 < \alpha < 2$, $2 < \alpha < 4$, $\alpha = 4$, and $4 < \alpha < 5.75$, respectively. This is illustrated in Figure 3 below, where $\lambda(\alpha)$ is the slope of the graph of F at $\bar{x}(\alpha)$.

When the steady-state $\bar{x}(\alpha)$ is globally stable, i.e., when $\alpha < 4$, the trajectory starting at any point always approaches it. In these cases, the qualitative dependence of the solution on the parameter α can be approximated by comparing the steady state for one value of α with that for another value of α. This comparative statics analysis can be carried out using the relation $\bar{x}(\alpha) = (\frac{\alpha}{1+\alpha})^2$. Note that $\bar{x}'(\alpha) = \frac{2\alpha}{(1+\alpha)^3} > 0$, so that increases in α can be expected to increase x_t, for sufficiently large t. This analysis is of no help when $\bar{x}$ is unstable, i.e., when $4 < \alpha < 5.75$.

In this region, if trajectories don't approach $\bar{x}$, but are bounded by 0 and 1, where do they go? The first task in discovering the answer is to examine (8), when α is in this region, for the possibility of stable periodic orbits which will attract nearby points into patterns of regular, bounded oscillation. This is now accomplished.

Periodic Orbit Structure: The 2-Period Cycle and its Harmonics

As α exceeds 4, the unstable equilibrium point bifurcates into two stable points of period two, i.e., into a stable periodic orbit of length 2. This can be seen by examining F^2, the second iterate of F, for nondegenerate fixed points, i.e., fixed points of F^2 which are not also fixed (equilibrium) points of F.

For $\alpha = 4.2$, Figure 4 shows the two nondegenerate fixed points of $F^2(x;4.2)$, labelled $\bar{x}_1^2$ and $\bar{x}_2^2$, as well as the degenerate fixed point $\bar{x}$. For $\alpha = 3.8$ and all other $\alpha < 4$, the 2-period orbit does not arise. Examination of the figure shows the slopes of $F(\bar{x}_1^2, 4.2)$ and $F^2(\bar{x}_2^2, 4.2)$ are both less than 1, which implies that the periodic orbit is stable.

It also appears that these two slopes are equal, and indeed this is so, as can be proven by a simple application of the chain rule [Samuelson (1972, page 390)]. Denoting the slope of $F^2(\bar{x}_i^2;\alpha)$ by $\lambda^2(\bar{x}_i^2(\alpha))$, compute

$$(10) \qquad \lambda^2(\bar{x}_i^2(\alpha)) = \frac{dF^2(\bar{x}_i^2;\alpha)}{dx} = \frac{dF(F(\bar{x}_i^2;\alpha))}{dx} = \prod_{i=1}^{2} \frac{dF(\bar{x}_i^2;\alpha)}{dx}; \; i=1, 2$$

as $F(\bar{x}_i^2;\alpha) = \bar{x}_j^2$.

Direct inspection of (10) proves that $\lambda^2(\bar{x}_1^2(\alpha)) = \lambda^2(\bar{x}_2^2(\alpha))$. Thus, one need only examine the slope of F^2 at either periodic point to determine the stability of the periodic orbit containing them.

Using (10) to evaluate $\lambda^2(\bar{x}(\alpha))$ (the slope of F^2 at the equilibrium point $\bar{x}$), compute

$$(11) \qquad \lambda^2(\bar{x}(\alpha)) = \frac{dF}{dx}(\bar{x}(\alpha))^2 \gtreqless 1 \text{ as } \left|\frac{dF}{dx}(\bar{x}(\alpha)) = \lambda(\alpha)\right| \gtreqless 1.$$

Therefore, $\lambda^2(\bar{x};\alpha)$ exceeds one if, and only if, $\bar{x}(\alpha)$ is unstable. As the geometry of Figure 4 makes clear, $\lambda^2(\bar{x};\alpha)$ must exceed one in order for nondegenerate 2-period points to exist. Thus, the two 2-period points appear when $\bar{x}(\alpha)$ is unstable, i.e., at $\alpha \geq 4$. This orbit is stable at first, but as α increases past 4, $|\lambda^2(\bar{x}_i^2(\alpha))|$ increases as well (compare $\lambda^2(\bar{x}_i^2, 4.2)$ with $\lambda^2(\bar{x}_i^2, 4.7)$ in Figure 4). For values of α in excess of about 4.8, the 2-period cycle becomes unstable, and each 2-period point bifurcates into two 4-period points, producing an (initially) stable cycle of length four denoted $\{\bar{x}_1^4, \bar{x}_2^4, \bar{x}_3^4, \bar{x}_4^4\}$. Figure 5 illustrates the phenomenon. Not surprisingly, the slope of F^4 at the 2-period points $\bar{x}_1^2(\alpha)$ and $\bar{x}_2^2(\alpha)$ must exceed one in order for the bifurcation to occur, and a similar application of the chain rule shows that this occurs when:

$$(12) \qquad \lambda^4(\bar{x}_i^2(\alpha)) = \frac{dF^4(\bar{x}_i^2;\alpha)}{dx} = \frac{dF^2(F^2(\bar{x}_i^2;\alpha))}{dx} = \frac{dF^2(\bar{x}_i^2;\alpha)^2}{dx} = \lambda^2(\bar{x}_i^2(\alpha))^2 > 1,$$

or when $|\lambda^2(\bar{x}_i^2(\alpha))| > 1$. Thus, the 4-period orbit first appears when α increases to the point where the 2-period orbit becomes unstable, as suggested earlier.

This <u>pitchfork bifurcation</u> process continues as α increases, producing nondegenerate orbits of lengths 2k; k = 2, ..., ∞. These orbits are called <u>harmonics</u> of the 2-period orbit. It is possible to show that all the harmonics occur prior to α reaching 5.540, although how much prior to this value is not known. Thus, the range of α, within which a stable orbit of length k first appears and later becomes unstable and bifurcates to a 2k-period orbit, decreases in length as α increases to a limiting value $\alpha_c < 5.540$.

The range of $\alpha_c < \alpha < 5.75$ is termed the <u>chaotic</u> region. As α enters this region, even stranger behavior occurs. For example, a 3-period orbit exists at $\alpha \cong 5.540$, illustrated in Figure 6 below. This, then, gives rise to orbits of periods 3k, k = 2, ..., ∞ via the pitchfork process just described. The 3-period orbit arises as the graph of F^3 in Figure 7 lowers itself enough to become tangent to the 45-degree line and is termed a <u>tangent bifurcation</u>. In fact, a remarkably simple theorem of Li and Yorke (1975) demonstrates that for any $F(x_t;\alpha)$ in which a nondegenerate 3-period orbit arises, there must also exist nondegenerate points of <u>all</u> periods, as well as an uncountable set of aperiodic, not even asymptotically periodic, points whose trajectories wander "randomly" throughout the domain of F.

The theorem is reprinted below.

Theorem 1 (Li and Yorke):

Let J be an interval and let $F:J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy

$$d \leqq a < b < c \text{ (or } d \geqq a > b > c\text{)}.$$

Then

T1: For every k = 1, 2, ..., there is a periodic point in J having period k.

Furthermore,

T2: There is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:

(A) For every $p, q \in S$ with $p \neq q$.

$$\text{(2.1)} \qquad \limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0$$

and

(2.2) $\lim\inf_{n\to\infty} |F^n(p)-F^n(q)| = 0.$

(B) For every $p\epsilon S$ and periodic point $q\epsilon J$,

$$\lim\sup_{n\to\infty} |F^n(p)-F^n(q)| > 0.$$

Note that the hypothesis of the theorem will be satisfied by the existence of a 3-period orbit.

Rather than present the proof of the theorem, which involves only elementary techniques of real analysis, the techniques used will be integrated into the following discussion of how one might count the number of k-period points for each k = 1, ..., ∞. As a by-product, T1 will be rigorously proven.

Counting the Periodic Points: Symbolic Dynamics

The method we use for counting the number of periodic points of various periods, and as a by-product proving T1, is termed symbolic dynamics and has been used by Smale and Williams (1976) and Guckenheimer, Oster, and Ipaktchi (1977) for this purpose. Their approach is followed here. Symbolic dynamics is also quite useful in establishing the nature of the aperiodic, chaotic behavior described precisely by T2 and has recently been applied in studying the chaotic nature of a class of two-player, noncooperative games by Rand (1978).

Let us examine the map F(x;5.540) of Figure 6 and embellished below in Figure 8. From Figure 8 it is evident that F[a,b] = [b,c], and that F[b,c] = [a,c] = [a,b] ∪ [b,c]. This can be seen by following the action of F on the end points a, b, and c, and using continuity and monotonicity to infer the rest.

Denote [a,b] = K and [b,c] = L. Then, with the notation:

(13)
$$F(K) = [F(a),F(b)] = [b,c] = L$$
$$\text{and } F^2(K) = F(L) = [F(c),F(b)] = [a,c] = K \cup L.$$

Figure 8 shows an equilibrium point $\overline{x}$ in L where F intersects the 45-degree line. A rigorous proof of this fact could be made by noting that F(L) = K U L $\supset$ L. As L is a compact interval and F is continuous, there ought to exist a compact subinterval of L, call it $Q_1 \subset L$, such that $F(Q_1) = L$.[2/] In our example, a glance at Figure 8 shows that $Q_1 = [b,q_1^u] \subset L$, by noting that F(b) = c and $F(q_1^u) = b$.

A simple argument will now be used to prove the existence of a unique fixed point $\overline{x}\varepsilon Q_1 \subset L$; $F(\overline{x}) = \overline{x}$. Define the function G(x): $Q_1 \rightarrow L$ by G(x) = x - F(x). Note that G(b) = b - c < 0, and $G(q_1^u) = q_1^u - b > 0$, as b is the lower limit of L. Therefore, by continuity of G, there must exist $\overline{x}\varepsilon Q_1$ such that $G(\overline{x}) = \overline{x} - F(\overline{x}) = 0$. Further, Figure 8 shows that F maps Q_1 monotonically to L. So, there must be only one such $\overline{x}\varepsilon L$. Of course, there are no equilibrium points other than the origin in K, as F(K) = L, and $K \cap L = b$, which is not an equilibrium point.

Suppose we wished to look for 2-period points in K or L. Note that:

(i) $F^2(K) = F(F(K)) = F(L) = K \cup L \supset K$ (in short: KLK),

(ii) $F^2(L) = F(F(L)) \supset F(F(L) \cap K) = F(K) = L$ (in short: LKL),

(iii) and $F^2(L) = F(F(L)) \supset F(F(L) \cap L) = F(L) \supset L$ (in short: LLL).

The information in (i) can be summarized by saying K covers itself once under the action of F^2, i.e., if the initial condition is in K, the only way to get back to K after two iterations of F is to first go to L and then back to K (denoted KLK). Similarly, the information in (ii) and (iii) means that there are two ways L covers itself under the action of F^2. One way is to start in L, go to K, and back to L (denoted LKL), and the other is to start in L, go to L again, and then stay in L once more (denoted LLL). If we could show that there had to be a fixed point of F^2 following each of these three "paths," then there would be three distinct

2-period points.[3/] One of these, of course, would be the equilibrium point $\bar{x}$ (a degenerate 2-period point) following the path LLL. For situations (i) and (ii) we can apply the previous argument to the continuous map F^2 finding compact subintervals $Q_1^2 \subset K$ and $Q_2^2 \subset L$ such that $F^2(Q_1^2) = K$ and $F^2(Q_2^2) = L$ (see Figure 9). Each contains a unique fixed point termed $\bar{x}_1^2$ and $\bar{x}_2^2$, respectively. These must be distinct, as their respective trajectories followed different paths (and the only common point to K and L is b, which is not a 2-period point). As they are both 2-period points, the nondegenerate 2-period cycle is $\{\bar{x}_1^2, \bar{x}_2^2\}$. Thus, there is only one 2-period orbit in L U K.

A similar technique is used to investigate the existence and number of periodic points (excluding the origin) of higher length. First, we count the number of "paths" of length k in L U K by following the transformation F. This task is made easier by considering the system as a "Markov chain" with "states" K and L and invariant transition law F. The "transition matrix" for this chain is then found to be:

$$(14) \qquad \begin{array}{c} \\ K \\ L \end{array} \begin{array}{c} \begin{array}{cc} K & L \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} .$$

where an entry of one indicates that F (interval at left) $\supset$ interval at top, and a zero indicates the absence of this condition. The number of paths of length k of the type $F^k(\cdot) \supset \ldots \supset (\cdot)$ which start and finish with the same interval $(\cdot)$ is found by computing trace $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k$, where the first component in the diagonal of the k^{th} power is the number of paths of length k connecting K and K, and the last component in the diagonal gives the number of paths connecting L and L. Thus, the trace, denoted N_k, is the number of all such paths of length k.

$$(15) \qquad N_k = \text{trace}\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k\right) = \left(\text{trace}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right)^k.$$

Trace is a similarity invariant, so one can diagonalize the matrix, raise it to the k^{th} power, and then compute the trace. The diagonalized matrix is

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^k$$

and computing find:

(16) $$N_k = \frac{1}{2^k}[(1+\sqrt{5})^k+(1-\sqrt{5})^k], \ k = 1, \ldots, \infty.$$

Computing (16), it appears that the sequence $\{N_k\}$ forms the Fibonacci numbers $\{1,3,4,7,11,18,29,\ldots\}$! In fact, this is indeed the case. For example, $N_3 = 4$, and the paths are KLLK, LKLL, LLKL, and LLLL.

The last step in counting the number of k-period points is to prove that each path of length k enumerated in (16) is associated with a unique, though possibly degenerate, k-period point so that the number of k-period points (degenerate and nondegenerate) in L U K equals N_k. Consider a path of length k, $F^k(\cdot) \supset \ldots \supset (\cdot)$, starting and ending with the same interval, K or L. As F^k is continuous and $(\cdot)$ is a compact interval, there is a compact subinterval Q_k of $(\cdot)$ such that $F^k(Q_k) = (\cdot)$. Then, via the same argument given earlier, there must exist a unique fixed point $\bar{x}^k$ in this interval. This k-period point is distinct from all other k-period points corresponding to the other paths of length k, as each follows a different path.[4/]

Thus, the number of k-period points equals N_k, the k^{th} Fibonacci number. Of course, a lot of these are degenerate. For example, for k = 3, there are $N_3 = 4$ periodic points. They are the equilibrium point $\bar{x}$ (following LLLL) and the three 3-period points comprising a 3-period orbit. For k = 4, there are $N_4 = 7$ periodic points. They are the equilibrium point $\bar{x}$, the two 2-period points $\bar{x}_1^2$ and $\bar{x}_2^2$, and four 4-period points comprising a nondegenerate 4-period orbit. Continuing the enumeration, it is obvious that there exists at least one nondegenerate orbit of period k for all k = 1, ..., ∞, thus proving T1.

What about intervals in [0,1] other than K and L? Might they contain periodic points? Partition the domain into the following sets:

$$[0,a),\ [a,b] = K,\ [b,c] = L,\ \text{and}\ (c,1].$$

A glance at Figure 8 shows there can be no k-period points in [0,a) other than the equilibrium point 0 as $F([0,a)) \subset [0,a) \cup K$ and is monotone increasing. Thus, F eventually repels all points in this range into K, subsequent to which no point ever returns to [0,a). They are absorbed into $K \cup L$ instead. Likewise, there are no periodic points in (c,1], as $F((c,1]) = [0,a)$, and we have just seen that all points in [0,a) become absorbed in $K \cup L$. This can be represented by looking at the transition matrix on the above "Markov Partition":

	[0,a)	K	L	[c,1]
[0,a)	1	1	0	0
K	0	0	1	0
L	0	1	1	0
(c,1]	1	0	0	0

Clearly, K and L are "absorbing states" of the process, and the search for periodic points (other than the origin) can be restricted to them.

An analysis similar to the above can be applied for any value of the parameter $\alpha > 5.540$, with the result that even more periodic points are present than for $\alpha = 5.540$.

Existence of Aperiodic Points: Chaos

Symbolic dynamics can also be used to prove the existence of aperiodic, not asymptotically periodic points. For example, any point $p \neq b$ following path KLKLLKLLLKLLLLK . . . is clearly aperiodic. The periodic points enumerated by (16) form a countable, and hence (Lebesgue) measure zero, subset of [0,1]. The aperiodic points are, therefore, uncountable. It is possible to demonstrate the existence of an uncountable set of points $S \subset [0,1]$ which are aperiodic and not asymptotically periodic. The formal properties of this set are stated in T2 of Li and Yorke's Theorem 1, reprinted earlier with proof omitted.[5/] This uncountable set has measure zero for many $\alpha < 5.75$. However, at $\alpha = 5.75$, F covers the interval [0,1] twice, with F[0,4/9] = F[4/9,1] = [0,1]. I will show that F(x;5.75) has no stable periodic orbits in [0,1], and that the (unstable) periodic orbits are dense in the interval [0,1]. These periodic orbits contain only a countable number of points. Trajectories emanating from other points approach an infinite set A with the following properties:

(i) A is invariant under F, i.e., F(A) = A

(ii) There exists p A such that $\{F^n(p);n=1,\ldots,\infty\}$ is dense in A.

(iii) There is a neighborhood N of A consisting of points whose trajectories tend asymptotically to A, i.e., $\lim_{n\to\infty} F^n(q)\varepsilon A$, for all $q\varepsilon N$.

A set A with properties (i)-(iii) is termed a "strange attractor," and is known to exist for some similar maps on the unit interval (e.g., f(x)=4x(1-x)). In both that case and F(x;5.75), the whole interval [0,1] is a strange attractor. Trajectories emanating on or near the strange attractor "randomly" wander around [0,1]. Perhaps the best one can hope for is that the long-run behavior of such a trajectory has nice statistical properties, e.g., that it has a long-run time average in a meaningful sense. Thus, starting from $x_0\varepsilon[0,1]$, one hopes that the <u>ergodic mean</u>

(17) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} F^i(x_0;\alpha);\ F^0(x_0) = x_0$$

exists for almost every x_0. Further, one would like to compute how often, upon averaging over a long length of time, the trajectory takes values in a given set B. Denoting the characteristic function on B by χ_B, <u>this mean sojourn time in B</u> is computed as:

(18) $$\lim_{n\to\infty} [\frac{1}{n} \sum_{i=0}^{n-1} \chi_B(F^i(x_0;\alpha))] = \eta[B;x_0].$$

Conditions under which these limits exist and can be feasibly computed are investigated through ergodic theory.

The individual ergodic theorem due to Birkhoff (1931) gives a useful condition for the existence and computation of (17) and (18). It implies that if there exists a measure η on [0,1] such that

(19) $$\eta(F^{-1}(B)) = \eta(B), \text{ for all measurable sets } B.$$

Then (17) exists for almost all x_0. Further, if (18) is independent of x_0 for almost every x_0, then (17) can be computed by:

(20) $$[\frac{1}{n} \sum_{i=0}^{n-1} F^i(x_0;\alpha)] = \int_0^1 d\eta, \text{ for almost all } x_0.$$

Equation (19) is summarized by saying "η is invariant under F" or "F preserves η," and (20) by "the time mean equals the space mean" or "F is ergodic."

Also, if g is any real valued η-integrable function on [0,1], thought of as a variable dependent on the state x_t,

(21) $$[\frac{1}{n} \sum_{i=0}^{n-1} g(F^i(x_0;\alpha))] = \int_0^1 g\,d\eta, \text{ for almost all } x_0.$$

Thus, if F is ergodic, (19) is computed by integrating χ_B over [0,1] with respect to the measure η.

Following a technique suggested by Stein and Ulam (1963), we try to show that F(x;5.75) on the unit interval is conjugate to the "triangle" map G on the unit interval given by:

(22) $$G(x) = \begin{cases} 2x; & 0 \le x \le 1/2 \\ 2(1-x); & 1/2 < x \le 1 \end{cases}$$

depicted in figure 10.

Conjugacy of F and G means that there exists a measurable transformation (i.e., bijective, measurable, with measurable inverse) h on the unit interval such that:

(23) $F = h^{-1}Gh.$

If F is conjugate to G, then from (22) $F^k(x) = h^{-1}G^k h(x)$, or $h(F^k(x)) = G^k(h(x))$. In particular, if for some x, h(x) is a k-period point of G, then $G^k(h(x)) = h(x) = h(F^k(x))$, and invertability of h implies $x = F^k(x)$, i.e., x is a k-period point of F. Likewise, if x in any k-period point of F, h(x) is a k-period point of G. Thus, the periodic orbits of F and G are in one-to-one correspondence. A simple indirect proof establishes the fact that any F conjugate to G can have no stable periodic points. To do so, we assume for purpose of contradiction that there exists a stable, k-period point x of F. Thus, we assume that $|F^{k'}(x)| < 1$. Conjugacy implies $h(F^k(x)) = G^k(h(x))$, for all x. Differentiating, find:

(24) $h'(F^k(x))F^{k'}(x) = G^{k'}(h(x))h'(x).$

But $x = F^k(x)$, and $h' \neq 0$ due to its bijectivity. Thus, (2.4) implies:

(25) $F^{k'}(x) = G^{k'}(h(x)),$

An application of the chain rule as was done in (11) and (12) yields:

(26) $G^{k'}(h(x)) = \prod_{i=0}^{k-1} G'(G^i(h(x))).$

Examination of (22) shows $G' = \begin{cases} 2 \text{ if } 0 \leq x \leq 1/2 \\ -2 \text{ if } 1/2 < x \leq 1 \end{cases}$. Thus, from (26) $|G^{k'}(h(x))| > 1$, thus implying h(x) is an unstable periodic point of G. By (25), $|F^{k'}(x)| > 1$, contradicting our assumption that x was stable. Therefore, there are no stable periodic points in G or F. Further, Guckenheimer, et al (1977) show for G that the unstable periodic points are dense in [0,1], and are thus dense for F via the one-one correspondence through h.

In addition, if F can be shown to be conjugate to G, then it is easy to find an invariant measure η for F. By inspection of G, it is clear that G preserves Lebesgue measure μ on [0,1]. If $F = h^{-1}Gh$, then it is easy to show that the induced measure η defined by $\eta(B) = \mu(h(B))$ is an invariant measure for F. Thus, the mean sojourn time (18) of F in the interval [0,x] can be computed as:

$$\eta([0,x]) = \mu(h([0,x])) = |h(x)-h(0)|. \quad (27)$$

Furthermore, if h is differentiable and monotone increasing, the calculation of (21) is simplified to:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(F^i(x_0;\alpha)) = \int_0^1 g\,d\eta = \int_0^1 g(x)h'(x)dx. \quad (28)$$

We have constructed a fast computer algorithm which implements Ulam's numerical technique for finding h. The results for F(x;5.75,a=1/2) and, more generally, for maps $F(x;\alpha^*,a)$ onto the unit interval are shown in Figure 11. For those maps, almost all trajectories are chaotic and a typical one is shown in Figure 12. Time averages of functions on the trajectories can be computed by numerically integrating (28).

Concluding Remarks

It is important to note that the qualitative dynamical results in this paper relied solely on certain geometric properties of the function family $F(x_t;\alpha,a)$, and not on its algebraic form. The process of successive pitchfork bifurcations, followed by a tangent bifurcation and chaos as the structural parameter "tuning" the nonlinearity varies, is generic to all "one-hump" difference equations whose graphs have the general appearance of Figure 2. For example, the families $F(x_t;\alpha) = \alpha x_t(1-x_t)$ and $F(x_t;\alpha) = x_t e^{\alpha(1-x_t)}$ both exhibit all the phenomena described herein.

The existence of a strange attractor and the necessity of statistical techniques to analyze fluctuating dynamics of trajectories on or near it do not arise in differential equation systems of dimensions one and two. We have seen that the phenomena does arise in our one-dimensional difference equation, and several authors including Beddington, Free, and Lawton (1975), and May (1976), have indicated that strange attractor-like phenomena requiring "less" non-linearity are far more likely in difference equation systems of dimensions two and above.

Footnotes

The author wishes to thank Paul O'Brien for ably providing the computer programming and moral support; K. S. Rolfe, Mary K. Steffenhagen, and Wenche Branden-N'Kjell, who produced the detailed graphics; and Sharon Schuerman, who patiently typed the hieroglyphics and script.

1/ See Adelman and Adelman (1959) for an example of this, by no means unique, practice.

2/ This is proven in lemma 0 of Li and Yorke (1978).

3/ The only point which possibly follows more than one path is $b = L \cap K$, and it follows none of these.

4/ Actually, there is one point which, at first thought, appears to be multipally counted, but which in fact is not. This point is $b = K \cap L$, which appears to be counted in paths of the following types:

(i) LKLLLKLLLKLL....

(ii) KKLKKKLKKKLK....

As b is a nondegenerate 3-period point, it is a fixed point of F^k, k = 3, 6, 9, ... The periodic orbits of b are of the form $\{b,c,a,b,b,c,a,b,...\}$, and are of type (i). Over counting would result if type (ii) paths were also listed in our enumeration, but the enumeration never listed the possibility of paths with consecutive K's. Thus, paths of type (ii) were not counted.

5/ For example, to prove T2, (2.1) one shows the existence of an uncountable set $S \subset J$ such that for any distinct p, $q \in S$, $F^n(p) \in L$ and $F^n(q) \in K$, for infinitely many n. Therefore, they essentially must stay a finite distance apart for most n.

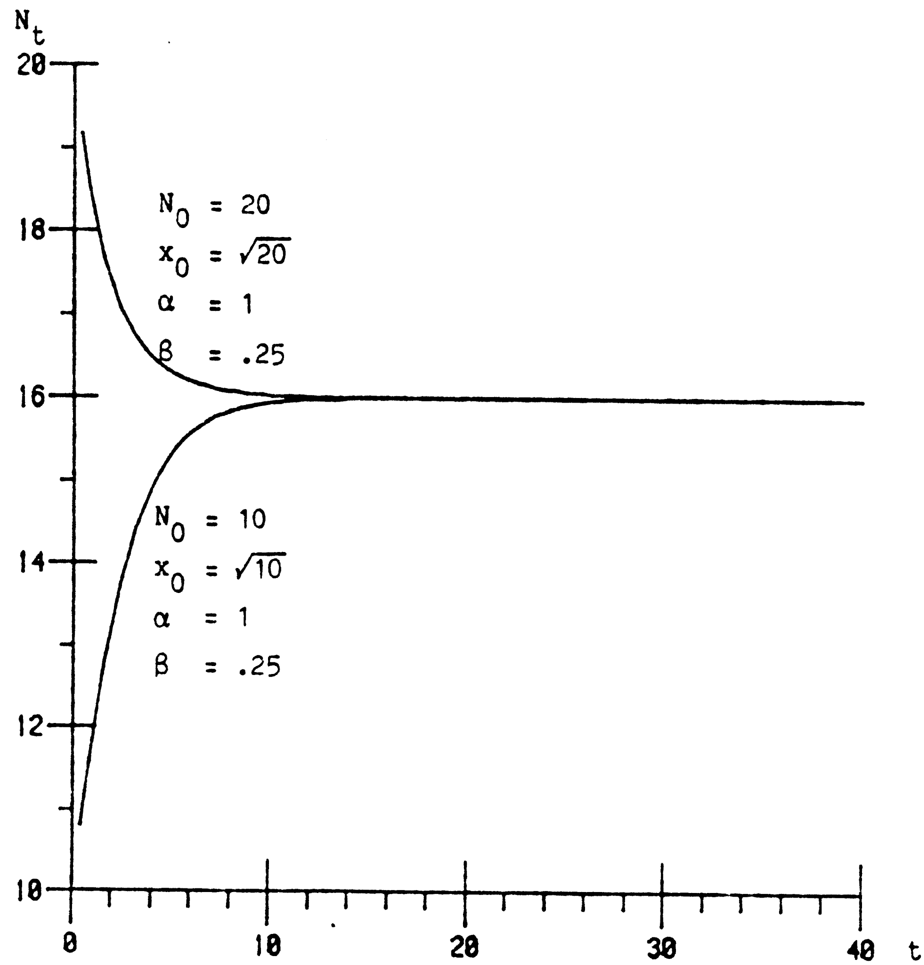

Figure 1: Continuous time model always approaches the steady state monotonically.

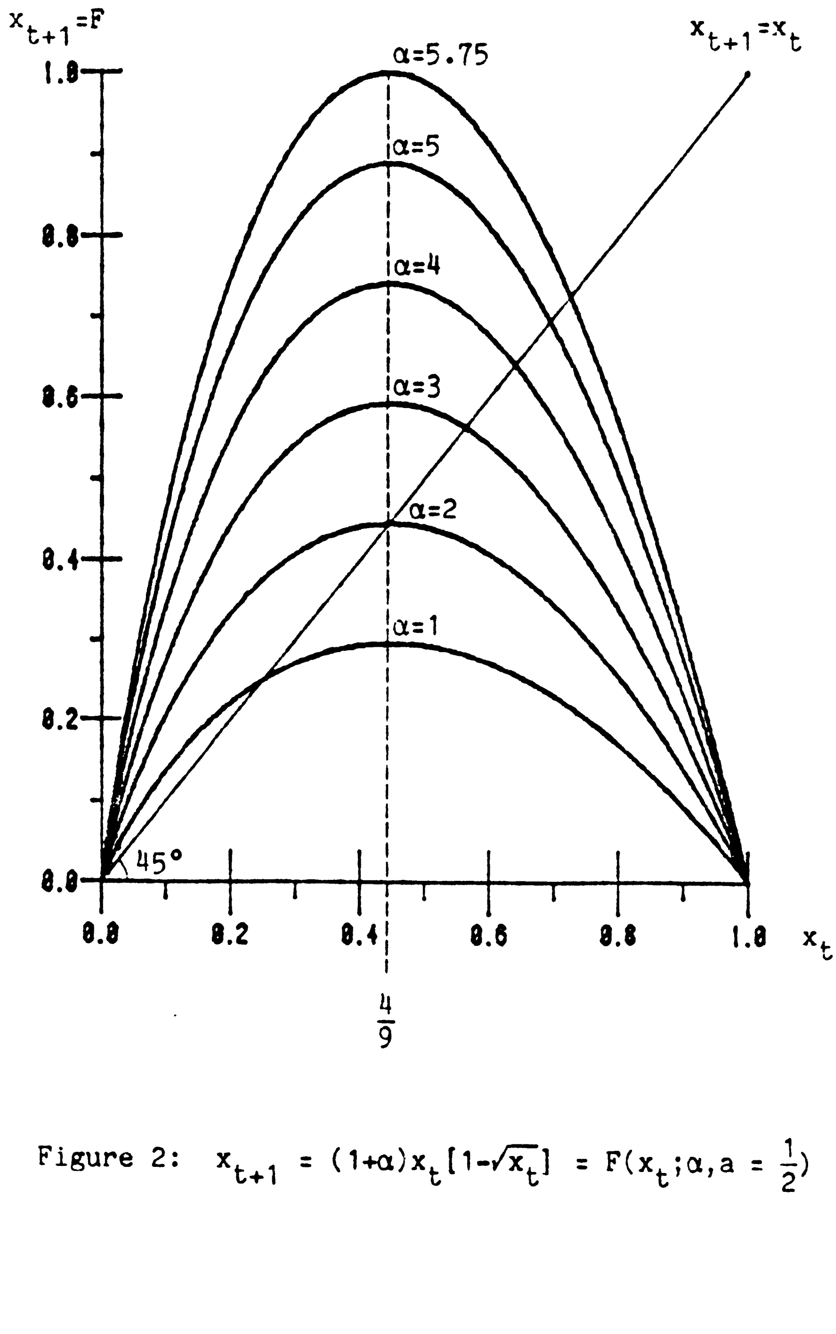

Figure 2: $x_{t+1} = (1+\alpha)x_t[1-\sqrt{x_t}] = F(x_t;\alpha, a = \frac{1}{2})$

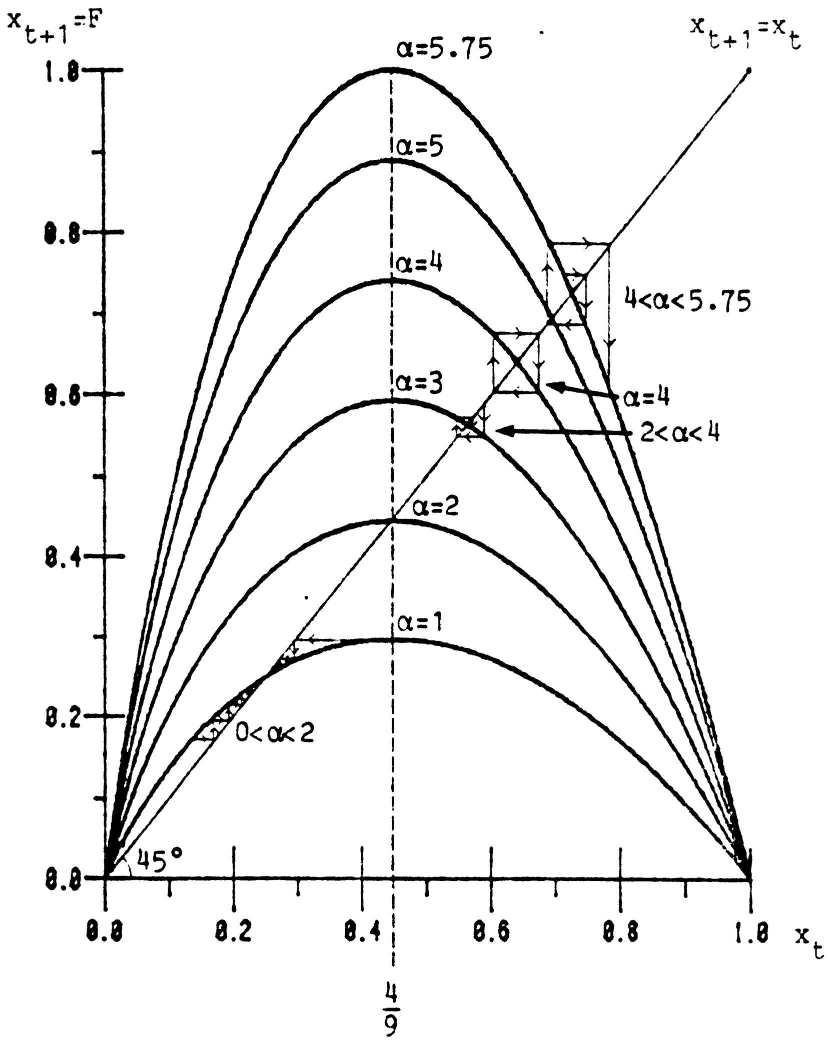

Figure 3: Stability of equilibrium

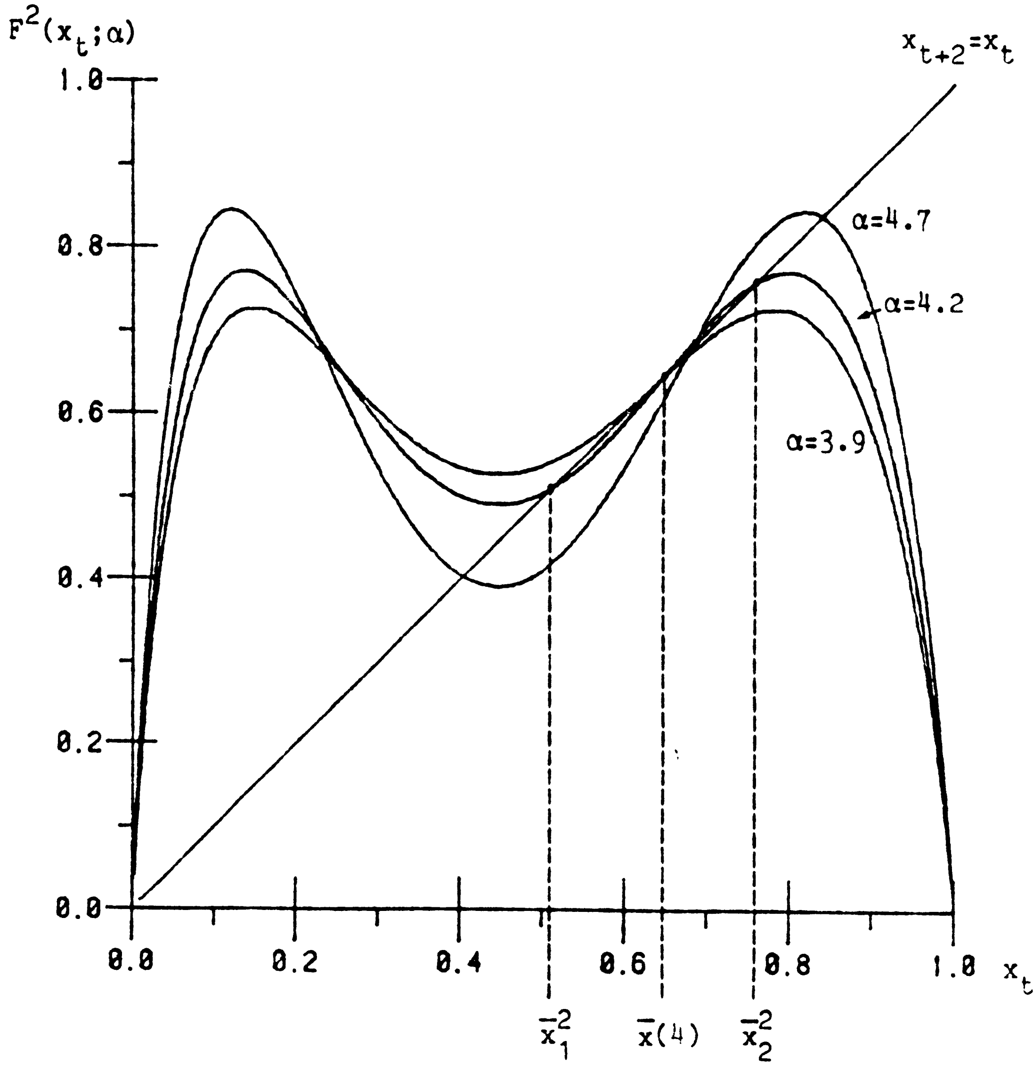

Figure 4: Bifurcation of $\bar{x}$ into a 2-period orbit $\{\bar{x}_1^2, \bar{x}_2^2\}$

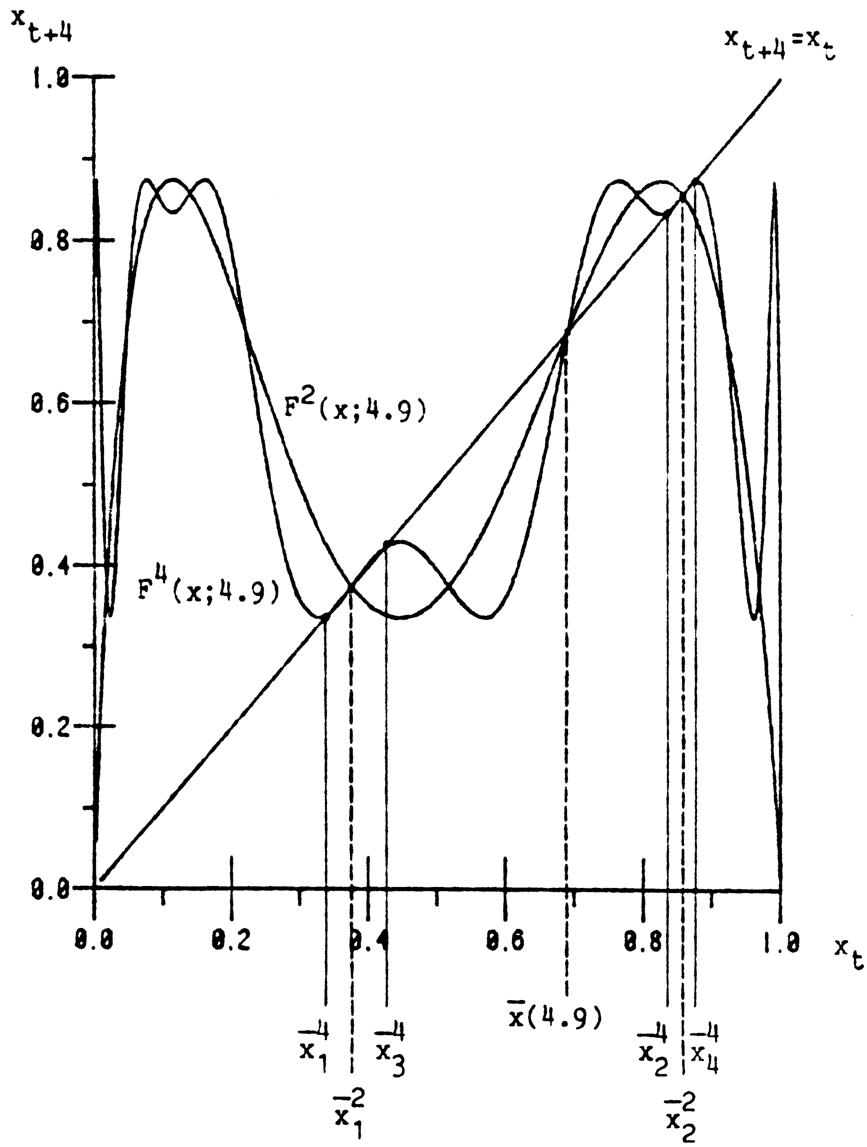

Figure 5: The 4-period orbit $\{\bar{x}_1^4, \bar{x}_2^4, \bar{x}_3^4, \bar{x}_4^4\}$ for α=4.9

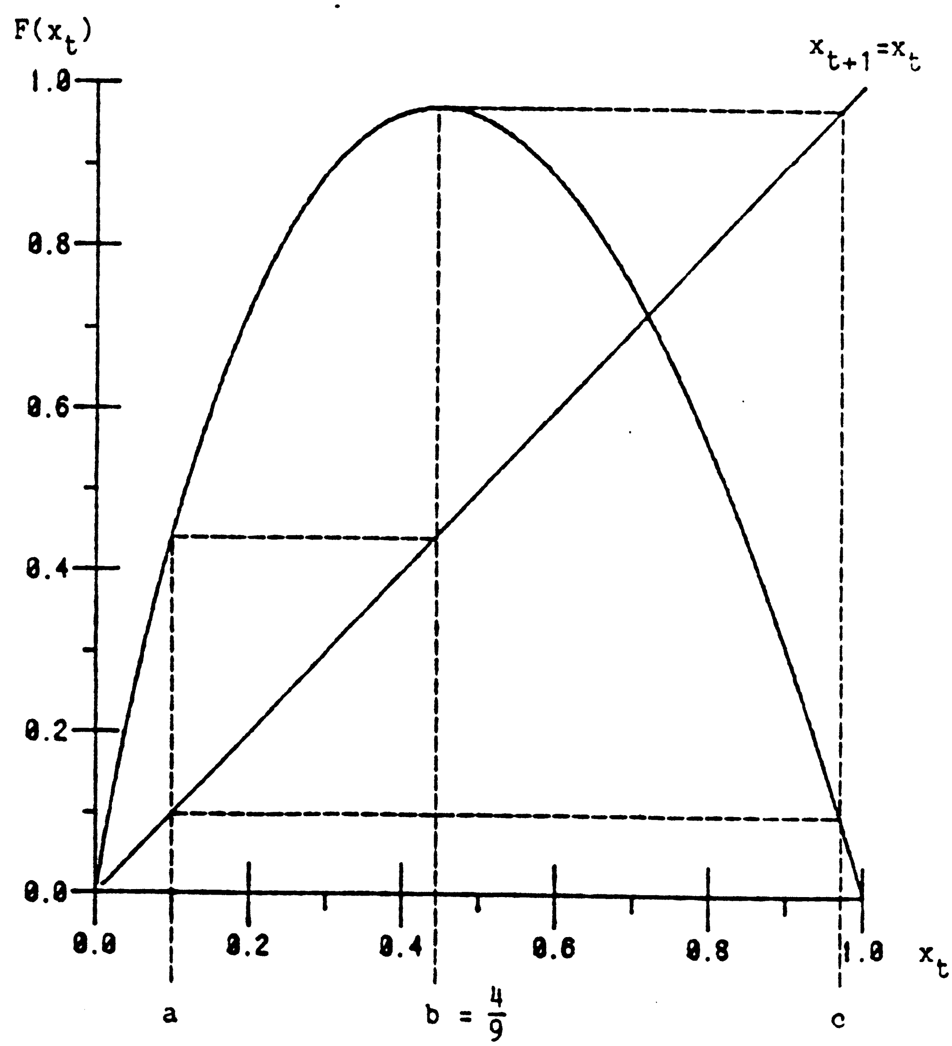

Figure 6: F(x;5.540) with 3-period orbit {a,b,c}

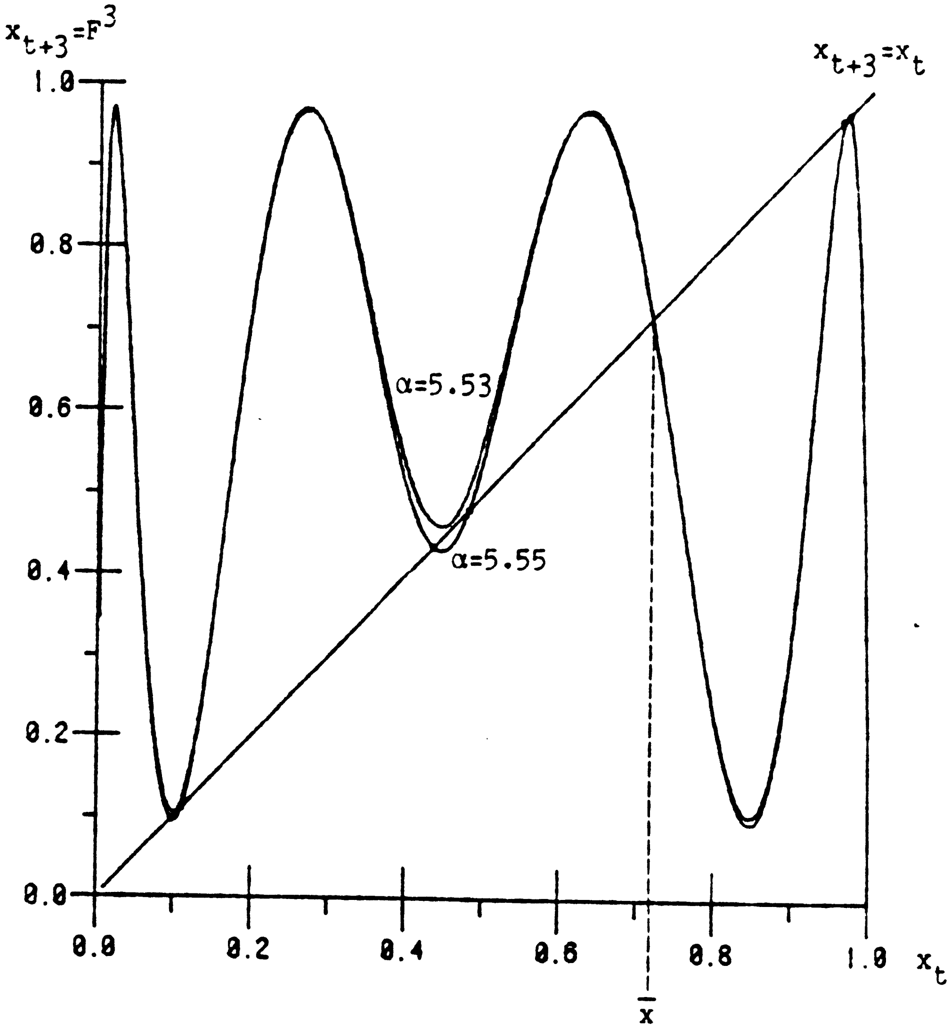

Figure 7: 3-period orbit $\{a,b,c\}$ arises by tangent bifurcation as α increases to 5.540. As α increases further, this orbit bifurcates into two 3-period orbits at the six points where $F(x;\alpha)$, $\alpha>5.540$ intersects the 45° line.

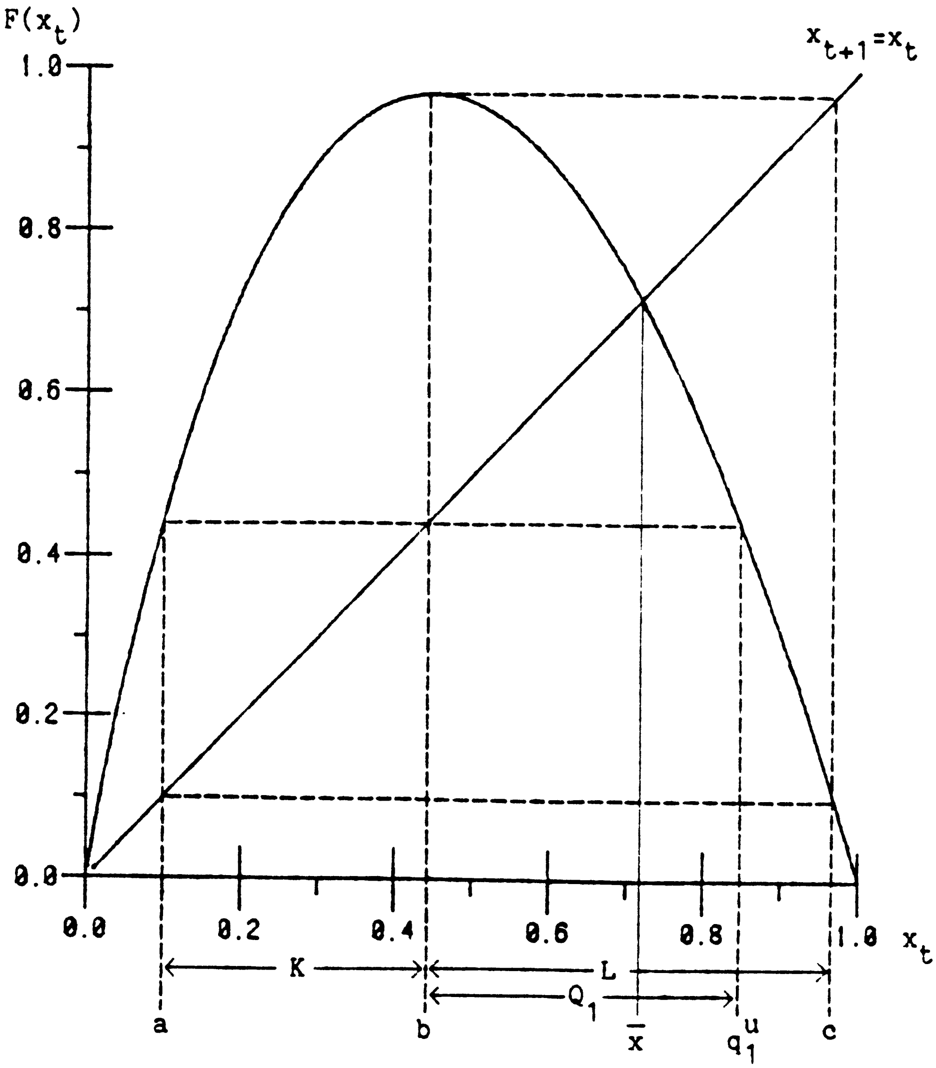

Figure 8: $F(x;5.540)$

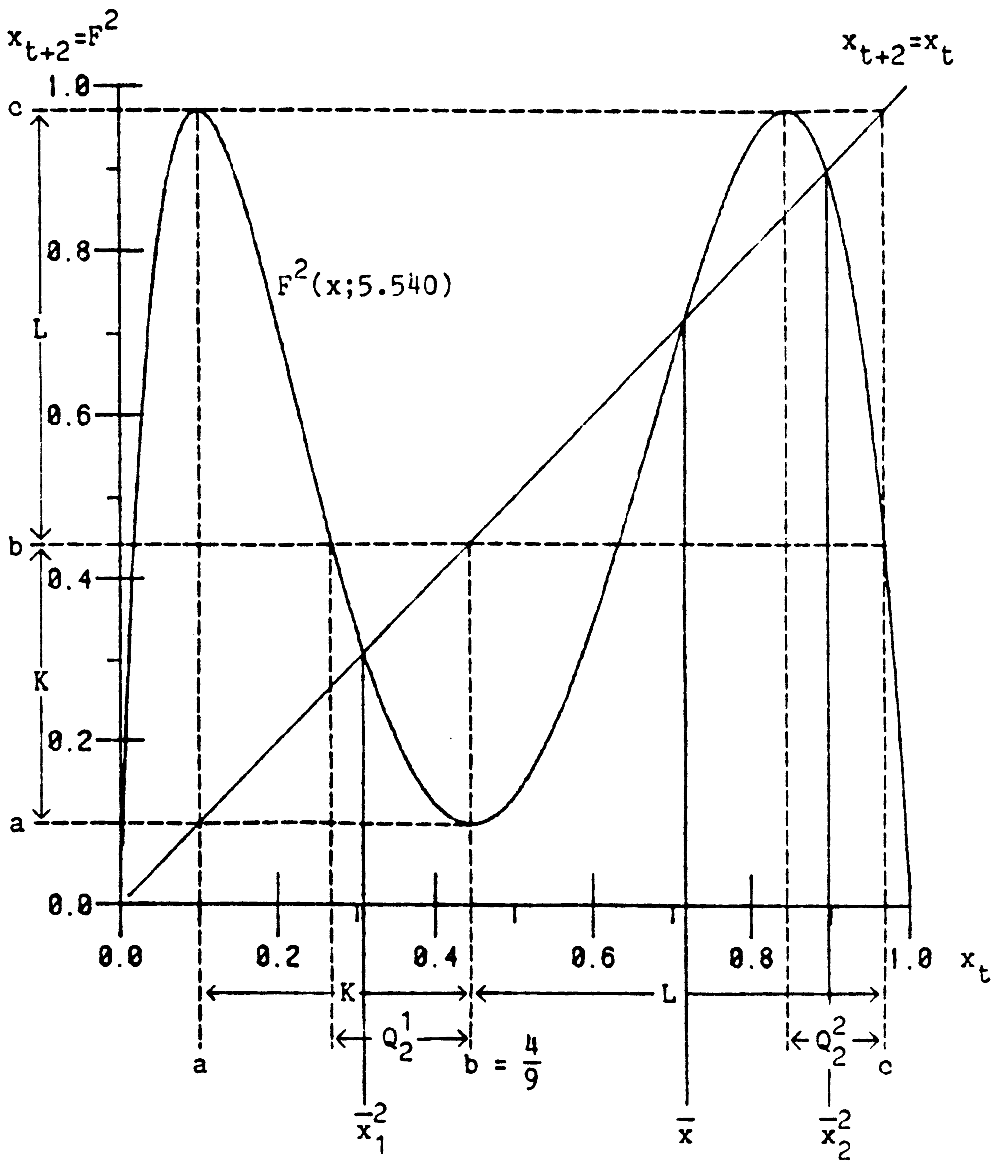

Figure 9: Existence of 2-period orbit $\{\bar{x}_1^2, \bar{x}_2^2\}$

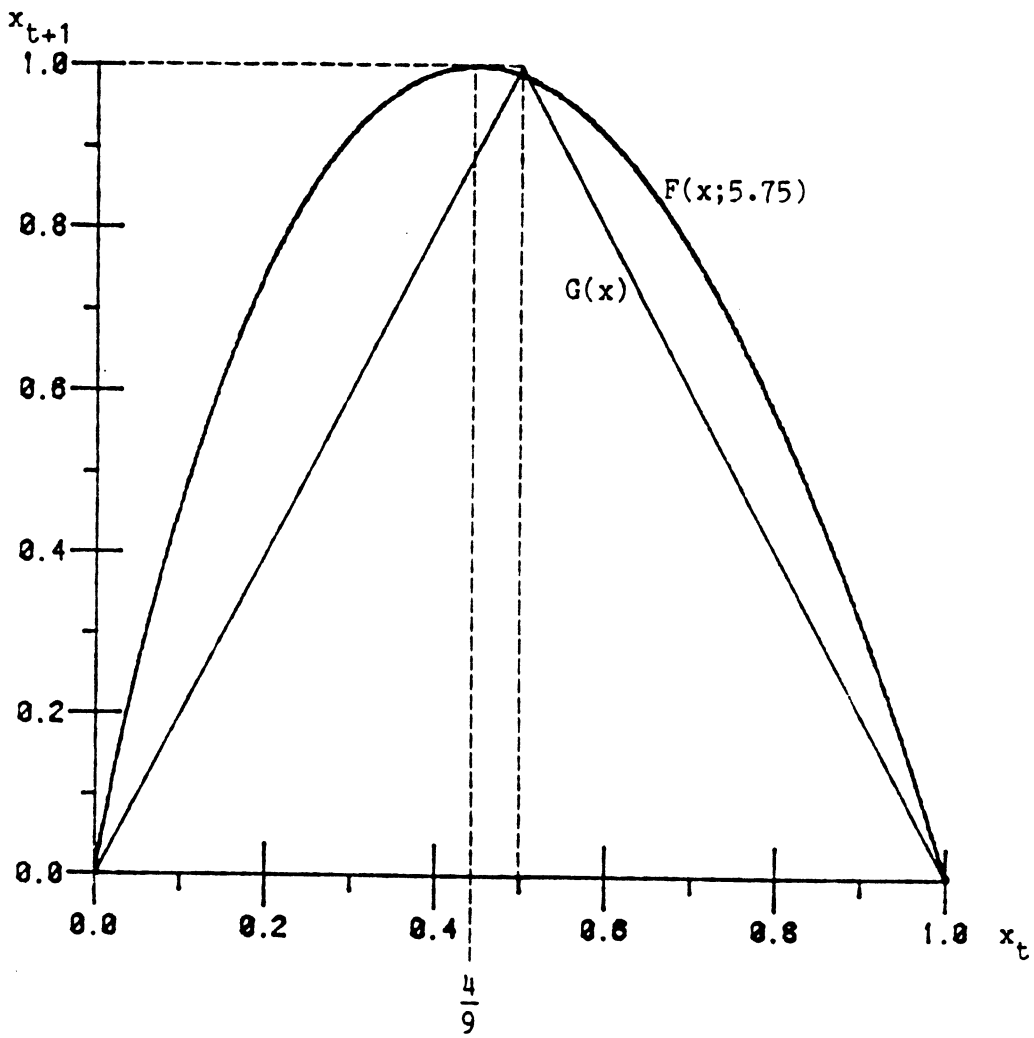

Figure 10

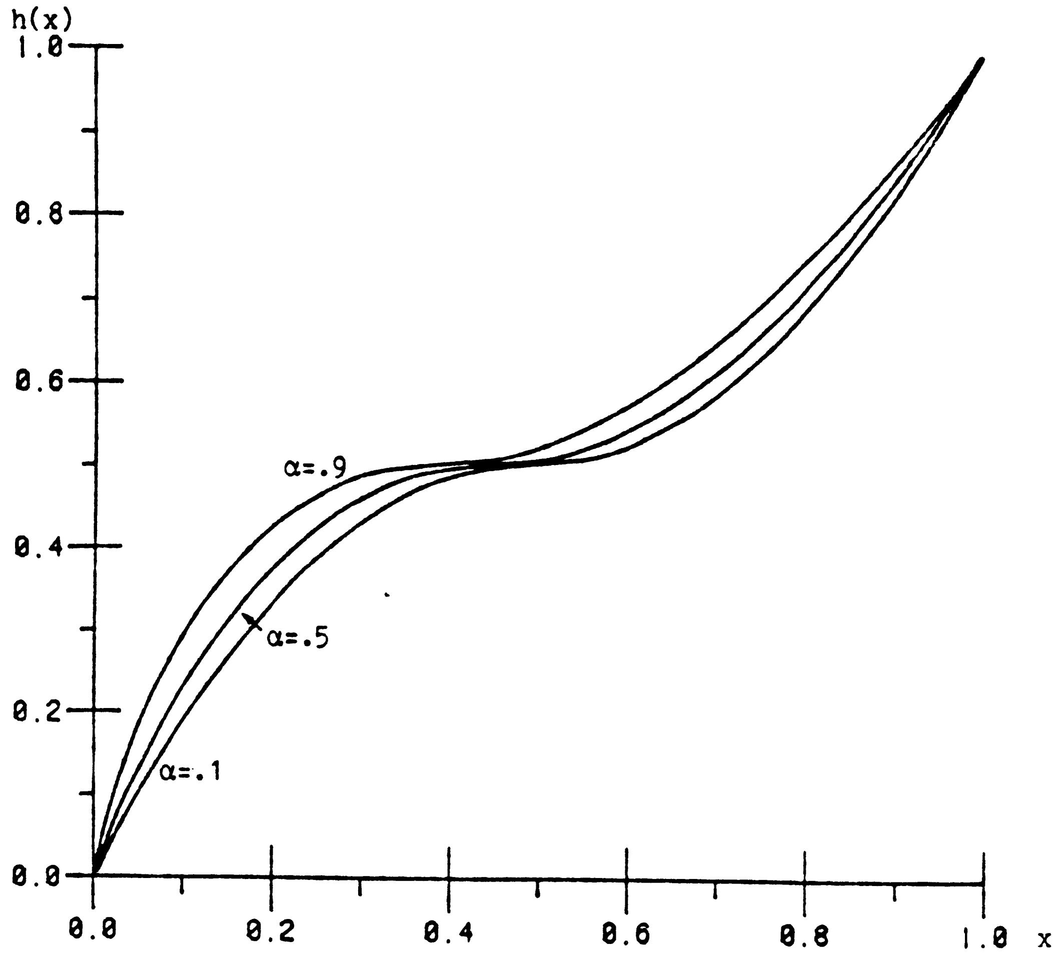

Figure 11: Mean sojourn time in interval [0,x] when F maps the interval onto itself

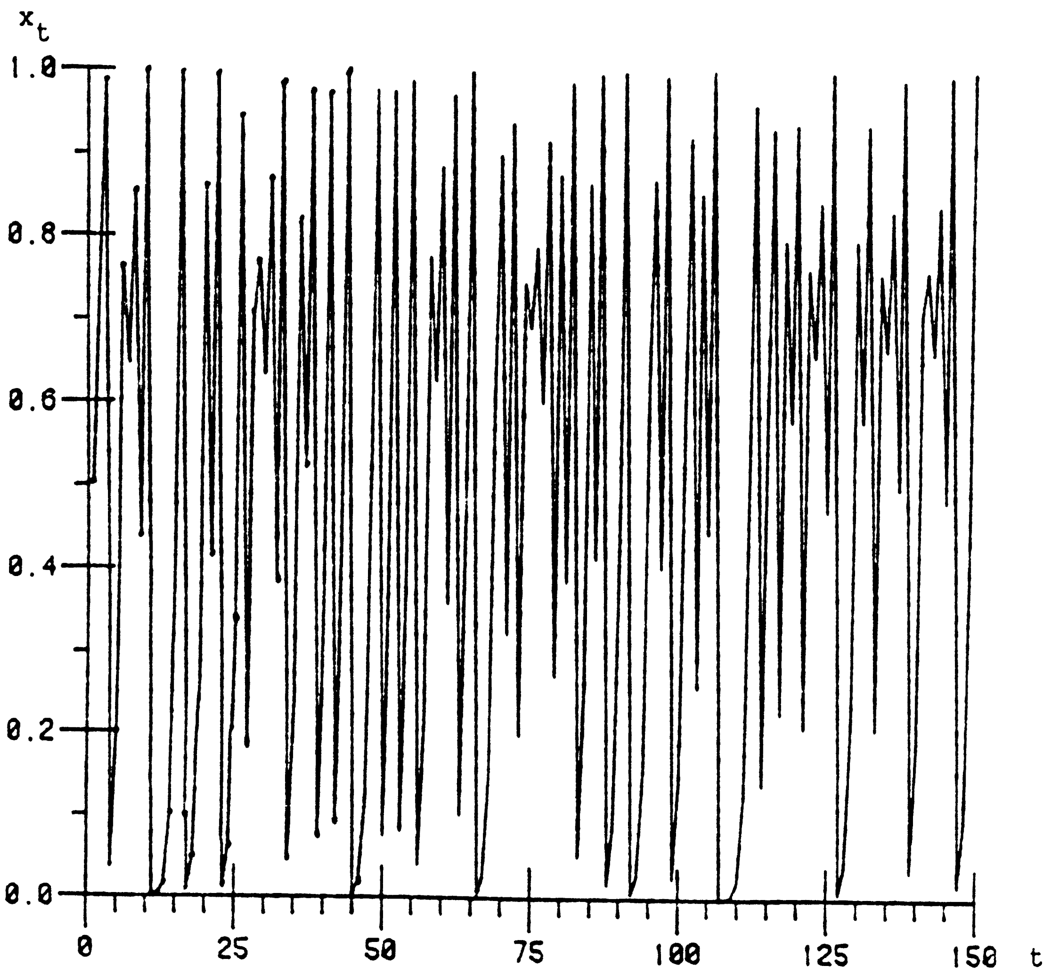

Figure 12: Trajectory of $x_{t+1} = 6.75x_t(1-\sqrt{x_t})$; $x_0 = \frac{1}{2}$

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