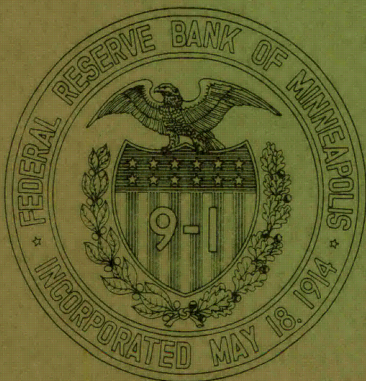# Working Paper

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Interpreting the Long-Run Relationship
Between Money and Prices in the Presence
of a Mundell-Tobin Effect

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1. Introduction

Lucas' (1978) elevation of the quantity theory to the "status of Natural Law" motivates this study of the quantity-theoretic propositions and their frequently discussed but empirically untested modifications. His ingenious but unstated use of Sims' (1972) approximation error formula to extract the long-run properties of time series for a single economy is analyzed in detail in the next section and several examples are presented. An important possible caveat to the quantity theory propositions noted by Lucas is the Mundell-Tobin effect of a monetary expansion: the resulting inflation lowers the real yield on money balances and hence induces an asset shift to real capital. Section 3 begins an examination of the importance of this effect in the context of Lucas' (1975) equilibrium model of the business cycle.

The use of the Lucas model necessarily makes this study similar to Fischer's (1979) examination of neutrality; the important difference is that the current study is of an explicitly stochastic economy. This modification allows the study of monetary interventions to which agent's decision rules are invariant, but raises a difficult analytical issue: the solution of stochastic difference equations which are not derived from optimum problems. To overcome this difficulty, a general, efficient method for the solution of stochastic difference equations is developed and discussed extensively in Section 4. Application of this method in Section 5 yields a model equilibrium embodying the standard cross-equation rational expectations restrictions which is then used in Section 6 to study the response patterns of the price level and capital stock to monetary innovations for different settings of the Mundell-Tobin effect. The

relation of these results to the Lucas empirical results is presented in Section 7, and some thoughts concerning further research are given in Section 8.

2. The Quantity Theory and the Mundell-Tobin Effect

One of the most familiar formulas of economics is the quantity equation

$$(2.1) \qquad M(t)V(t) = P(t)Y(t)$$

where $P(t)$ is the price level at time t, $Y(t)$ is real output at t, and $V(t)$ is the velocity of turnover per period of the money stock $M(t)$. Unadulterated, (2.1) is a truism: the value of output must equal the effective volume of nominal assets used to purchase it. Constrained, (2.1) has been used for modeling purposes. In the simplest incarnation of (2.1), velocity is assumed constant: $V(t) = V \forall t$. The resultant form

$$(2.2) \qquad \frac{M(t)}{P(t)} = kY(t)$$

is enshrined on the pages of beginning and intermediate textbooks as an element of an "economic model." (2.2) indicates that an increase in the nominal money stock must either increase the price level, output, or both. In more sophisticated usages, a version of (2.1) arises as a characteristic of equilibrium. For instance, in the simplest versions of the classical model, a cleared labor market serves to fix $Y(t)$, and equilibrium is characterized by

$$(2.3) \qquad \frac{dM(t)}{M(t)} = \frac{dP(t)}{P(t)}$$

which is produced by (2.2) when $Y(t) = \overline{Y}$. (2.3) is the mathematical formulation of the quantity theory of money: a jump in the money stock at t induces a proportional jump in the price level at t, and has no real effects. In explicitly dynamic frameworks, (2.3) often appears as

$$(2.4) \qquad \frac{\dot{M}(t)}{M(t)} = \frac{\dot{P}(t)}{P(t)}$$

so that an increase in the rate of growth of the money stock causes an equal increase in the rate of inflation.

Of course, the quantity theory relation is not a characteristic of all models of economic behavior. Models which are as defensible as the classical model which produced (2.3), but in which (2.3) does not hold, include simple versions of the Keynesian model and Tobin's (1955) dynamic aggregative model.[1] Indeed, it was the failure of the simple, ad hoc, deterministic, continuous time classical model to conform to empirical observations which led to the development of the Keynesian framework.

But the quantity theory will not die. The current generation Keynesian macroeconometric models have failed on a scale similar to the supposed inability of the classical model to explain the unemployment of the 1930s.[2] As a result, some researchers, led by Robert Lucas and Thomas Sargent, have resurrected the classical model in a sophisticated, explicitly stochastic, dynamic form.[3] Hence, the quantity theory is again receiving attention, though not necessarily in the form (2.4). Economists now explicitly recognize that a variety of forces might cause (2.4) to hold only an average or not at all. But Lucas (1978) suggests that one ought to take a version of (2.4) seriously.

Lucas argues that the quantity theoretic propositions[4] are unique among propositions of monetary economics in that they "possess that combination of theoretical coherence and empirical verification which merits the status of Natural Law" (1978, page 1). By "theoretical" coherence, Lucas means that formulas like (2.4) arise as necessary conditions for equilibrium in well-posed models of economies in which agents optimize and markets clear, as in Sidrauski (1967a, 1967b). Equally importantly, this general equilibrium theorizing points to possible qualifications of the quantity theory.

The most obvious <u>caveat</u> stems from the possibility of a Mundell (1963)-Tobin (1965) effect of an increase in the money stock: the resultant inflation lowers the yield on money holdings and induces an asset shift to real capital. In the presence of a Mundell-Tobin effect, an increase in the rate of money creation would cause inflation to rise by a lesser amount, and would have real effects. In some models, a Lucas (1973)-type Phillips curve can produce a Mundell-Tobin effect, so the effect has received some attention. But in standard models, the effect, if present, is typically thought of as a second-order one, though, in an important class of models stemming from the work of Samuelson (1958), the Mundell-Tobin effect is the primary one.[5/]

By "empirical verification," Lucas means the variety of studies appearing to confirm (2.4). One such study cited by Lucas is Vogel's (1974) study of average inflation and money growth in Latin America. Vogel found that average inflation and money creation rates for a cross section of Latin American countries lay on a 45 degree line. This approach is typical: in most studies of (2.4) the data are a cross section of time series for several countries. But there are obvious comparability issues in such an approach, and a natural question is: how does one study (2.4) for a single economy? In essence, Lucas does this by studying very long distributed lag regressions of inflation on money creation, and his pictures tend to confirm (2.4). The method is novel and somewhat subtle, and an understanding of his results requires some discussion of his approach.

Lucas took time series for money creation (μ_t) and inflation (π_t) and applied a two-sided filter to each. For instance, he replaced the time series $\{\mu_t\}$ by[6/]

(2.5) $\{\mu_t(\beta)\} = \{(\frac{1-\beta}{1+\beta}) \sum\limits_{k=-\infty}^{\infty} |\beta|^k \mu_{t-k}\}$

with $0 < \beta < 1$. He then generated plots of the pairs $(\mu_t(\beta), \pi_t(\beta))$ for various values of β, and found that for β near one, the filtered observations approximately fell on a 45 degree line; i.e., his graphical method discovered R^2 and α near one in the regression[7]

(2.6) $\pi_t(\beta) = \alpha\mu_t(\beta) + \varepsilon_t.$

In order to judge his method and results, it is necessary to review some results from projection theory.

Consider the projection of π_t on the entire μ_t process:

(2.7) $\pi_t = \sum\limits_{k=-\infty}^{\infty} \gamma_k \mu_{t-k} + \eta_t$

where η_t satisfies $E_t\mu_{t-k}\eta_t = 0 \,\,\forall\, k$. Defining $S_\pi(e^{-iw})$ and $S_\mu(e^{-iw})$ as the spectral densities of $\{\pi_t\}$ and $\{\mu_t\}$, the coefficients γ_k can be recovered by (inverse) Fourier transforming

(2.8) $\gamma(e^{-iw}) = S_{\pi\mu}(e^{-iw})/S_\mu(e^{-iw})$

where $S_{\pi\mu}(e^{-iw})$ is the cross spectrum of (π_t, μ_t).[8] The "frequency domain R^2" obtained by using (2.8) is given by the coherence:

(2.9) $\text{coh}(e^{-iw}) = \dfrac{|S_{\pi\mu}(e^{-iw})|^2}{S_\pi(e^{-iw})S_\mu(e^{-iw})}.$

In practice, (2.7) is impossible to estimate: it contains an infinite number of parameters. Some restriction of $\{\gamma_k\}$ is necessary. Though there are many such restrictions, by far the most common is truncation, i.e., $\gamma_k = 0 \,\,\forall\, k > N_1 > 0$, $\gamma_k = 0 \,\,\forall\, k < N_2 < 0$. In any event, let $\{\gamma_k'\}$ be a restricted version of $\{\gamma_k\}$ so that $\{\gamma_k'\} \neq \{\gamma_k\}$ for any such restriction. The restricted regression to be implemented is

(2.10) $\pi_t = \sum\limits_{k=-\infty}^{\infty} \gamma_k' \mu_{t-k} + u_t.$

Least squares chooses $\{\hat{\gamma}_k'\}$ to minimize the sum of squared residuals $\sum\limits_{t=0}^{T} u_t^2.$ Again turning to population arguments, this sum of squared residuals is proportional to the area under the spectrum of u_t, $S_u(e^{-iw})$. Substituting (2.7) into (2.10) and rearranging yields

(2.11) $u_t = \sum\limits_{k=-\infty}^{\infty} (\gamma_k - \gamma_k') \mu_{t-k} + \eta_t.$

From (2.11), the spectrum of u_t is

(2.12) $S_u(e^{-iw}) = |\gamma(e^{-iw}) - \gamma'(e^{-iw})|^2 S_\mu(e^{-iw}) + \dfrac{\sigma_\eta^2}{2\pi}$

where $\gamma(e^{-iw}) = \sum\limits_{k=-\infty}^{\infty} \gamma_k e^{-iwk}$. Hence, least squares chooses $\{\gamma_k'\}$ to minimize

$$\int_{-\pi}^{\pi} |\gamma(e^{-iw}) - \gamma'(e^{-iw})|^2 S_\mu(e^{-iw}) dw$$

which is Sims' (1972) approximation error formula. Clearly, the approximation will be "best" at frequencies w over which $S_\mu(e^{-iw})$ is the largest. Since

$$w \cdot \text{period} = 2\pi,$$

good "long-run" approximations require relatively high power in $\{\mu_t\}$ at low frequencies. If the long-run properties of $\{\gamma_k\}$ are important, it may be useful to filter μ_t with a filter which has large power at low frequencies.[9/]

Lucas notes that the filter in (2.5) has Fourier transform

(2.13) $\beta(e^{-iw}) = \dfrac{(1-\beta)^2}{1+\beta^2-2\beta\cos w}$

which, for large β, concentrates most of the power at low frequencies. His graphical method forces the α of (2.6) to approximate $\{\gamma_k\}_{-\infty}^{\infty}$ of (2.7), so

surely approximation error is present, and Sims' formula (2.12) is useful: Lucas' methods amount to choosing α to minimize

$$\int_{-\pi}^{\pi} |\gamma(e^{-iw})-\alpha|^2 S_{\mu(\beta)}(e^{-iw})dw.$$

When $S_{\mu(0)}(e^{-iw})$ has power at low frequencies, $S_{\mu(\bullet)}(e^{-iw})$ will, for β near one, have most of its power concentrated at low frequencies. Indeed, $S_{\mu(1)}(e^{-iw})$ will be $S_{\mu(0)}(e^{-iw})$ at zero frequency and zero elsewhere. Thus, when $S_{\mu(\beta)}(e^{-iw})$ is itself low-pass, $\hat{\alpha}$ will be a good approximation of $\gamma(1)$, the sum of lag coefficients in (2.7). The quality of the estimate is governed by $coh(w)$ for small w, the coherence at low frequencies of μ_t and π_t.

Examples of the types of calculations performed by Lucas[10] are given in Figures 2.1-2.7. The figures illustrate the effects of seven filters on $\pi_t(\beta)$ and $\mu_t(\beta)$. Since the coherence of π_t and μ_t is unaffected by filtering, this quantity is presented only once, in Panel a of Figure 2.1. In the following figures, Panel a illustrates the gains[11] of the filter used throughout that figure. The absolute value of the Fourier transform of $\{\gamma_k\}$,[12]

$$|\gamma(e^{-iw})| = \frac{|S_{\pi(0)\mu(0)}(e^{-iw})|}{S_{\mu(0)}(e^{-iw})},$$

is also unaffected by symmetric[13] filtering, but is presented in each figure as a reminder of what the α of (2.6) is approximating. Panels b and d present spectra of $\mu_t(\beta)$ and $\pi_t(\beta)$, while Panel e displays a scatter plot of $\mu_t(\beta)$ and $\pi_t(\beta)$.

The data are 289 monthly observations from December 1953 to December 1977, not seasonally adjusted, on M1 and the Consumer Price Index.

The spectra, the transfer modulus, and the coherence were calculated as follows. First, means were extracted from the logarithms of M1 and the CPI. First differences were then taken to obtain μ_t and π_t. These series were regressed against a consant, 24 own lags, and 11 seasonal dummies; i.e.,

$$\pi_t = \sum_{j=1}^{24} \gamma_j^{\pi} \pi_{t-j} + \text{constant} + \text{seasonal dummies} + \text{residual}$$

$$\mu_t = \sum_{j=1}^{24} \gamma_j^{\mu} \mu_{t-j} + \text{constant} + \text{seasonal dummies} + \text{residual}.$$

Denote the whitening filters $1 - \sum_{j=1}^{24} \gamma_j^{\pi} L^i$ and $1 - \sum_{j=1}^{24} \gamma_j^{\mu} L^j$ by $f^{\pi}(L)$ and $f^{\mu}(L)$. The residuals from these regressions were then Fourier transformed to obtain $\tilde{\mu}^W$ and $\tilde{\pi}^W$,[14/] e.g.;

$$\tilde{\mu}^W\left(\frac{2\pi j}{288}\right) = \sum_{t=1}^{288} \mu_t^W e^{-\frac{2\pi i j t}{288}} \qquad\qquad j = 0, 1, \ldots, 287.$$

The periodograms and cross periodogram were obtained as $\tilde{\mu}^W(\cdot)\overline{\tilde{\mu}^W(\cdot)}/(2\pi\cdot264)$, $\tilde{\mu}^W(\cdot)\overline{\tilde{\mu}^W(\cdot)}/(2\pi\cdot264)$, and $\tilde{\pi}^W(\cdot)\overline{\tilde{\pi}^W(\cdot)}/(2\pi\cdot264)$. These quantities were smoothed using a triangular spectral window with a base of 7 ordinates to obtain the "prewhitened" spectral estimates $S\mu^W(\cdot)$, $S\pi^W(\cdot)$, and $S\pi^W\mu^W(\cdot)$. The coherence of Panel a of Figure 2.1 was computed as

$$\frac{|S\pi^W\mu^W(\cdot)|^2}{S\pi^W(\cdot)S\mu^W(\cdot)},$$

while the tranfer modulus appearing in Panel c of all figures was computed as

$$\frac{|S\pi^W\mu^W(\cdot)|}{S\mu^W(\cdot)} \frac{|f^{\mu}(\cdot)|}{|f^{\pi}(\cdot)|}$$

Denote the gain of the low-pass filter[15/] by $|\ell_{(\beta)}(e^{-iw})|^2$, and that of the quarterly averaging filter $(L^{-1}+1+L)/3$ by $|q(e^{-iw})|^2$. The spectra presented in the several figures were computed as

$$\frac{S\pi^W(\cdot)}{|f^\pi(\cdot)|^2}\,|\ell_{(\beta)}(\cdot)|^2|q(\cdot)|^2,\ \frac{S\mu^W(\cdot)}{|f^\mu(\cdot)|^2}\,|\ell_{\beta(\cdot)}|^2|q(\cdot)|^2.$$

The data for the Panel e scatterplots, $\mu_t(\beta)$ and $\pi_t(\beta)$, were calculated as follows. First, μ_t and π_t were Fourier transformed. The transforms of the series were then multiplied by the Fourier transforms $\ell_{(\beta)}(e^{-iw})$ and $q(e^{-iw})$ of the low-pass and quarterly-average filters. The resulting quantities were then inverse Fourier transformed to yield $\mu_t(\beta)$ and $\pi_t(\beta)$.[17/]

In Figure 2.1, β of (2.5) is zero. Hence, the filter treats all frequencies identically: it is as if no filter were applied. Therefore, the spectra of Panels b and d are for the unfiltered data. Notice that although money growth displays seasonals evidenced by peaks in the spectrum at seasonal frequencies, inflation has no peak in its spectrum except that at zero frequency. The scatter plot of the raw data in Panel e does not reveal any systematic relationship. Indeed, the estimated[18/] α from (2.6) is 0.02.

In Figure 2.2 the filter is beginning to take effect. The higher frequency peak in the spectrum of money growth has been sharply reduced. A similar reduction of high frequency power in inflation has occurred, but it is difficult to see. But (2.5) with $\beta = 0.5$ does not appear to extract much from the data: the scatter plot Panel e is still disorderly and $\hat{\alpha}$ is only 0.08.

But in Figure 2.3, order begins to emerge. The seasonal in money growth has been removed relative to the peak at zero frequency. This is

reflected in Panel e, where now $\hat{\alpha} = 0.87$ and the points begin to fall on a line.

Finally, in Figure 2.4, the spectra are essentially narrow spikes at low frequency. In Panel e, $\hat{\alpha} = 0.99$ and there is a clear relationship between $\mu_t(0.99)$ and $\pi_t(0.99)$. Apparently, one can reject the hypothesis[19/] that the sum of lag coefficients $\sum_{k=-\infty}^{\infty} \gamma_k$ of (2.7) is zero. Because π_t and μ_t are highly coherent near zero frequency and the transfer modulus is near one there, the estimate of $\sum_{k=-\infty}^{\infty} \gamma_k$ near one appears to be a good one.

There are several points to be made about these figures. First, as (2.13) indicates, obtaining a good estimate of $\sum_{k=-\infty}^{\infty} \gamma_k$ by estimation of α in (2.6) requires a sharp reduction in power at high frequencies. This is made clear in Figures 2.5 through 2.7 where the calculations of Figures 2.2-2.4 are reproduced with a slightly different filter. In Figures 2.5-2.7, $\pi_t(\beta)$ and $\mu_t(\beta)$ were computed as

$$\mu_t(\beta) = (1+\beta L)^2 \mu_t$$

$$\pi_t(\beta) = (1+\beta L)^2 \pi_t$$

with $\beta = 0.5$, 0.9, and 0.95. As Panels a indicate, $(1+\beta L)^2$ is a low-pass filter, though not nearly so low pass as $\sum_{k=-\infty}^{\infty} \beta^{|k|} L^k$. As a result, the scatterplot Panels e show no order: the estimated α's are near zero.

Second, $\hat{\alpha}$ of (2.6) is an estimate of $\sum_{k=-\infty}^{\infty} \gamma_k$, not $\sum_{k=0}^{\infty} \gamma_k$. It is only the case that $\gamma_k = 0$ for $k < 0$ when $\{\pi_t\}$ fails to Granger (1969)-cause $\{\mu_t\}$. It is of interest to note that if $\{\pi_t\}$ fails to Granger-cause $\{\mu_t\}$, then $\{\pi_t(\beta)\}$ fails to Granger-cause $\{\mu_t(\beta)\}$. This is because $\pi_t(\beta)$ and $\mu_t(\beta)$ are calculated from π_t and μ_t by applying the same filter.

Third, though the filtering procedure leaves substantial serial correlation in the residuals of (2.6),[20] there should be no correction for it. This point is made by Sims (1972), and rests on the fact that sharp low frequency power in $\mu_t(\beta)$ is necessary for a good estimate $\hat{\alpha}$ in (2.6). In this case, $\pi_t(\beta)$ is also low pass, and for large β, the residuals in (2.6) will also have this property; they will be serially correlated.

Finally, Panel e of Figure 2.4 indicates that for the long run, the quantity theory (2.4) holds. But in what sense? Does figure 2.4 reject the "hypothesis" that the Mundell-Tobin effect is present? Is Figure 2.4 compatible with a Lucas (1973)-type Philips curve?

The answer to these questions requires some structure. To this end, a model first proposed by Lucas (1975) will be used. Lucas' interest was in generating equilibrium business cycles by concealing certain information from agents. Though similar in spirit to Lucas, the usage here more closely resembles that of Fischer (1979).[21] There the intent was to study the responses of real variables and prices to certain deterministic monetary interventions. The development below will allow these issues, as well as the Mundell-Tobin and Phillips effects, to be addressed.

3. <u>The Model</u>

A discussion of the model's theoretical foundations will be facilitated by having some notation and a structure in hand. Hence, the model will be presented first, with its underpinnings to be discussed below. Also to be postponed is the discussion of a version of the model exhibiting Phillips curve effects.

Let y_t denote the log of output of t; k_t, the log of the capital stock at t; r_t, the log of the real return to capital at t; m_t, the log of the money stock at t; and p_t, the log of the price level at t. Output is generated according to

$$(3.1) \qquad y_t = \delta_0' + \delta_1' k_t \qquad \qquad \delta_1' > 0$$

while capital's real return follows

$$(3.2) \qquad r_t = \delta_0 - \delta_1 k_t \qquad \qquad \delta_1 > 0$$

(3.1) and (3.2) can be thought of as arising from a setting in which markets are competitive and workers supply their services inelastically to firms whose technology is Cobb-Douglas in capital and labor.[22]

There are two assets in the model, real capital and real balances. The demand for capital is governed by

$$(3.3) \qquad k_{t+1} = \alpha_0' + \alpha_1 E_t r_{t+1} + \alpha_2 (E_t p_{t+1} - p_t) + \alpha_3 k_t$$

where $\alpha_1 > \alpha_2 > 0$; $\alpha_3 \epsilon (0,1)$, and $E_t x$, the mathematical expectation[23] of x conditioned on information known at t, is defined by $E_t x \equiv E(x|\Omega_t)$. The information set Ω_t includes current (date t) and past values of all the variables listed above as well as knowledge of the structure of the model. The demand for real balances is given by[24]

(3.4) $\quad m_t - p_t = \beta_0' - \beta_1 E_t r_{t+1} - \beta_2 (E_t p_{t+1} - p_t) + \beta_3 k_t$

with $\beta_2 > \beta_1 > 0$ and $\beta_3 \epsilon (0,1)$.

The capital stock is fixed at date t and has a one-period gestation; i.e., date t output from the production process (3.1) is divided among new capital (date t+1), consumption, and government expenditures which are financed totally by money creation. The sequence of money supplies $\{m_t\}_{t=-\infty}^{\infty}$ is taken to be a linearly regular[25/] covariance stationary stochastic process with moving average representation

(3.5) $\quad m_t = \sum_{k=0}^{\infty} h_k \epsilon_{t-k}$

where

$$\sum_{k=0}^{\infty} h_k^2 < \infty, \quad \epsilon_t = m_t - E(m_t | m_{t-1}, m_{t-2}, \ldots]$$

i.e., $\{\epsilon_t\}$ is fundamental for $\{m_t\}$. (3.5) expresses m_t as the convolution of the two sequences $\{h_k\}$ and $\{\epsilon_k\}$. With the understanding that $h(L) = \sum_{k=0}^{\infty} h_k L^k$, where the lag operator L is defined by $L^n x_t \equiv x_{t-n}$, (3.5) can also be written

(3.5') $\quad m_t = h(L) \epsilon_t.$

In addition, assume $\{m_t\}$ possesses an autoregressive representation[26/]

$$H(L) m_t \equiv h(L)^{-1} m_t = \epsilon_t.$$

The interesting system dynamics are completely determined by (3.3), (3.4), and (3.5). The demand for capital, given in (3.3), is positively related to capital's own expected return $E_t r_{t+1}$ and inversely related to the expected return to money holdings $p_t - E_t p_{t+1}$. Clearly,

when $\alpha_2 = 0$, there is no Mundell-Tobin effect. Thus, α_2 measures the magnitude of nonneutralities. Similarly, the demand for end-of-period real balances, given in (3.4), is positively related to the own expected return and negatively related to the expected return to real capital.

There are a number of ways to view (3.3) and (3.4), but they cannot be viewed as necessary conditions for equilibrium in an environment in which markets clear and agent's preference functionals are explicitly stated. Lucas "makes virtue of analytical necessity"[27/] by directly postulating versions of the two equations, while noting that one's intuition is aided by the artificial separation of the agent's choice problem implied by (3.3) and (3.4). In Fischer's setup, a number of agents are born at each date and live for two periods. The young supply labor, the old supply capital. Labor's share of output is divided between consumption and saving in the form of real balances and capital. This savings decision is embodied in the (again, postulated) equations (3.3) and (3.4).

Two slightly more mechanical "justifications" for (3.3) and (3.4) can be advanced. First, their simultaneous solution gives rise to a restricted vector autoregression which is "integrable"; i.e., by a theorem of Mosca and Zappa (1979) a quadratic objective functional can be found which, when maximized, produces (3.3) and (3.4) as first-order necessary conditions. This is only mildly comforting in the present context because, given the above assumptions, the resultant functional is quadratic, and the constants linear, in the logarithms of the variables of interest. Also, the objective functional thus obtained is correct only[28/] if p_t is a decision variable; an assumption inconsistent with the concept of competitive markets which clear at each date. Second, (3.3) and (3.4) can be thought of as log-linear approximations[29/] to the nonlinear Euler equations which might arise as necessary conditions for maxima in nonquadratic

optimum problems. Indeed, (3.3) and (3.4) look very much like the Taylor expansions given in Lucas (1972,1980a, and 1980b). According to this view, (3.3) and (3.4) should not be perceived as demand functions, but their parametric solutions could be. For instance, the solution of (3.3) for k_{t+1} with $\{p_t\}$ taken exogenously mimics the demand function for capital which appears in Sargent's (1979) linear quadratic rendition of the Lucas-Prescott (1971) model of investment under uncertainty. Though separate solutions of (3.3) and (3.4) can be regarded as decision rules, the simultaneous solution of the pair of equations cannot: (3.3) and (3.4), taken together, describe the evolution of $\{k_t\}$ and $\{p_t\}$ in equilibrium.

(3.3) and (3.4) compose a cumbersome mathematical object, albeit one that occurs frequently in economics. Consequently, methods for solving systems like (3.3) and (3.4) have been dependent on the particular problem under study. One of the purposes of this paper is to describe a general, easily applied solution method for these systems. In order to compare this solution method to the standard procedures, it is useful to briefly review the approaches adopted by Lucas and Fischer.

Lucas begins[30] by noting that the state of the system (3.3) and (3.4) at date t is completely described by (m_t, k_t). Then the solution for (k_{t+1}, p_t) should be a linear[31] function of the state:

$$(3.6) \qquad k_{t+1} = \pi_{10} + \pi_{11} k_t + \pi_{12} m_t$$

$$(3.7) \qquad p_t = \pi_{20} + \pi_{21} k_t + \pi_{22} m_t.$$

The unknown π_{ij}'s are found by substituting (3.1), (3.2), (3.6), (3.7), and an expression for the money process, $m_{t+1} - m_t = \mu$ (constant), into (3.3) and (3.4), and equating coefficients on constants and state variables. This yields six nonlinear equations which determine the π_{ij}'s uniquely.[32]

Lucas found $\pi_{12} = 0$ and $\pi_{22} = 1$. Hence, once-and-for-all changes in the <u>level</u> of the money stock leave real balances and the capital stock unchanged. However, π_{10} depends on μ, so changes in the rate of growth of money do have real effects.

Several observations about the Lucas solution are in order. First, note that (3.6) and (3.7) do not express the unknowns k_{t+1} and p_t solely in terms of their own history and the driving process m_t. For Lucas' purposes, this is no fault, but if one is interested in the separate dynamics of $\{k_t\}$ and $\{p_t\}$, an alternative solution might be more appropriate. Second, the guesses (3.6) and (3.7) were made judiciously. In particular, the number of lagged money stock terms is determined by the order of the difference equation generating money: had money been characterized by a second-order (deterministic) difference equation, (3.6) and (3.7) would have included terms in m_{t-1}. Finally, it is disquieting that determination of the π_{ij}'s requires the solution of nonlinear equations, the number of which increases at twice the rate of increases in the order of the difference equation generating the driving process. Complicated money processes could make the Lucas solution method intractible.

Fischer's intent was to study the responses of prices and capital to a general class of changes in money, and his solution procedure differs from Lucas'. Fischer begins with the parametric solution of (3.3) for k_{t+1} in terms of k_t and past expected rates of inflation. This equation, along with (3.2), is then substituted into (3.4) to yield

$$(3.8) \qquad p_t = b_0 + b_1 p_{t-1} + b_2 E_t p_{t+1} + b_3 E_{t-1} p_t + b_4 m_t + b_5 m_{t-1}.$$

The "guess" at the solution[33/] for p_t is given by

$$(3.9) \qquad p_t = \delta + \lambda p_{t-1} + \sum_{i=0}^{\infty} \pi_i E_t m_{t+i} + \sum_{i=0}^{\infty} \theta_i E_{t-1} m_{t-1+i}.$$

The π_i's and θ_i's are determined by using (3.9) to obtain expressions for $E_t p_{t+1}$ and $E_{t-1} p_t$, substituting these into (3.8), and then equating coefficients on like quantities in (3.8) and (3.9). It turns out that (3.9) can be written as[34]

$$(3.9') \qquad p_t = \delta + \lambda p_{t-1} + a \sum_{i=0}^{\infty} b^i E_t m_{t+1} + c \sum_{i=0}^{\infty} d^i E_{t-1} m_{t+i} + e m_{t-1}.$$

Given a process for the money supply, say (3.5), the Hansen-Sargent (1980) formula[35] can be employed to convert (3.9') into an expression containing only current and past values of the money stock. However, Fischer is interested in processes for the money stock of the form

$$(3.10) \qquad m_t = \begin{cases} 0 & t \neq t_0 \\ 1 & t = t_0, \end{cases}$$

a "transitory" increase in m_t at t_0,[36] and

$$(3.11) \qquad m_t = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0, \end{cases}$$

a "permanent" increase in $\dot{m}_t$ at t_0, not in processes for $\{m_t\}$ like that given in (3.5).[37] By using (3.10) and (3.11) directly in (3.9), Fischer is able to draw pictures displaying the response patterns of prices to changes in the money stock.

As in Lucas' solution, Fischer's guess (3.9) is a judicious one which is by no means obvious. An examination of the forms of the guesses (3.6), (3.7), and (3.9) indicates that one must possess good prior knowledge about solutions to employ this method of undetermined coefficients. Even given this prior knowledge, Lucas and Fischer found it necessary to solve a variety of other problems before the conjectured coefficients could be found. While the ingenuity of the above solutions cannot be discounted, one hopes that more mechanical solution procedures exist.

For the purposes of this study, it would be enough to use (3.9), various versions of (3.5), and the Hansen-Sargent formula to generate "reduced forms" for the price level and the capital stock.[38/] It would then be a simple matter to investigate the responses of p_t and k_{t+1} to innovations in the money stock under alternative settings of the Mundell-Tobin effect. However, it will prove useful to adopt a slightly different solution strategy. This method, like the Lucas and Fischer approaches, is a method of undetermined coefficients. Yet, unlike the procedures above, there will be only one undetermined coefficient.[39/] It is not a costless simplification, though. The intuition and ingenuity of the Lucas and Fischer solutions will be traded for investment in mathematical tools, with the hope that the acquired tools will prove useful in other problems.

4. A Solution Strategy

The approach taken here relies on the fact that one can, without loss of information, "travel" from the space of square-summable sequences[40] to the space of Lebesque square-integrable functions defined on the unit circle. That is, the two spaces are related by an invertible transformation. This transformation, known as the "z-transform" can be described as follows.

Consider the space of square summable sequences $\{x_k\}_{k=-\infty}^{\infty}$, i.e., $\sum_{k=-\infty}^{\infty} |x_k|^2 < \infty$.[41] The z-transform of a member $\{x_k\}$ of this space is defined by[42]

$$x(z) = \sum_{k=-\infty}^{\infty} x_k z^k$$

where z is a complex number. Notice that x(z) exists at least on the unit circle ($|z|=1$):

$$x(z) = \sum_{k=-\infty}^{\infty} x_k z^k$$

$$\leq \sum_{k=-\infty}^{\infty} |x_k z^k| = \sum_{k=-\infty}^{\infty} |x_k||z|^k$$

$$= \sum_{k=-\infty}^{\infty} |x_k| < \infty.$$

In addition, if $x_k = 0$ for $k < 0$,

$$x(z) = \sum_{k=0}^{\infty} x_k z^k.$$

In this case, x(z) exists on the unit disk, i.e., x(z) exists for all $|z| \leq 1$. This can be demonstrated as follows. Write $z = re^{-iw}$ where $r = \sqrt{Re(z)^2 + Im(z)^2}$ and $w = \tan^{-1}(\frac{Im(z)}{Re(z)})$. Then

$$x(z) = \sum_{k=0}^{\infty} x_k r^k e^{-ikw}$$

$$\leq \sum_{k=0}^{\infty} |x_k| r^k$$

which converges for all $r = |z| \leq 1$. The sequence $\{x_k\}_{k=-\infty}^{\infty}$ can be recovered from its transform according to[43]

$$x_k = \frac{1}{2\pi i} \oint_\Gamma x(z) z^{-(k+1)} dz$$

where $\oint$ denotes contour integral and Γ denotes any contour in the annulus of convergence.[44] A version of the Riesz-Fischer theorem indicates that

$$\left| \frac{1}{2\pi i} \oint_\Gamma |x(z)|^2 \frac{dz}{z} \right| < \infty.$$

So the z-transform is a bijective transformation from the space of square summable sequences to the space of square integrable complex valued functions on the unit circle. Hence, knowledge of a sequence is equivalent to knowledge of its z-transform and the notation $x(z) \overset{z}{\leftrightarrow} \{x_k\}$ has an obvious meaning.

When $\{x_t\}$ is a zero-mean covariance stationary stochastic process with moving average representation $x_t = \sum_{k=0}^{\infty} \beta_k \epsilon_{t-k}$, Ex_t^2 is given by

$$Ex_t^2 = E\left(\sum_{k=0}^{\infty} \beta_k \epsilon_{t-k} \sum_{k=0}^{\infty} \beta_k \epsilon_{t-k} \right)$$

$$= \sum_{k=0}^{\infty} \beta_k^2 E\epsilon_{t-k}^2$$

$$= E(\epsilon_t)^2 \sum_{k=0}^{\infty} \beta_k^2$$

which is finite only if $\sum_{k=0}^{\infty} \beta_k^2 < \infty$.[45] Hence, there is an intimate link between the space of square summable sequences and that of covariance stationary processes.

It is useful to note that the z-transform:

a. Is isometric (preserves linear structure). If

$$\sum_{k=-\infty}^{\infty} |x_k|^2 < \infty, \quad \sum_{k=-\infty}^{\infty} |y_k|^2 < \infty, \quad x(z) \overset{z}{\leftrightarrow} \{x_k\},$$

$$y(z) \overset{z}{\leftrightarrow} \{y_k\}, \text{ and } \alpha \epsilon R^1,$$

then

$$x(z) + y(z) = \sum_{k=-\infty}^{\infty} (x_k + y_k) z^k$$

and

$$\alpha x(z) = \sum_{k=-\infty}^{\infty} \alpha x_k z^k,$$

i.e.,

$$x(z) + y(z) \overset{z}{\leftrightarrow} \{x_k + y_k\}$$

$$\alpha x(z) \overset{z}{\leftrightarrow} \{x_k\}.$$

b.   Is isomorphic (preserves "distance").

$$\left\{ \frac{1}{2\pi i} \oint_\Gamma |x(z) - y(z)|^2 \frac{dz}{z} \right\}^{1/2} = \left\{ \sum_{k=-\infty}^{\infty} |x_k - y_k|^2 \right\}^{1/2}.$$

c.   Is easily led or lagged,

$$z^{-n} x(z) = \sum_{k=-\infty}^{\infty} x_{k+n} z^k$$

i.e.,

$$z^{-n} x(z) \overset{z}{\leftrightarrow} \{x_{k+n}\}.$$

d.   Produces the convolution relation

$$x(z) y(z) \overset{z}{\leftrightarrow} \left\{ \sum_{k=-\infty}^{\infty} x_k y_{t-k} \right\}_{t=-\infty}^{\infty}.$$

e.   Is analytic for one-sided sequences. If $x_k = 0$ for $k < 0$ and $\sum_{k=0}^{\infty} |x_k|^2 < \infty$, then $x(z)$ is analytic inside the unit disk.

Property d is well known and provides the primary basis for the use of the z-transform: it simplifies the manipulation of objects like the

moving average (3.5) above. Property e follows from the fact that a power series can be integrated term by term inside its circle of convergence. Because z^n is entire, this integral is zero. By a converse of the Cauchy-Goursat theorem known as Morera's theorem, the power series is then analytic on its convergent disk. The property is useful in prediction problems and is exploited extensively in the solution strategy to be discussed below.

Before turning to the solution of difference equations, it will be useful to examine the inversion operation. For the most part, the sequences of interest will be one-sided. Hence, property e will play a major role in the discussion of the inversion of the z-transform.

There are many ways to invert the z-transform. Of the methods to be discussed, all but one rely on the reduction of a complicated transform to the sum of a number of very simple transforms. For the following discussion, make the identification

$$x(z) = \sum_{k=0}^{\infty} x_k z^k$$

so that attention will be restricted to one-sided sequences. In addition, assume $x(z)$ exists for all $|z| \leq 1 + \delta$, $\delta > 0$. The following draws heavily on Chapter 4 of Gabel and Roberts (1973). The interested reader should consult the original.

1. Inspection

If $x(z)$ is a polynomial in nonnegative powers of z, the sequence $\{x_k\}_{k=0}^{\infty}$ can be found directly. For instance, consider

$$x(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots = \sum_{j=0}^{\infty} a_j z^j.$$

Then clearly

$$x_0 = a_0$$

$$x_1 = a_1$$

.

.

.

or

$$\{x_k\}_{k=0}^{\infty} = \{a_0, a_1, a_2, \ldots\}.$$

2.   Polynomial Long Division

When $x(z)$ is a ratio of polynomials (a rational function) the sequence $\{x_k\}$ can sometimes be found by performing the required division. For instance, consider the z-transform

$$x(z) = \frac{1 - \frac{26}{24}z + \frac{9}{24}z^2 - \frac{1}{24}z^3}{1 - \frac{1}{4}z}.$$

Performing the division,

$$
\require{enclose}
\begin{array}{r}
1 - \frac{5}{6}z + \frac{1}{6}z^2 \\[2pt]
1 - \frac{1}{4}z \enclose{longdiv}{1 - \frac{26}{24}z + \frac{9}{24}z^2 - \frac{1}{24}z^3} \\[2pt]
\underline{1 - \frac{1}{4}z } \\[2pt]
- \frac{20}{24}z + \frac{9}{24}z^2 - \frac{1}{24}z^3 \\[2pt]
\underline{- \frac{20}{24}z + \frac{5}{24}z^2 } \\[2pt]
\frac{4}{24}z^2 - \frac{1}{24}z^3 \\[2pt]
\underline{\frac{4}{24}z^2 - \frac{1}{24}z^2} \\[2pt]
0
\end{array}
$$

By inspection,

$$x_0 = 1$$

$$x_1 = -\frac{5}{6}$$

$$x_2 = \frac{1}{6}$$

$$x_j = 0 \qquad \forall \; j > 2.$$

It will frequently be the case that this division will yield a remainder polynomial; i.e., let

$$x(z) = \frac{P(z)}{Q(z)}$$

with $P(z) = \sum_{j=0}^{m} P_j z^j$, $Q(z) = \sum_{j=0}^{n} q_j z^j$, and $m > n$. Then, the result of the division might be

$$x(z) = \sum_{j=0}^{m-n} a_j z^j + \frac{\displaystyle\sum_{j=0}^{n-1} b_j z^j}{\displaystyle\sum_{j=0}^{n} q_j z^j} .$$

Here another inversion method, the method of partial fractions, is useful.

3. Partial Fractions

This method relies on the fact that when $|\lambda| < 1$,

$$\frac{1}{1-\lambda z} = \sum_{j=0}^{\infty} \lambda^j z^j$$

converges, and when $|\lambda| > 1$,

$$\frac{1}{1-\lambda z} = -(1/\lambda z) \sum_{j=0}^{\infty} \lambda^{-j} z^{-j}$$

is convergent. Consider the transform

$$x(z) = (1-\tfrac{1}{2}z)^{-1}(1-\tfrac{1}{3}z)^{-1}.$$

Write x(z) in the partial fractions expansion

$$x(z) = \frac{A}{1-\frac{1}{2}z} + \frac{B}{1-\frac{1}{3}z}$$

or

$$x(z) = \frac{2A}{2-z} + \frac{3B}{3-z}.$$

Then

$$(2-z)x(z) = 2A + \frac{2-z}{3-z}(3B).$$

So that

$$(2-z)x(z)\Big|_{z=2} = 2A.$$

Similarly,

$$(3-z)x(z)\Big|_{z=3} = 3B.$$

Hence,

$$2A = \frac{2(2-z)}{(2-z)(1-\frac{1}{3}z)}\Big|_{z=2} = \frac{2}{1-\frac{2}{3}},$$

$$A = 3.$$

Similarly,

$$3B = \frac{3(3-z)}{(3-z)(1-\frac{1}{2}z)}\Big|_{z=3} = \frac{3}{1-\frac{3}{2}}$$

$$B = -2.$$

Therefore,

$$x(z) = \frac{3}{1-\frac{1}{2}z} - \frac{2}{1-\frac{1}{3}z}.$$

From above,

$$x(z) = 3\left(\sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j z^j\right) - 2\left(\sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j z^j\right)$$

$$= \sum_{j=0}^{\infty} \left(3\left(\frac{1}{2}\right)^j - 2\left(\frac{1}{3}\right)^j\right) z^j.$$

Hence,

$$x_k = 3\left(\frac{1}{2}\right)^k - 2\left(\frac{1}{3}\right)^k \qquad k \geq 0.$$

In most cases, the method of partial fractions is the easiest to apply.

4.   The Inversion Integral

As noted above, the sequence $\{x_k\}$ can be recovered from its transform according to

$$x_k = \frac{1}{2\pi i} \oint_\Gamma x(z) z^{-(k+1)} dz.$$

In the cases considered here, the contour Γ is the unit circle U. Notice that if $z = e^{-iw}$, $x(z)$ is the Fourier transform of $\{x_k\}$, and the inverse Fourier transform

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(e^{-iw}) e^{iwk} dw$$

can be used to recover the sequence $\{x_k\}$. This is computationally convenient, since very rapid FORTRAN Fourier transform routines exist. In some cases, though, the inversion integral can be used directly. Take the example above,

$$x(z) = \frac{1 - \frac{26}{24}z + \frac{9}{24}z^2 - \frac{1}{24}z^3}{1 - \frac{1}{4}z}.$$

From the inversion formula,

$$x_0 = \frac{1}{2\pi i} \oint_U x(z) z^{-1} dz.$$

Notice $x(z)z^{-1}$ has one isolated singularity, $z = 0$, inside U. The residue of $x(z)z^{-1}$ at $z = 0$ is

$$\lim_{z \to 0} z(x(z)z^{-1}) = x(0) = 1.$$

Hence, by the residue theorem, $x_0 = 1$. The next term is

$$x_1 = \frac{1}{2\pi i}\S_U x(z)z^{-2}dz.$$

Here the integrand has a second-order pole at 0. The residue of the integrand at $z = 0$ is $x'(0)$. Now

$$x'(z) = \frac{(1-\frac{1}{4}z)(-\frac{26}{24}+\frac{18}{24}z-\frac{3}{24}z^2) - (1-\frac{26}{24}z+\frac{9}{24}z^2-\frac{1}{24}z^3)(-\frac{1}{4})}{(1-\frac{1}{4}z)^2}$$

so that

$$x'(0) = \frac{(1)(-\frac{26}{24}) + \frac{1}{4}}{(1)^2} = -\frac{20}{24} = -\frac{5}{6}$$

Hence, $x_1 = -\frac{5}{6}$. Similarly, $\frac{x''(0)}{2} = x_2$, and in general, $\frac{x^{(m)}(0)}{m!} = x_m$. This is simply a special case of the following general theorem.

Theorem. If the z-transform $x(z)$ is analytic on the unit disk,

$$\{x_m\} = \{\frac{d^m x(z)}{dz^m} \frac{1}{m!}\} \qquad m > 0$$

$$x_0 = x(0).$$

Though it is mathematically elegant, this result emphasizes the fact that z-transforms can be difficult to invert. In some cases, one need only refer to a table of transform pairs to "invert" the transform in question. One very important member of such a table is the Hansen-Sargent formula (HS1):

$$\{ \sum_{j=0}^{\infty} \lambda^j E_t m_{t+k} \} \underset{\leftrightarrow}{z} (\frac{1-\lambda z^{-1} h(\lambda) h(z)^{-1}}{1-\lambda z^{-1}}) m(z)$$

where $m_t = \sum_{j=0}^{\infty} h_j m_{t-j}$. This formula is used frequently in the solution procedures which can now be discussed.

Consider the difference equation

$$(4.1) \qquad E_t x_{t+1} - (\lambda_1 + \lambda_2) x_t + \lambda_1 \lambda_2 x_{t-1} = y_t$$

where $0 < |\lambda_1| < 1 < |\lambda_2|$ and y_t is a linearily regular covariance stationary stochastic process with moving average representation

$$y_t = \sum_{k=0}^{\infty} \beta_k \epsilon_{t-k} = \beta(L) \epsilon_t$$

and autoregressive representation

$$\delta(L) y_t = \epsilon_t, \quad \epsilon_t = y_t - E(y_t | y_{t-1}, y_{t-2}, \ldots).$$

The information set implicit in (4.1) consists of current and past values of y_t, or, equivalently, current and past values of ϵ_t.

Since $\{y_t\}$ is the only driving process, guess at a solution of the form

$$x_t = \gamma \epsilon_t + \sum_{i=0}^{\infty} f_i \epsilon_{t-i-1} = (\gamma + Lf(L)) \epsilon_t = g(L) \epsilon_t$$

$$= (\gamma + Lf(L)) \delta(L) y_t.$$

As it turns out, γ is the single undetermined coefficient. For purposes of solution, $\{x_t\}$ may be thought of as one-sided: $x_k = 0$, $k > t$, and $\{E_k x_{k+1}\}$ is clearly one-sided. Make the identification

$$\sum_{k=0}^{\infty} x_k z^k = x(z), \quad \sum_{k=0}^{\infty} \epsilon_k z^k = \epsilon(z)$$

and note that

$$x(z) = (\gamma+zf(z))\epsilon(z).$$

Write

$$E_t x_{t+1} = \sum_{j=0}^{\infty} \alpha_j \epsilon_{t-j} = \alpha(L)\epsilon_{t-j}.$$

By the famous Wiener-Kolmogorov formula,[46]

$$\alpha(z) = \left[\frac{\alpha+zf(z)}{z}\right]_+$$

where the annihilation operator $[\ \ ]_+$ means "ignore negative powers of z."
But

$$\left[\frac{\gamma+zf(z)}{z}\right]_+ = \left[\gamma z^{-1}+f(z)\right]_+ = f(z).$$

Then, using properties a - d above, the z-transform of (4.1) can be written

$$(4.1') \qquad f(z)\epsilon(z) - (\lambda_1+\lambda_2)(\gamma+zf(z))\epsilon(z) + \lambda_1\lambda_2 z(\gamma+zf(z))\epsilon(z) = \beta(z)\epsilon(z).$$

(4.1') must hold for all realizations of $\{\epsilon_t\}$, so

$$f(z) - (\lambda_1+\lambda_2)(\gamma+zf(z)) + \lambda_1\lambda_2 z(\gamma+zf(z)) = \beta(z).$$

Collecting terms, write

$$f(z)[(1-\lambda_1 z)(1-\lambda_2 z)] = \beta(z) + (\lambda_1+\lambda_2)\gamma-\lambda_1\lambda_2 z\gamma$$

or

$$(4.2) \qquad f(z) = \frac{\beta(z) + z^{-1}[-(1-\lambda_1 z)(1-\lambda_2 z)+1]\gamma}{(1-\lambda_1 z)(1-\lambda_2 z)}.$$

$f(z)$, being the z-transform of a one-sided, square-summable sequence, must be analytic on the unit disk by property e. But $f(\cdot)$ has an isolated singularity[47] at λ_2^{-1} which is inside the unit circle. Fortunately, the

free parameter γ can be chosen to make λ_2^{-1} a removable singularity, or, equivalently, to make the residue of f at λ_2^{-1} equal to zero. Hence, the proper choice of γ guarantees the required analyticity of f. This can be demonstrated as follows.

From (4.2), the residue of f at λ_2^{-1} is

$$\lim_{z \to \lambda_2^{-1}} (1-\lambda_2 z) f(z) = \frac{\beta(\lambda_2^{-1}) + \lambda_2 [-(1-\lambda_1 \lambda_2^{-1})(1-1)+1]\gamma}{(1-\lambda_1 \lambda_2^{-1})}$$

$$= \frac{\beta(\lambda_2^{-1}) + \lambda_2 \gamma}{1-\lambda_1 \lambda_2^{-1}}.$$

For this quantity to equal zero, $\gamma = -\lambda_2^{-1}\beta(\lambda_2^{-1})$. Then,

$$g(z) =$$

$$\frac{-\lambda_2^{-1}\beta(\lambda_2^{-1})(1-\lambda_1 z)(1-\lambda_2 z) + z\beta(z) - \lambda_2^{-1}\beta(\lambda_2^{-1})(-(1-\lambda_1 z)(1-\lambda_2 z)+1)}{(1-\lambda_1 z)(1-\lambda_2 z)}$$

$$(4.3) \qquad = \frac{z\beta(z) - \lambda_2^{-1}\beta(\lambda_2^{-1})}{(1-\lambda_1 z)(1-\lambda_2 z)}$$

Hence,

$$(1-\lambda_1 z)g(z) = \frac{\beta(z) - \lambda_2^{-1}z^{-1}\beta(\lambda_2^{-1})}{z^{-1} - \lambda_2}$$

$$= \frac{-1}{\lambda_2} \left(\frac{\beta(z)-\lambda_2^{-1}z^{-1}\beta(\lambda_2^{-1})}{1 - \lambda_2^{-1}z^{-1}} \right)$$

Using property c, this can be written

$$x_t = \lambda_1 x_{t-1} - \frac{1}{\lambda_2}\left(\frac{\beta(L)-\lambda_2^{-1}L^{-1}\beta(\lambda_2^{-1})}{1-\lambda_2^{-1}L^{-1}} \right)\varepsilon_t$$

or

$$(4.4) \qquad x_t = \lambda_1 x_{t-1} - \lambda_2^{-1}\left(\frac{1-\lambda_2^{-1}L^{-1}\beta(\lambda_2^{-1})\delta(L)}{1-\lambda_2^{-1}L^{-1}} \right)y_t.$$

According to Hansen and Sargent (1980) (hereafter HS2),when

$$\delta(L) = \delta_0 + \delta_1 L + \delta_2 L^2 + \ldots + \delta_n L^n,$$

(4.4) can be written as

$$(4.5) \qquad x_t = \lambda_1 x_{t-1} - \lambda_2^{-1} \delta(\lambda_2^{-1})^{-1} [1 + \sum_{m=1}^{n-1} \{ \sum_{k=m+1}^{n} \lambda_2^{k-m} \delta_k \} L^m] y_t$$

giving x_t in terms of its own past and current and past values of y_t.

The method used to obtain (4.4) is essentially a "frequency domain" version of the method first proposed by Muth (1960) and later used by Taylor (1977), but is most closely related to the methods of Saracoglu and Sargent (1978) and Saracoglu (1977). Futia (1979a,1979b) has developed and used these methods to slightly different ends. This frequency domain undetermined coefficients method has three virtues. First, only one coefficient must be determined, regardless of the complexity of the moving average representation for the driving process. This is in contrast to the "time domain" methods used by Lucas and Fischer which, in general, require the determination of infinitely many coefficients. Second, for many purposes, the transition from the easily obtained (4.3) to the more complex (4.5) need not be made. For instance, all of the interesting frequency domain behavior of $\{x_t\}$ can be determined by using $z = e^{-iw}$ and viewing (4.3) as a function on $[-\pi,\pi]$. Time domain methods, however, are constrained to find (4.5) first, obtaining (4.3) as a transform. Clearly, (4.3) is easier to obtain than (4.5). Third, the above method sidesteps a difficult certainty-equivalence[48/] issue; (4.1) can be thought of as an Euler equation for a stochastic linear quadratic dynamic optimum problem[49/] where x_t is a decision variable and y_t a state variable. Properties of the objective function allow the researcher to ignore the expectation in

(4.1), treat $\{y_t\}$ as known, and solve the deterministic (certainty-equivalent) version of (4.3) to obtain

$$x_t = \lambda_1 x_{t-1} + \sum_{k=0}^{\infty} \lambda_2^{-k} y_{t+k}$$

At this stage, expectations are taken to produce

$$x_t = \lambda_1 x_{t-1} + \sum_{k=0}^{\infty} \lambda_2^{-k} E_t y_{t+k}.$$

Successive applications of the two formulas of Hansen and Sargent yield (4.4) and (4.5). This procedure is easy to apply, although it is somewhat cumbersome. But it is often the case that expressions like (4.1) are postulated directly without the benefit of a parent optimum problem. Clearly, solution of (4.1) does not require the certainty-equivalence property, but attempts to use it on slightly more complicated problems may be unsuccessful. For instance, a straight-forward application of certainty-equivalent procedures to the system formed by (3.3) and (3.4) gave a correct answer for the deterministic problem, but the decisions as to how to incorporate expectations were very difficult.[50/]

Presented below are two additional examples of the ideas sketched above. These examples are taken from Sargent (1979). Comparisons of his manipulations to those to appear below indicate that one can indeed trade ingenuity for algebra.

Consider Cagan's (1956) portfolio balance equation

$$(4.6) \qquad m_t - p_t = \alpha(E_t p_{t+1} - p_t), \qquad \alpha < 0$$

where m and p denote the logarithms of the money stock and the price level. According to the above procedure, assume $\{m_t\}$ possesses autoregressive and moving average representations

$$\delta(L)m_t = \epsilon_t, \quad m_t = \beta(L)\epsilon_t, \quad \epsilon_t = m_t - E(m_t|m_{t-1},\ldots).$$

Postulate

$$p_t = g(L)\epsilon_t = (\gamma + Lf(L))\epsilon_t, \quad f(L) = \sum_{i=0}^{\infty} f_i L^i.$$

From the z-transform of both sides of (4.6), there arises the equality

$$\beta(z) - (\gamma + zf(z)) = (f(z) - (\gamma + zf(z))).$$

Then

$$f(z)(-\alpha - z + \alpha z) = -\beta(z) + \gamma - \alpha\gamma$$

or

$$f(z)(\alpha + (1-\alpha)z) = \beta(z) - \gamma(1-\alpha),$$

whence

$$f(z) = \frac{1}{\alpha} \frac{\beta(z) - \gamma(1-\alpha)}{1 - \frac{\alpha-1}{\alpha}z}.$$

Now $\alpha < 0$ implies $(\alpha-1)/\alpha > 1$. Hence, $f(\cdot)$ has an isolated singularity at $\alpha/(\alpha-1) < 1$. The residue there is

$$\lim_{z \to \frac{\alpha}{\alpha-1}} (1 - \frac{\alpha-1}{\alpha}z)f(z)$$

$$= \beta(\frac{\alpha}{\alpha-1}) - \gamma(1-\alpha).$$

This will be zero when $\gamma = (1-\alpha)^{-1}\beta(\frac{\alpha}{\alpha-1})$. $f(\cdot)$ is (as required) analytic on the unit disk for this value of γ. Then

$$g(z) = \frac{\frac{-\alpha}{(\alpha-1)}\beta(\frac{\alpha}{\alpha-1})(1 - \frac{\alpha-1}{\alpha}z) + z\beta(z) - \frac{(\alpha-1)z\beta(\frac{\alpha}{\alpha-1})}{(\alpha-1)}}{(1 - \frac{\alpha-1}{\alpha}z)}$$

$$= \frac{\dfrac{-\beta(\frac{\alpha}{\alpha-1})(\alpha-(\alpha-1)z)}{(\alpha-1)} - \dfrac{(\alpha-1)z\beta(\frac{\alpha}{\alpha-1})}{(\alpha-1)} + z\beta(z)}{\alpha(1-\frac{\alpha-1}{\alpha}z)}$$

$$= \frac{\dfrac{-\alpha}{\alpha-1}\beta(\frac{\alpha}{\alpha-1}) + z\beta(z)}{\alpha(1-\frac{\alpha-1}{\alpha}z)}$$

$$= \frac{\beta(z) - \dfrac{\alpha}{\alpha-1}z^{-1}\beta(\frac{\alpha}{\alpha-1})}{\alpha z^{-1} - (\alpha-1)}$$

$$= \frac{1}{1-\alpha}[\frac{\beta(z) - \dfrac{\alpha}{\alpha-1}z^{-1}\beta(\frac{\alpha}{\alpha-1})}{1 - \dfrac{\alpha}{\alpha-1}z^{-1}}] .$$

Hence,

$$p_t = \frac{1}{1-\alpha}[\frac{1 - \dfrac{\alpha}{\alpha-1}L^{-1}\beta(\frac{\alpha}{\alpha-1})\beta(L)^{-1}}{1 - \dfrac{\alpha}{\alpha-1}L^{-1}}]m_t$$

which is the result of applying the Hansen-Sargent formula, HS1, to

$$(4.7) \qquad p_t = \frac{1}{1-\alpha}\sum_{k=0}^{\infty}(\frac{\alpha}{\alpha-1})^k E_t m_{t+k},$$

the expression obtained by Sargent (1979, p. 269).

The final example is interesting because two distinct information sets appear. The model is a modification of one used by Muth (1961). P_t is the price of a commodity at t, C_t the demand for consumption, I_t the inventory of the commodity, Y_t the commodity's output, and x_t is a linearily regular covariance stationary stochastic process with moving average representation

$$x_t = d(L)\epsilon_t$$

and autoregressive representation

$$d(L)^{-1}x_t = \epsilon_t.$$

x_t might represent the effect of weather on supply. The model is

$$C_t = \beta P_t \qquad\qquad \beta > 0 \qquad\qquad \text{(demand curve)}$$

$$Y_t = \gamma E_{t-1} P_t + x_t \quad \gamma > 0 \qquad\qquad \text{(supply curve)}$$

$$I_t = \alpha(E_t P_{t+1} - P_t) \quad \alpha > 0 \qquad\qquad \text{(inventory demand)}$$

$$Y_t = C_t + (I_t - I_{t-1}). \qquad\qquad \text{(market clearing)}$$

Substitution of the first three equations into the fourth yields

$$(\gamma+\alpha)E_{t-1}P_t + (\alpha+\beta)P_t - \alpha E_t P_{t+1} - \alpha P_{t-1} = -x_t.$$

Assume P_t can be written as

$$P_t = g(L)\epsilon_t = (r+Lf(L))\epsilon_t, \quad f(L) = \sum_{k=0}^{\infty} f_i L^i.$$

The z-transform of the above equation must then satisfy

$$(\gamma+\alpha)zf(z) + (\alpha+\beta)(r+zf(z)) - \alpha f(z) - \alpha z(r+zf(z)) = -d(z).$$

A little manipulation yields

$$f(z) = \frac{d(z) + (\alpha+\beta)r - \alpha rz}{\alpha(1-\lambda_1 z)(1-\lambda_2 z)}$$

where λ_1 and λ_2 are the roots of $1 - \dfrac{(2\alpha+\beta+\gamma)}{\alpha}y + y^2 = 0$. λ_1 and λ_2 satisfy

$$0 < \lambda_1 < 1 < \lambda_2$$

$$\lambda_1 + \lambda_2 = (2\alpha+\beta+\gamma)$$

$$\lambda_1\lambda_2 = 1.$$

For $f(\cdot)$ to be analytic inside the unit circle, its residue at $z = \lambda_2^{-1}$ must

be zero. Hence

$$\text{Res}(f(\cdot),\lambda_2^{-1}) = \lim_{z\to\lambda_2^{-1}}(1-\lambda_2 z)f(z)$$

$$= d(\lambda_2^{-1}) + (\alpha+\beta)r - \alpha r\lambda_2^{-1}$$

must equal zero. This occurs when

$$r = \frac{d(\lambda_1)}{\alpha\lambda_1 - (\alpha+\beta)}.$$

Then

$$g(z) = r + zf(z)$$

$$= \frac{\dfrac{(\alpha-(\alpha+\gamma)z)}{\alpha\lambda_1 - (\alpha+\beta)}d(\lambda_1) + zd(z)}{(1-\lambda_1 z)(1-\lambda_2 z)}$$

For most purposes, this $g(z)$ is an appropriate form for the solution. It can be shown that $g(z)$ emerges after application of HS1 to

$$P_t = \lambda_1 P_{t-1} + \frac{1}{\alpha+\beta-\alpha\lambda_1} [(\gamma+\alpha)\frac{\lambda_1}{\alpha}\sum_{k=0}^{\infty}\lambda^k E_{t-1}x_{t+k} - \sum_{k=0}^{\infty}\lambda^k E_t M_{t+k}]$$

which is the correct solution for the commodity price. It is _not_ the "solution" obtained by Sargent; though it was not intended to be, his solution is essentially the inverse transform of $f(z)$, not of $g(z)$. This is suggestive: certainty-equivalent procedures sometimes fail. While it is the case that these procedures combined with extensive checks will eventually lead to the correct solution, the method presented above leads directly to the frequency domain representation of the desired result.

With the method of frequency domain undetermined coefficients firmly in hand, the system (3.3), (3.4), and (3.5) can be easily solved. In the following development, an expression like (4.3) will be called a "solution." The relation between z-transforms and Fourier transforms will then be exploited to examine the behavior of the system.

5. Frequency Domain Solution of the Model

For the present purpose, the model consists of (3.2), (3.3), (3.4), and (3.5'). Under the assumption that all variables are measured in deviations from their means, the four equations can be written

(3.2) $r_t = -\delta_1 k_t$

(3.3) $k_{t+1} = \alpha_1 E_t r_{t+1} + \alpha_2 (E_t p_{t+1} - p_t) + \alpha_3 k_t$ $\alpha_1 > \alpha_2 \geq 0$, $\alpha_3 \in (0,1)$

(3.4) $m_t - p_t = -\beta_1 E_t r_{t+1} - \beta_2 (E_t p_{t+1} - p_t) + \beta_3 k_t$ $\beta_2 > \beta_1 > 0$, $\beta_3 \in (0,1)$.

(3.5') $m_t = \sum_{k=0}^{\infty} h_k \varepsilon_{t-k}$, $\varepsilon_t = m_t - E(m_t | m_{t-1}, m_{t-2}, \dots)$, $h_0 = 1$.

To obtain a more convenient representation, use (3.2) to find

$$E_t r_{t+1} = -\delta_1 E_t k_{t+1}.$$

But since k_{t+1} is known at t,

$$E_t r_{t+1} = -\delta_1 k_{t+1}.$$

Substituting this into (3.3) and (3.4) yields

$$k_{t+1} = -\alpha_1 \delta_1 k_{t+1} + \alpha_2 (E_t p_{t+1} - p_t) + \alpha_3 k_t$$

$$m_t - p_t = \beta_1 \delta_1 k_{t+1} - \beta_2 (E_t p_{t+1} - p_t) + \alpha_3 k_t,$$

which can be rearranged to read

(5.1) $\begin{bmatrix} (1+\alpha_1 \delta_1) & -\alpha_2 \\ -\beta_1 \delta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ E_t p_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_3 & -\alpha_2 \\ \beta_3 & (1+\beta_2) \end{bmatrix} \begin{bmatrix} k_t \\ p_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ m_t \end{bmatrix}.$

According to the analysis in Section 4, assume

$$k_{t+1} = \sum_{k=0}^{\infty} l_k \varepsilon_{t-k} = l(L)\varepsilon_t$$

(5.2)

$$p_t = \sum_{k=0}^{\infty} g_k \varepsilon_{t-k} = g(L)\varepsilon_t,$$

where $g(L) = \gamma + Lf(L)$ with $f(L) = \sum_{k=0}^{\infty} f_k L^k$. γ is the single coefficient to be determined. Now take z-transforms of both sides of (5.1) to obtain the condition

$$\begin{bmatrix} 1+\alpha_1\delta_1 & -\alpha_2 \\ -\beta_1\delta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} l(z) \\ f(z) \end{bmatrix} = \begin{bmatrix} \alpha_3 & -\alpha_2 \\ \beta_3 & (1+\beta_2) \end{bmatrix} \begin{bmatrix} zl(z) \\ \gamma+zf(z) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ h(z) \end{bmatrix}$$

which can be written

$$A(z) \begin{bmatrix} l(z) \\ f(z) \end{bmatrix} = \begin{bmatrix} -\alpha_2\gamma \\ (1+\beta_2)\gamma-h(z) \end{bmatrix}$$

with

$$A(z) = \begin{bmatrix} 1+\alpha_1\delta_1-\alpha_3z & -\alpha_2+\alpha_2z \\ -\beta_1\delta_1-\beta_3z & \beta_2-(1+\beta_2)z \end{bmatrix}.$$

Then invert $A(z)$ to obtain

(5.3)
$$\begin{bmatrix} l(z) \\ f(z) \end{bmatrix} = \frac{1}{\det A(z)} \begin{bmatrix} \beta_2-(1+\beta_2)z & \beta_1\delta_1+\beta_3z \\ \alpha_2-\alpha_2z & 1+\alpha_1\delta_1-\alpha_3z \end{bmatrix} \begin{bmatrix} -\alpha_2\gamma \\ (1+\beta_2)\gamma-h(z) \end{bmatrix}.$$

It is shown in Appendix A that $\det A(z) = \Delta(1-\lambda_1 z)(1-\lambda_2 z)$ where $0 < \lambda_1 < 1 < \lambda_2$ and $\Delta > 0$. Hence, (5.2) can be written

(5.4)
$$\begin{bmatrix} l(z) \\ f(z) \end{bmatrix} = \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)} \left\{ \begin{bmatrix} \beta_2 & \alpha_2 \\ \beta_1\delta_1 & 1+\alpha_1\delta_1 \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} -(1+\beta_2) & \beta_3 \\ -\alpha_2 & -\alpha_3 \end{bmatrix} z \right\} \left\{ \begin{matrix} -\alpha_2\gamma \\ (1+\beta_2)\gamma-h(z) \end{matrix} \right\}.$$

$l(z)$ and $f(z)$ are z-transforms of one-sided square summable sequences and, thus, are analytic on the unit disk. But clearly, (5.4) has an isolated singularity inside the unit circle at $z = \lambda_2^{-1}$. γ is chosen to make this singularity removable, i.e., γ is chosen to set the residues of $f(\cdot)$ and $l(\cdot)$ at λ_2^{-1} to zero:

$$\lim_{z \to \lambda_2^{-1}} (1-\lambda_2 z) \begin{bmatrix} l(z) \\ f(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Appendix B gives this value of γ[51/] as $(1-\lambda_2^{-1})h(\lambda_2^{-1})$. The frequency domain solution[52/] of the system (1.1)-(1.5) is obtained by substituting $(1-\lambda_2^{-1})h(\lambda_2^{-1})$ for γ in (3.4):

$$(5.5) \quad \begin{bmatrix} l(z) \\ f(z) \end{bmatrix} = \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)} \left\{ \begin{bmatrix} \beta_2 & \alpha_2 \\ \beta_1\delta_1 & 1+\alpha_1\delta_1 \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} -(1+\beta_2) & \beta_3 \\ -\alpha_2 & -\alpha_3 \end{bmatrix} z \right\} \left\{ \begin{array}{l} -\alpha_2(1-\lambda_2^{-1})h(\lambda_2^{-1}) \\ (1+\beta_2)(1-\lambda_2^{-1})h(\lambda_2^{-1})-h(z) \end{array} \right\} .$$

In Appendix C, formulas analogous to Fischer's (3.9) are derived. The resultant expressions are:

$$(5.6) \quad k_{t+1} = \lambda_1 k_t + \frac{\alpha_2}{\Delta\lambda_2} \sum_{k=0}^{\infty} \lambda_2^{-k} E_t (1-L)m_{t+1+k}$$

$$(5.7) \quad p_t = \lambda_1 p_{t-1} + \frac{1}{\Delta\lambda_2(1-\lambda_2)} \left\{ (1-\lambda_1 L) \sum_{k=0}^{\infty} \lambda_2^{-k} E_t (1+\alpha_1\delta_1-\alpha_3 L)m_{t+1+k} \right.$$

$$\left. - (1+\alpha_1\delta_1-\alpha_3 L) \sum_{k=0}^{\infty} \lambda_2^{-k} E_t (1-L)m_{t+1+k} \right\} .$$

Operational time domain solutions containing only current and past values of the money stock are derived in Appendix D. The resultant expressions are:

(5.8) $k_{t+1} = \lambda_1 k_t +$

$$\frac{\alpha_2}{\Delta\lambda_2}\left\{\lambda_2 + (1-\lambda_2)h(\lambda_2^{-1}) + (1-\lambda_2)h(\lambda_2^{-1})\sum_{m=1}^{n-1}\left[\sum_{k=m+1}^{n}\lambda_2^{m-k}H_k\right]L^m\right\}m_t.$$

(5.9) $p_t = \lambda_1 p_{t-1} + \dfrac{(1+\alpha_1\delta_1-\alpha_3)}{\Delta\lambda_2(1-\lambda_2)}h(\lambda_2^{-1})m_t$

$$+ \frac{1}{\Delta\lambda_2(1-\lambda_2)}\left\{(1+\alpha_1\delta_1-\alpha_3)h(\lambda_2^{-1})(\sum_{k=2}^{n}\lambda_2^{1-k}H_k - a) - \right.$$

$$\left.\lambda_2(\alpha_3-\lambda_1(1+\alpha_1\delta_1))\right\}m_{t-1}$$

$$+ \frac{h(\lambda_2^{-1})}{\Delta\lambda_2(1-\lambda_2)}\left\{\sum_{m=0}^{n-3}\left[(1+\alpha_1\delta_1-\alpha_3)\left(\sum_{k=m+3}^{n}\lambda_2^{m-k}H_k - \right.\right.\right.$$

$$\left.\left.\left. a\sum_{k=m+2}^{n}\lambda_2^{m-k}H_k\right)\right]L^m\right\}m_{t-2}$$

$$- \frac{(1+\alpha_1\delta_1-\alpha_3)a}{\Delta\lambda_2(1-\lambda_2)}h(\lambda_2^{-1})H_n m_{t-n}.$$

It is a general characteristic of systems like these that the autoregressive parameters in (5.8) and (5.9) are the same. This point is emphasized in Zellner and Palm (1974).

In the absence of a Tobin effect ($\alpha_2 = 0$), the model simplifies greatly. From (5.6),

$$k_{t+1} = \lambda_1 k_t.$$

Since $|\lambda_1| < 1$, the capital stock approaches its mean value, zero, and innovations in the money stock have no real effects. In Appendix E it is shown that (5.7) simplifies to

(5.9') $p_t = \dfrac{1}{1+\beta_2}\displaystyle\sum_{k=0}^{\infty}(\dfrac{\beta_2}{1+\beta_2})^k E_t m_{t+k},$

which is the same as (4.7), the "solution" for Cagan's portfolio balance equation. That the behavior of k_{t+1} and p_t becomes so simple should not be surprising since $\alpha_2 = 0$ destroys the link between (3.3) and (3.4).

The investigation of the system's behavior when $\alpha_2 \neq 0$ is more involved. However, (5.5) makes easy the computation of numerous illuminating quantities such as spectra, gains, phases, and moving average and distributed lag coefficients.

The __spectrum__ of a process is an orthogonal decomposition of its variance by frequency. The spectrum of the money stock is, from (3.5'), proportional[53] to $|h(e^{-iw})|^2$; i.e.,

$$(5.10a) \quad S_m(e^{-iw}) = \sigma_\epsilon^2 |h(e^{-iw})|^2.$$

Accordingly,

$$(5.10b) \quad S_k(e^{-iw}) = \sigma_\epsilon^2 |1(e^{-iw})|^2$$

$$(5.10c) \quad S_p(e^{iw}) = \sigma_\epsilon^2 |g(e^{-iw})|^2.$$

The variances of these series can be obtained via a well-known property of spectra:

$$var(m) = \sigma_\epsilon^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(e^{-iw})|^2 dw$$

$$var(k) = \sigma_\epsilon^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |1(e^{-iw})|^2 dw$$

$$var(p) = \sigma_\epsilon^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(e^{-iw})|^2 dw.$$

From Section 2,

$$frequency \times period = 2\pi.$$

Hence, (5.10) can be used to obtain the variances corresponding to "short-," "medium-," and "long-" runs.

The <u>gain</u> of one process over another measures how the amplitude in the latter contributes to the amplitude in the former by frequency. This quantity is the modulus of the cross spectrum. In the current case, it is given by

$$(5.11a) \quad G_{km}(e^{-iw}) = \sigma_\epsilon^2 |1(e^{-iw})h(e^{iw})|$$

and

$$(5.11b) \quad G_{pm}(e^{-iw}) = \sigma_\epsilon^2 |g(e^{-iw})h(e^{iw})|.$$

The <u>phase lead</u> of one process over another is a frequency-by-frequency account of the number of radians by which cycles in the first series lead those in the second series. This quantity, the principal argument of the cross spectrum, is given here by

$$(5.12a) \quad Ph_{km}(e^{-iw}) = \tan^{-1}(\frac{Im(1(e^{-iw})h(e^{iw}))}{Re(1(e^{-iw})h(e^{iw}))}).$$

$$(5.12b) \quad Ph_{mp}(e^{-iw}) = \tan^{-1}(\frac{Im(g(e^{-iw})h(e^{iw}))}{Re(g(e^{-iw})h(e^{iw}))}).$$

Since $Ph_{\cdot m}(e^{-iw}) = Ph_{\cdot m}(e^{-i(w+2\pi)})$, (5.12) can only indicate whether the series are "in-phase" or "out-of-phase."

A quantity closely related to the gain, the <u>transfer modulus</u>, is given by

$$(5.13a) \quad T_{km}(e^{-iw}) = |1(e^{-iw})/h(e^{-iw})|$$

$$(5.13b) \quad T_{pm}(e^{-iw}) = |g(e^{-iw})/h(e^{-iw})|.$$

(5.13a) and (5.13b) are the moduli of the Fourier transforms of the distributed lag coefficients of capital and prices on money. The sum of lag coefficients from these regressions is obtained by evaluating $T_{\cdot m}(e^{-iw})$ at $w = 0$.

To obtain the distributed lag coefficients themselves, simply inverse Fourier transform the ratios used to compute (5.13). To fix these notions, consider the distributed lag regressions

$$(5.14a) \quad k_{t+1} = \sum_{k=-\infty}^{\infty} \psi_k m_{t-k} + \eta_t \qquad E\eta_t m_{t-s} = 0 \; \forall \; s$$

$$(5.14b) \quad p_t = \sum_{k=0}^{\infty} \xi_k m_{t-k} + \nu_t \qquad E\nu_t m_{t-s} = 0 \; \forall \; s.$$

Then

$$(5.15a) \quad \psi_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{l(e^{-iw})}{h(e^{-iw})} e^{iwk} dw$$

$$(5.15b) \quad \xi_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(e^{-iw})}{h(e^{-iw})} e^{iwk} dw.$$

Because $h_k = l_k = g_k = 0 \; \forall \; k < 0$, $\psi_k = 0 = \xi_k \; \forall \; k < 0$, so that the regressions (5.14) are <u>one-sided</u> in current and past values of the money stock. This is equivalent with the statement that k_{t+1} and p_t fail to <u>Granger-cause</u> m_t.

The moving average coefficients l_i and g_i are obtained by inverse Fourier transforming (5.5);[54/] i.e.,

$$l_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} l(e^{-iw}) e^{iwk} dw$$

$$g_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{-iw}) e^{iwk} dw.$$

These coefficients measure the responses of k_{t+1} and p_t to unit to unit impulses in m_t. To take an example, assume $\{g_k\}$ turns out to be

$(1,.9,.81,\ldots)$.

A sequence $\{\varepsilon_t\}$ displaying a unit impulse in m_t is

$(0,\ldots,0,1,0,0,0,\ldots)$.

Then from (5.2),

$$p_t = \sum_{k=0}^{\infty} g_k \varepsilon_{t-k}$$

the response in $\{p_t\}$ can be found:

$$p_t = g_0 \varepsilon_t + g_1 \varepsilon_{t-1} + \cdots = g_0 = 1$$

$$p_{t+1} = g_0 \varepsilon_{t+1} + g_1 \varepsilon_t + \cdots = g_1 = .9,$$

etc. Of course, the inversion techniques described in the previous section could have been used to recover those parameters; the Fourier transform technique is simply easier to apply.

6. Examples of System Behavior

Examples of (5.10)-(5.15) are given in Figures 6.1-6.5. Five different money supply processes generated these figures. The processes were:

FIGURE 6.1: $m_t = \varepsilon_t$

FIGURE 6.2: $m_t = (1-.5L)^{-1}\varepsilon_t$

FIGURE 6.3: $m_t = (1-.9L)^{-1}\varepsilon_t$

FIGURE 6.4: $m_t = (1-.9999L)^{-1}(1-.5L)\varepsilon_t$

FIGURE 6.5: $m_t = (1-.9999L)^{-1}(1-.9L)\varepsilon_t$

where $\{\varepsilon_t\}$ has variance equal to one, and, in each case, is fundamental for $\{m_t\}$. Hence, Figure 6.1 displays results for a white noise money supply, Figures 6.2 and 6.3 for an autoregressive money process, and Figures 6.4 and 6.5 for a "transient-permanent" money process. The processes used in Figures 6.4 and 6.5 are special cases of a process used by Muth (1960)[55/]. To show that this process exhibits transient and permanent behavior, consider the process

$$m_t = \frac{1-\lambda L}{1-\rho L}\varepsilon_t$$

with $0 < \lambda, \rho < 1$. This can be written

$$m_t = (1-\lambda L)(1+\rho L+\rho L^2+\ldots)\varepsilon_t$$

$$= (1-\lambda L)(\varepsilon_t+\rho\varepsilon_{t-1}+\ldots)$$

$$= \varepsilon_t + (\rho-\lambda)\varepsilon_{t-1} + (\rho^2-\lambda)\varepsilon_{t-2} + \ldots.$$

Now allow $\rho \rightarrow 1$. Then

$$m_t = \varepsilon_t + (1-\lambda)(\varepsilon_{t-1}+\varepsilon_{t-2}+\ldots).$$

Consider an "innovation sequence," i.e.,

$$\varepsilon_k = \begin{cases} 1 & k = t \\ 0 & k \neq t \end{cases}.$$

Then

$$m_t = \varepsilon_t + (1-\lambda)(\varepsilon_{t-1}+\varepsilon_{t-2}+\ldots) = 1$$

$$m_{t+1} = \varepsilon_{t+1} + (1-\lambda)(\varepsilon_t+\varepsilon_{t-1}+\ldots) = 1 - \lambda$$

$$m_{t+2} = \varepsilon_{t+2} + (1-\lambda)(\varepsilon_{t+1}+\varepsilon_t+\varepsilon_{t-1}+\ldots) = 1 - \lambda$$

$$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \qquad\qquad\qquad\qquad\qquad \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$$

Thus, the innovation ε_t is transmitted directly to m_t, displaying a "transient" increase in m. But the innovation also causes m_{t+1}, m_{t+2}, ..., to increase by $1 - \lambda < 1$, displaying a "permanent" increase in m.

The five figures share two additional common elements. First, recalling that the model can be written

$$(1+\alpha_1\delta_1)k_{t+1} - \alpha_2 E_t p_{t+1} = \alpha_3 k_t - \alpha_2 p_t$$

$$- \beta_1\delta_1 k_{t+1} + \beta_2 E_t p_{t+1} = \beta_3 k_t + (1+\beta_2)p_t - m_t,$$

in all cases $\alpha_1 = 1$, $\alpha_3 = .5$, $\beta_1 = .5$, $\beta_2 = 1$, $\beta_3 = .5$, and $\delta_1 = 1$. Second, the Mundell-Tobin effect parameter α_2 was varied between 0 and 1. The values were

$\alpha_2 = 0$ (solid line)

$\alpha_2 = 0.25$ (dotted line)

$\alpha_2 = 0.5$ (dash-dot)

$\alpha_2 = 0.75$ (dashed line)

$\alpha_2 = 1.0$ (long dash).

Figures 6.1a and 6.1b display the moving average coefficients and corresponding spectrum (from 5.10) for the white noise money process. The spectrum is flat, indicating that all frequencies contribute equally to the "noise in m."[56/] This is in contrast to the processes generated by the moving average coefficients displayed in Figures 6.2a and 6.3a. Those coefficients die out at the exponential rates .5 and .9, respectively. The corresponding spectra are smooth, with peaks at zero frequency. In contra-distinction, the spectra for the transient-permanent processes presented in Figures 6.4b and 6.5b show what approximates a "spike" at zero frequency. These several money processes generate the price and capital behavior of Panels c - 1 of the five figures.

Panels c and d display the spectra of prices and capital. These spectra are in one sense similar to, and one sense different from, the money spectra. The difference is that in each figure, the Panel d spectrum of capital shows a uniformly smaller amplitude than the Panel b spectrum of money, indicating that, in this system, capital possesses a smaller variance than the money supply. A similar result obtains for the price levels generated by the white noise and autoregressive money supplies, as indicated in Panel c of Figures 6.1, 6.2, and 6.3. However, the price spectra for the transient-permanent processes displayed in the c panel of Figures 6.4 and 6.5 appear to enclose a greater area than the money

spectra: for this class of money processes, the price level variance exceeds that of the money supply.

The similarities of the money and prices-capital spectra are in their shapes. For instance, when $\alpha_2 = 0$, the flat money spectrum of 6.1b leads to a flat price spectrum in 6.1c and an identically zero capital spectrum in 6.1d.[57/] Also, the spectra are all maximal at zero frequency. The spectra of prices and capital do not display peaks at the business cycle frequencies, although the spectra of inflation and investment would. This does not mean that prices and capital would not show interesting cyclical behavior. Sargent (1979) notes that business cycles can be thought of as high pairwise coherence of important aggregates at the business cycle frequencies. The series in this model have this property, but in a trivial way: the absence of "noise" in (3.3) and (3.4) makes money, prices, and capital perfectly coherent not only at the business cycle frequencies, but at all frequencies. In Lucas' (1975) use, the model does generate cycles, but an additional device is used: information is concealed from agents.

Another common interesting feature of these spectra is that their amplitudes increase uniformly in α_2. Because the variance of a process is equal to the area under its spectrum, it is clear that the Mundell-Tobin effect is a variance-inducing one. In addition, the k-step-ahead forecast error variance is proportional to the variance of the process; e.g., for the process (3.5'), the k-step-ahead forecast error variance $\sigma_\epsilon^2(k)$ is given by

$$\sigma_\epsilon^2(k) = \sigma_\epsilon^2 \sum_{j=0}^{k-1} h_j^2.$$

Therefore, the Mundell-Tobin effect decreases the accuracy of capital stock and price level predictions.

The responses of prices and capital to unit innovations ("surprises") in money are displayed in the moving average panels e and i of Figures 6.1-6.5. Notice that in 6.1e these coefficients mimic those of money when $\alpha_2 = 0$. Hence, in the absence of a Mundell-Tobin effect, a shock in money is felt immediately in prices with no persistence. When α_2 is not zero, prices settle back to the preshock level, but only after several periods. Figures 6.2 and 6.3 indicate that a shock from an autoregressive money process persists indefinitely, with prices returning to the preshock level only in the limit. The price behavior of Figures 6.4 and 6.5 is markedly different. There the money innovation causes a contemporaneous jump in the price level, but prices settle to a permanently higher level than that characterizing the preshock period.[58/]

The impulse responses of capital are never positive. Of course, when $\alpha_2 = 0$, the capital stock and the money supply are independent: the moving average coefficients in 6.1i-6.5i are zero. The qualitative behavior of the capital stock in response to a money innovation is similar for the five processes. In each case, the money shock induces an initial drop in capital. Thereafter, capital increases back to its preinnovation level.

The price-capital response patterns of Figures 6.1e, i-6.5e, i are similar to those found by Fischer (1979). The results most comparable to his are displayed in Figures 6.4 and 6.5. The results there, like Fischer's, indicate that a transient-permanent innovation in money produces both transient and permanent price changes, but only transient changes in capital. The virtues of the results of Figures 6.4 and 6.5 over Fischer's are that the figures presented here were generated in a system

where randomness plays a crucial role,[59] and that the transient and permanent changes in the money stock are not artificially separated.

The implications of the system's dynamics for regressions of prices and capital on money are displayed in the distributed lag panels f and j of the five figures. Panel f indicates that the contemporaneous coefficient in a distributed lag of prices on money is the largest, and that the lag coefficients die out rapidly. For the white noise and autoregressive money processes with $\alpha_2 = 0$, this "dieing out" is immediate: nonzero lag coefficients are zero. In Figure 6.1 this occurs because white noise money produces white noise prices, and no lags appear because money is fundamental for prices. Only zero order lags appear in Figures 6.2 and 6.3 because, by HS2, the order of the lags on money in (5.9') is one less than that in the univariate money autoregression. In all figures, distributed lag coefficients beyond the first are nonzero when the Mundell-Tobin effect is nonzero. As is evident in Figures 6.4 and 6.5, even when $\alpha_2 = 0$ and the system dichotomizes, it is possible to obtain nontrivial lag distributions by complicating the money process.

Except when $\alpha_2 = 0$, the distributed lag coefficients of capital on money presented in Panel j are always initially negative. In Figures 6.1-6.3 where no persistence of the money shock exists, these coefficients are never positive. But in Figures 6.4 and 6.5, the first lag coefficient[60] is negative while all the rest are positive, damping smoothly to zero after a peak at lag three.

The moduli of the Fourier transform of the lag distributions are displayed in Panels g and k. As noted in the previous section, the zero frequency value of this quantity gives the absolute value of the sum of the lag coefficients. The figures indicate that prices are always positively

affected by money, and more so the greater the Mundell-Tobin effect for the white noise and autoregressive money processes. But for the transient-permanent money processes of Figures 6.4 and 6.5, the sum of lag coefficients is essentially the same, independent of the Mundell-Tobin effect. In addition, this sum is one: all of the increase in money is reflected in prices. Indeed, the sum of coefficients in the capital regressions is zero for these money processes.

The lead of money over capital and prices by frequency is depicted in the phase plots of Panels h and 1.[61/] The figures indicate that the lead of money over prices is small at all frequencies. As expected, money and prices are perfectly in phase for $\alpha_2 = 0$ in Figures 6.1, 6.2, and 6.3. The interesting feature of Panel 1 is that money and capital are very nearly 180 degrees (3.14159 radians) out of phase at all frequencies for all settings of the Mundell-Tobin effect.

By way of summary, it is useful to delineate the most interesting characteristics of the system under study. First, prices always respond positively to money innovations, capital never does. Second, the variance in capital and prices as well as their responsiveness to money shocks increases with the magnitude of the Mundell-Tobin effect. Third, "permanent" changes in money lead to "permanent" changes in prices but not in the capital stock. Finally, the Mundell-Tobin effect does not influence the sum of lag coefficients in regressions of prices and capital on money when money is generated by the "transient-permanent" process. The sum is essentially one for prices, zero for capital.

7. <u>Implications for Interpreting Lucas' Empirical Results</u>

Some care must be exercised in using the model of Section 3 to address the empirical issues raised in Section 2. The reason is that the objects π_t, μ_t of Section 2 are the first differences of the objects p_t, m_t of Section 3.[62] Hence, to translate the Section 6 spectra of prices and money to Section 2 form, the spectra must be multiplied by the gain of the filter (1-L). This gain is zero at zero frequency, four at frequency π, and is concave on the interval $(0,\pi)$. The modified Section 6 spectra will, therefore, display peaks not at frequency zero but at $w > 0$. This seems to rule out the simple processes of Figures 6.1-6.3: the modified spectra do not resemble those of Section 2.

But there is a similarity in the spectra of the two sections.[63] The Section 2 spectra look as though the underlying processes might be "transient-permanent." For instance, it appears as though

$$\pi_t = \lim_{\rho \to 1} \frac{1-\lambda L}{(1-\rho L)} \epsilon_t,$$

A $\{p_t\}$ process which generates this π_t is

$$p_t = \lim_{\rho \to 1} \frac{(1-\lambda L)}{(1-\rho L)^2} \epsilon_t,$$

whose spectral density is shaped like the ones in Figures 6.4 and 6.5. That is, there is a rough correspondence between the Section 2 spectra and those of Section 6, appropriately modified. Thus, there is some, albeit casual, evidence in Section 2 for the model of Section 3.

In one very fundamental sense, it does not matter that the series in Section 2 are differenced and those of Section 3 are not: differencing has no effect on the transfer modulus (Panels g and k, Figures 6.1-6.5) and, hence, none on the sum of lag coefficients estimated in 2.6. Thus,

according to the results of Section 6, an estimate of the sum of lag coefficients in (2.7) does <u>not</u>, in general, say anything about the magnitude of the Mundell-Tobin effect.

8.   Thoughts on Further Research

Some idea about the appropriateness of the Section 3 model can be obtained by regenerating the Section 6 pictures for a money process estimated from the data.   One way to generate the moving average coefficients for such a money process is to invert an estimated univariate autoregressive representation.   Another method is the direct estimation of moving average coefficients via frequency-domain maximum likelihood procedures.   This is not simple, and will require some work.

One can view the model of Section 3 as embedding a Lucas (1973)-type Phillips curve if one assumes that α_2 is strictly positive.   As in Fischer (1979), this Phillips curve effect arises when the difference between actual and expected prices is "tacked on" to the output equation 3.1.   A more well-founded approach based on, say, a theory of the labor market is somewhat subtle but of interest.

Although the methods of Section 2 provide substantial insight into the long-run relationship between money and prices, the results of Sections 6 and 7 indicate that more information is necessary before Figure 2.4 can be interpreted as evidence of the quantity theory.   In particular, Panel g of Figures 6.4 and 6.5 indicates that in the Lucas model a unit sum of lag coefficients in a prices on money regression is consistent with any setting of the Mundell-Tobin effect when the autoregressive representation for money possesses a unit root.   Thus, Figure 2.4 combined with evidence that money's autoregressive representation has no roots near one would be evidence against the Mundell-Tobin effect.   Unfortunately, such evidence is difficult to obtain.   The problem, discussed by Sims (1974), is that under the least squares metric, a given lag distribution can be approximated arbitrarily well even when one of its roots is fixed a priori.

Though the Sims paper indicates some methods to circumvent this problem, it appears that direct estiamtion of α_2 may be more fruitful. This approach requires the joint estimation of (5.8), (5.9), and (3.5).[64/] But (5.8) and (5.9) are exact relationships; there is no error term. Hence, estimation requires a certain rethinking of (3.3) and (3.4). A theory giving rise to versions of (3.3) and (3.4) and the presence of, and a deep theory for, error terms in (5.8) and (5.9) would be of interest.

Appendix A

The text contains the expression

(A1) $\det A(z) = (1+\alpha_1\delta_1)\beta_2 - \alpha_2\beta_1\delta_1$

$- [(1+\alpha_1\delta_1)(1+\beta_2)-\alpha_2\beta_1\delta_1+\alpha_2\beta_3+\alpha_3\beta_2]z$

$+ [\alpha_3+\alpha_2\beta_3+\alpha_3\beta_2]z^2.$

Let

$\Delta = (1+\alpha_1\delta_1)\beta_2 - \alpha_2\beta_1\delta_1$

$a = (\alpha_2\beta_3+\alpha_3\beta_2)/\Delta$

$b = \alpha_2/\Delta$

$c = ((1+\alpha_1\delta_1)\beta_3+\alpha_3\beta_1\delta_1)/\Delta$

$d = ((1+\alpha_1\delta_1)(1+\beta_2)-\alpha_2\beta_1\delta_1)/\Delta = (1+\alpha_1\delta_1+\Delta)/\Delta.$

From these definitions follow the following lemmata:

<u>Lemma A1</u>: $\det A(z) = \Delta(1-(a+d)z+(ad-bc)z^2).$

Proof: Clearly, the constant in (A1) is Δ and the coefficient on

z is $-\Delta(a+d)$. It remains to be shown that $ad - bc =$

$(\alpha_3+\alpha_2\beta_3+\alpha_3\beta_2)/\Delta.$

Now

$$a + d = (\alpha_2\beta_3+\alpha_3\beta_2+(1+\alpha_1\delta_1)+\Delta)/\Delta$$

$$ad = (\alpha_2\beta_3+\alpha_3\beta_2)(1+\alpha_1\delta_1+\Delta)/\Delta^2$$

$$= (\alpha_2\beta_3+\alpha_3\beta_2+\alpha_1\alpha_2\beta_3\delta_1+\alpha_1\alpha_3\beta_2\delta_1+\alpha_2\beta_3\Delta+\alpha_3\beta_2\Delta)/\Delta$$

$$bc = (\alpha_2\alpha_3\beta_1\delta_1+\alpha_2\beta_3+\alpha_1\alpha_2\beta_3\delta_1)/\Delta^2.$$

Then

$$ad - bc = (\alpha_3\beta_2+\alpha_1\alpha_3\beta_2\delta_1+\alpha_2\beta_3\Delta+\alpha_3\beta_2\Delta-\alpha_2\alpha_3\beta_1\delta_1)/\Delta^2$$

$$= (\alpha_3\beta_2(1+\alpha_1\delta_1)-\alpha_2\alpha_3\beta_1\delta_1+\alpha_2\beta_3\Delta+\alpha_3\beta_2\Delta)/\Delta^2$$

$$= (\alpha_3\Delta+\alpha_2\beta_3\Delta+\alpha_3\beta_2\Delta)/\Delta^2$$

$$= (\alpha_3+\alpha_2\beta_3+\alpha_3\beta_2)/\Delta$$

as was to be shown.

<u>Lemma A2</u>: $\Delta > 0$, $d > 1$.

    Proof: Write $\Delta = \beta_2 + \delta_1(\alpha_1\beta_2-\alpha_2\beta_1)$. But in the text it was assumed that $\alpha_1 > \alpha_2 \geq 0$, $\beta_2 > \beta_1 > 0$, $\delta_1 > 0$, and $\alpha_2, \beta_3 \epsilon (0,1)$. Then $\alpha_1\beta_2 - \alpha_2\beta_1 > 0$, and $\Delta > 0$ follows. Then clearly $d = 1 + (1+\alpha_1\delta_1)/\Delta > 1$.

<u>Lemma A3</u>: Write $\det A(z) = \Delta(1-\lambda_1 z)(1-\lambda_2 z)$ where $\lambda_1 + \lambda_2 = a + d$, $\lambda_1\lambda_2 = ad - bc$. Then λ_1 and λ_2 can be ordered as $0 < \lambda_1 < 1 < \lambda_2$.

    Proof: Using Lemma A1, write

$$\lambda_2 = (a+d)/2 + (1/2)\sqrt{(a+d)^2 - 4(ad-bc)}$$

$$= (a+d)/2 + (1/2)\sqrt{(a-d)^2 + 4bc}.$$

Since b and c are positive, λ_2 (and hence λ_1) is real. Then

$$\lambda_2 > (a+d)/2 + (1/2)\sqrt{(a-d)^2}$$

$$= (a+d)/2 + |a-d|/2$$

$$= \begin{cases} a & a > d \\ d & d > a \end{cases}.$$

Hence, $\lambda_2 > \max(a,d) \geq d$. But by Lemma A2, $d > 1$. Then $\lambda_2 > 1$. Above, it was shown that

$$ad - bc = \lambda_1\lambda_2 = a + \alpha_3/\Delta.$$

The assumptions above, the definition of a, and Lemma A2 give $\lambda_1\lambda_2 > 0$. Clearly, then, $\lambda_1 > 0$. Now

$$\lambda_1 + \lambda_2 - \lambda_1\lambda_2 = \lambda_1(1-\lambda_2) + \lambda_2 = a + d - (ad-bc).$$

But from above,

$$a + d - (ad-bc) = (1+\alpha_1\delta_1+\Delta-\alpha_3)/\Delta$$

$$= (1-\alpha_3+\alpha_1\delta_1)/\Delta + 1.$$

Then, since $\alpha_3 \epsilon (0,1)$,

$$\lambda_1 + \lambda_2 - \lambda_1\lambda_2 > 1$$

$$\lambda_1(1-\lambda_2) > 1 - \lambda_2$$

and

$$\lambda_1 < 1$$

since $1 - \lambda_2 < 0$ from above. Hence, there obtains the ordering

$$0 < \lambda_1 < 1 < \lambda_2.$$

In addition to the above lemmata, we can now demonstrate the following

Facts:

A1. $\lambda_1 + \dfrac{1 + \alpha_1\delta_1 - \alpha_3/\lambda_2}{(1-1/\lambda_2)\Delta} = d.$

Demonstration: From above,

$$\lambda_1 + \lambda_2 - \lambda_1\lambda_2 = 1 + (1+\alpha_1\delta_1-\alpha_3)/\Delta.$$

Hence,

$$\lambda_1(1-\lambda_2) = (1+\alpha_1\delta_1-\alpha_3)/\Delta + 1 - \lambda_2$$

$$\lambda_1 = (1+\alpha_1\delta_1-\alpha_3+\Delta-\lambda_2\Delta)/\Delta(1-\lambda_2)$$

$$= -((1+\alpha_1\delta_1+\Delta-\alpha_3)/\lambda_2-\Delta)/\Delta(1-1/\lambda_2).$$

Then

$$\lambda_1 + (1+\alpha_1\delta_1-\alpha_3/\lambda_2)/(1-1/\lambda_2)\Delta$$

$$= (1+\alpha_1\delta_1-\alpha_3/\lambda_2-((1+\alpha_1\delta_1+\Delta-\alpha_3)/\lambda_2-\Delta))/\Delta(1-1/\lambda_2)$$

$$= ((1+\alpha_1\delta_1+\Delta)(1-1/\lambda_2)-\alpha_3/\lambda_2+\alpha_3/\lambda_2)/\Delta(1-1/\lambda_2)$$

$$= (1+\alpha_1\delta_1+\Delta)(1-1/\lambda_2)/\Delta(1-1/\lambda_2)$$

$$= (1+\alpha_1\delta_1+\Delta)/\Delta$$

$$= d.$$

A2. $(d\lambda_1-\lambda_1^2-\lambda_1\lambda_2+a)/(1-\lambda_1) = a.$

Demonstration: Recall that $\lambda_1 + \lambda_2 = a + d$ (Lemma A3). Then

$$(d\lambda_1-\lambda_1^2-\lambda_1\lambda_2+a)/(1-\lambda_1) = (\lambda_1(d-\lambda_1-\lambda_2)+a)/(1-\lambda_1)$$

$$= (\lambda_1(d-(a+d))+a)/(1-\lambda_1)$$

$$= a(1-\lambda_1)/(1-\lambda_1)$$

$$= a.$$

A3. $\lambda_2(1+\alpha_1\delta_1) - \alpha_3 = \Delta(d-\lambda_1)(\lambda_2-1).$

Demonstration: From above, $(1+\alpha_1\delta_1) = \Delta(d-1)$ and
$\lambda_1 + \lambda_2 - \lambda_1\lambda_2 - d = -\alpha_3/\Delta$ (Lemma A3). Then

$$\lambda_2(1+\alpha_1\delta_1) - \alpha_3 = \lambda_2\Delta(d-1) + \Delta(\lambda_1+\lambda_2-\lambda_1\lambda_2-d)$$

$$= \Delta(\lambda_2 d+\lambda_1-\lambda_1\lambda_2-d)$$

$$= \Delta(\lambda_2(d-\lambda_1)-(d-\lambda_1))$$

$$= \Delta(\lambda_2-1)(d-\lambda_1).$$

A4. If $\alpha_2 = 0$,

(i) $\Delta = (1+\alpha_1\delta_1)\beta_2$

(ii) $a = \alpha_3/(1+\alpha_1\delta_1)$

(iii) $d = (1+\beta_2)/\beta_2$

(iv) $\lambda_1 = a, \lambda_2 = d.$

Demonstration: (i) is obvious from the definitions. Given (i), (ii) and (iii) are obvious. To show (iv), notice $\alpha_2 = 0$ implies $c = 0$. Then by lemmata A1 and A2, det $A(z) = \Delta(1-az)(1-dz) = \Delta(1-\lambda_1 z)(1-\lambda_2 z)$. Then since $a < 1$ (or $d > 1$) we have $\lambda_1 = a, \lambda_2 = d$.

Appendix B

In the text there appears the expression

$$(\text{B1}) \quad \begin{bmatrix} 1(z) \\ f(z) \end{bmatrix} = \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)} \left\{ \begin{bmatrix} \beta_2 & \alpha_2 \\ \beta_1\delta_1 & 1+\alpha_1\delta_1 \end{bmatrix} + \begin{bmatrix} -(1+\beta_2) & -\alpha_2 \\ \beta_3 & -\alpha_3 \end{bmatrix} z \right\}$$

$$\cdot \begin{bmatrix} -\alpha_2\gamma \\ (1+\beta_2)\gamma - h(z) \end{bmatrix} .$$

Sought is the γ for which the residue of $[1(z)f(z)]'$ at $z = \lambda_2^{-1}$ is zero. This residue is

$$\lim_{z \to \lambda_2^{-1}} (1-\lambda_2 z) \begin{bmatrix} 1(z) \\ f(z) \end{bmatrix}$$

$$= \frac{1}{\Delta(1-\frac{\lambda_1}{\lambda_2})} \left\{ \begin{bmatrix} \beta_2 & \alpha_2 \\ \beta_1\delta_1 & 1+\alpha_1\delta_1 \end{bmatrix} + \begin{bmatrix} -(1+\beta_2) & -\alpha_2 \\ \beta_3 & -\alpha_3 \end{bmatrix} \frac{1}{\lambda_2} \right\} \begin{bmatrix} -\alpha_2\gamma \\ (1+\beta_2)\gamma - h(\frac{1}{\lambda_2}) \end{bmatrix}$$

$$= \frac{\lambda_2}{\Delta(\lambda_2-\lambda_1)} \left\{ \begin{matrix} \beta_2 - \dfrac{1+\beta_2}{\lambda_2} & \alpha_2 - \dfrac{\alpha_2}{\lambda_2} \\[2mm] \beta_1\delta_1 + \dfrac{\beta_3}{\lambda_2} & 1 + \alpha_1\delta_1 - \dfrac{\alpha_3}{\lambda_2} \end{matrix} \right\} \left\{ \begin{matrix} -\alpha_2\gamma \\[2mm] (1+\beta_2)\gamma - h(\frac{1}{\lambda_2}) \end{matrix} \right\}$$

$$= \frac{1}{\Delta(\lambda_2-\lambda_1)} \left\{ \begin{array}{l} (\lambda_2\beta_2-1-\beta_2)(-\alpha_2\gamma) + (\alpha_2\gamma_2-\alpha_2)[(1+\beta_2)\gamma-h(\frac{1}{\lambda_2})] \\[2ex] (\lambda_2\beta_1\delta_1+\beta_3)(-\alpha_2\gamma) + (\lambda_2(1+\alpha_1\delta_1)-\alpha_3)[(1+\beta_2)\gamma-h(\frac{1}{\lambda_2})] \end{array} \right\}$$

$$= \frac{1}{\Delta(\lambda_2-\lambda_1)} \left\{ \begin{array}{l} -\alpha_2\gamma\lambda_2\beta_2+(1+\beta_2)\alpha_2\gamma+\lambda_2\alpha_2(1+\beta_2)\gamma-\alpha_2(1+\beta_2)\gamma \\[2ex] \qquad\qquad -(\alpha_2\lambda_2-\alpha_2)h(\frac{1}{\lambda_2}) \\[3ex] -\alpha_2\beta_1\delta_1\lambda_2\gamma-\alpha_2\beta_3\gamma+\lambda_2(1+\alpha_1\delta_1)(1+\beta_2)\gamma - \alpha_3(1+\beta_2)\gamma \\[2ex] \qquad\qquad -(\lambda_2(1+\alpha_1\delta_1)-\alpha_3)h(\frac{1}{\lambda_2}) \end{array} \right\}$$

$$= \frac{1}{\Delta(\lambda_2-\lambda_1)} \left\{ \begin{array}{l} \alpha_2\lambda_2\gamma-(\alpha_2\lambda_2-\alpha_2)h(\frac{1}{\lambda_2}) \\[3ex] \lambda_2\gamma[-\alpha_2\beta_1\delta_1+(1+\beta_2)(1+\alpha_1\delta_1)]-\gamma[\alpha_2\beta_3+(1+\beta_2)\alpha_3] \\[2ex] \qquad\qquad -(\lambda_2(1+\alpha_1\delta_1)-\alpha_3)h(\frac{1}{\lambda_2}) \end{array} \right\}$$

$$= \frac{1}{\Delta(\lambda_2-\lambda_1)} \left\{ \begin{array}{l} \alpha_2(\gamma\lambda_2-(\lambda_2-1)h(\frac{1}{\lambda_2})) \\[3ex] \lambda_2\gamma\Delta d-\gamma\Delta(ad-bc)-(\lambda_2(1+\alpha_1\delta_1)-\alpha_3)h(\frac{1}{\lambda_2}) \end{array} \right\}$$

by virtue of Lemma A1 and the definitions given there. From Appendix A, $\lambda_1\lambda_2$ = ad - bc. In addition, Fact A3 gives $\lambda_2(1+\alpha_1\delta_1)$ - α_3 = $(d-\lambda_1)(\lambda_2-1)\Delta$. Then

$$\lim_{z \to \lambda_2^{-1}} (1-\lambda_2 z) \begin{bmatrix} 1(z) \\ f(z) \end{bmatrix} = \frac{\lambda_2}{\Delta(\lambda_2 - \lambda_1)} \begin{bmatrix} \alpha_2(\gamma - \dfrac{\lambda_2 - 1}{\lambda_2} h(\dfrac{1}{\lambda_2})) \\ \\ \gamma \Delta d - \gamma \Delta \lambda_1 - (d - \lambda_1)(\dfrac{\lambda_2 - 1}{\lambda_2}) h(\dfrac{1}{\lambda_2}) \end{bmatrix}$$

$$= \frac{\lambda_2}{\lambda_2 - \lambda_1} \begin{bmatrix} \alpha_2 \Delta(\gamma - \dfrac{\lambda_2 - 1}{\lambda_2} h(\dfrac{1}{\lambda_2})) \\ \\ (d - \lambda_1)(\gamma - \dfrac{\lambda_2 - 1}{\lambda_2}) h(\dfrac{1}{\lambda_2})) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ when } \gamma = (1 - \lambda_2^{-1}) h(\lambda_2^{-1}).$$

Appendix C

In this appendix, (5.5) will be used to derive (5.6) and (5.7).

From (5.5)

$$
1(z) = \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)}[-\alpha_2\gamma(\beta_2-(1+\beta_2)z)+\alpha_2(1-z)((1+\beta_2)\gamma-h(z))]
$$

$$
= \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)}[-\alpha_2\beta_2\gamma+\alpha_2(1+\beta_2)\gamma+\gamma\alpha_2(1+\beta_2)z-\alpha_2(1+\beta_2)\gamma z
$$

$$
-\alpha_2(1-z)h(z)]
$$

$$
= \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)}[\alpha_2\gamma-\alpha_2(1-z)h(z)]
$$

$$
= \frac{\alpha_2}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)}[(1-\lambda_2^{-1})h(\lambda_2^{-1})-h(z)+zh(z)]
$$

$$
= \frac{\alpha_2}{\Delta(1-\lambda_1 z)}\left(\frac{-\lambda_2^{-1}z^{-1}}{1-\lambda_2^{-1}z^{-1}}\right)(h(\lambda_2^{-1})-h(z)-\lambda_2^{-1}h(\lambda_2^{-1})+zh(z))
$$

$$
= \frac{\alpha_2}{\Delta(1-\lambda_1 z)}\left(\frac{1}{\lambda_2}\right)\left(\frac{h(z)-h(\lambda_2^{-1})}{z-\lambda_2^{-1}}\right) - \frac{\alpha_2}{\Delta(1-\lambda_1 z)}\left(\frac{1}{\lambda_2}\right)
$$

$$
\cdot\left(\frac{h(z)-\lambda_2^{-1}z^{-1}h(\lambda_2^{-1})}{1-\lambda_2^{-1}z^{-1}}\right).
$$

Using the Hansen-Sargent formula HS1, this can be written

$$k_{t+1} = \frac{\alpha_2}{\Delta(1-\lambda_1 L)}(\frac{1}{\lambda_2}) \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+1+k} - \frac{\alpha_2}{\Delta(1-\lambda_1 L)}(\frac{1}{\lambda_2})$$

$$\cdot \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k}$$

or

$$(5.6) \qquad k_{t+1} = \lambda_1 k_t + \frac{\alpha_2}{\Delta \lambda_2} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-L) m_{t+1+k}.$$

To find the solution for p_t; from (5.5),

$$f(z) = \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)} [(\beta_1 \delta_1 + \beta_3 z)(-\gamma \alpha_2) + (1+\alpha_1 \delta_1 - \alpha_3 z)$$

$$\cdot ((1+\beta_2)\gamma - h(z))]$$

$$= \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)} [-\alpha_2 \beta_1 \delta_1 \gamma - \alpha_2 \beta_3 \gamma z + (1+\alpha_1 \delta_1)\gamma + (1+\alpha_1 \delta_1)\beta_2 \gamma$$

$$-\alpha_3(1+\beta_2)\gamma z - (1+\alpha_1 \delta_1 - \alpha_3 z)h(z)]$$

$$(C1) \qquad = \frac{1}{\Delta(1-\lambda_1 z)(1-\lambda_2 z)} [\Delta \gamma (d - \lambda_1 \lambda_2 z) - (1+\alpha_1 \delta_1 - \alpha_3 z)h(z)]$$

by Appendix A. By Fact A1, $d = \lambda_1 + \dfrac{1+\alpha_1 \delta_1 - \dfrac{\alpha_3}{\lambda_2}}{\Delta(1-\lambda_2^{-1})}$. Hence,

$$f(z) = \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)} [\gamma(\lambda_1 + \frac{1+\alpha_1 \delta_1 - \dfrac{\alpha_3}{\lambda_2} - \lambda_1 \lambda_2 z}{\Delta(1-\lambda_2^{-1})})$$

$$- (\frac{1+\alpha_1 \delta_1 - \alpha_3 z}{\Delta})h(z)]$$

$$= \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)}[(1-\lambda_2^{-1})h(\lambda_2^{-1})(\lambda_1(1-\lambda_2 z) + \frac{1+\alpha_1\delta_1 - \frac{\alpha_3}{\lambda_2}}{\Delta(1-\lambda_2^{-1})})$$

$$- \frac{(1+\alpha_1\delta_1 - \alpha_3 z)}{\Delta}h(z)]$$

$$= \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)}[\gamma\lambda_1(1-\lambda_2 z) + \frac{1+\alpha_1\delta_1 - \alpha_3\lambda_2^{-1}}{\Delta}h(\lambda_2^{-1})$$

$$- \frac{(1+\alpha_1\delta_1 - \lambda_2 z(1+\alpha_1\delta_1) + \lambda_2 z(1+\alpha_1\delta_1) - \alpha_3 z h(z))}{\Delta}]$$

$$= \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)}[\gamma\lambda_1(1-\lambda_2 z) - \Delta^{-1}(1-\lambda_2 z)(1+\alpha_1\delta_1)h(z)$$

$$- \lambda_2^{-1}\Delta^{-1}((1+\alpha_1\delta_1)\lambda_2 - \alpha_3)(\lambda_2 z h(z) - h(\lambda_2^{-1}))]$$

$$= \frac{1}{(1-\lambda_1 z)}[\gamma\lambda_1 - \Delta^{-1}(1+\alpha_1\delta_1)h(z) - \Delta^{-1}((1+\alpha_1\delta_1)\lambda_2 - \alpha_3)z$$

$$\cdot \frac{[h(z) - \lambda_2^{-1}z^{-1}h(\lambda_2^{-1})]}{1-\lambda_2 z}]$$

$$= \frac{1}{(1-\lambda_1 z)}[\gamma\lambda_1 - \Delta^{-1}(1+\alpha_1\delta_1)h(z) + \Delta^{-1}\lambda_2^{-1}((1+\alpha_1\delta_1)\lambda_2 - \alpha_3)$$

$$\cdot [\frac{h(z) - \lambda_2^{-1}z^{-1}h(\lambda_2^{-1})}{1-\lambda_2^{-1}z^{-1}}]].$$

Therefore

$$f(z) = \gamma\lambda_1 + \lambda_1 z f(z) - \Delta^{-1}(1+\alpha_1\delta_1)h(z) + \Delta^{-1}\lambda_2^{-1}((1+\alpha_1\delta_1)\lambda_2 - \alpha_3)$$

$$\cdot \frac{h(z) - \lambda_2^{-1}z^{-1}h(\lambda_2^{-1})}{1-\lambda_2^{-1}z^{-1}}$$

$$= \lambda_1 g(z) - \Delta^{-1}(1+\alpha_1\delta_1)h(z) + \Delta^{-1}\lambda_2^{-1}((1+\alpha_1\delta_1)\lambda_2-\alpha_3)$$

$$\cdot \frac{h(z)-\lambda_2^{-1}z^{-1}h(\lambda_2^{-1})}{1-\lambda_2^{-1}z^{-1}}$$

which gives, by the Hansen-Sargent formula HS1,

$$E_t p_{t+1} = \lambda_1 p_t - \Delta^{-1}(1+\alpha_1\delta_1)m_t + \Delta^{-1}\lambda_2^{-1}((1+\alpha_1\delta_1)\lambda_2-\alpha_3)$$

$$\cdot \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t(m_{t+k})$$

$$= \lambda_1 p_t - \Delta^{-1}(1+\alpha_1\delta_1)m_t + \Delta^{-1}(1+\alpha_1\delta_1) \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k}$$

$$- \alpha_3\Delta^{-1}\lambda_2^{-1} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k}$$

$$= \lambda_1 p_t + (1+\alpha_1\delta_1)\Delta^{-1}\lambda_2^{-1} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+1+k}$$

$$- \alpha_3\Delta^{-1}\lambda_2^{-1} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k}$$

$$(C2) \qquad E_t p_{t+1} = \lambda_1 p_t + \Delta^{-1}\lambda_2^{-1} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t(1+\alpha_1\delta_1-\alpha_3 L)m_{t+1+k}.$$

To find the solution for $\{p_t\}$, substitute (5.6) and (C2) into (5.1).
Now (5.1) is equivalent to

$$- \{\frac{1+\alpha_1\delta_1-\alpha_3 L}{\alpha_2} k_{t+1} - E_t p_{t+1}\} = p_t.$$

Then, from (5.6) and (C2),

$$P_t = \lambda_1 P_t + \Delta^{-1}\lambda_2^{-1} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1+\alpha_1\delta_1-\alpha_3 L) m_{t+1+k}$$

$$- \Delta^{-1}\lambda_2^{-1} \frac{1+\alpha_1\delta_1-\alpha_3 L}{1-\lambda_1 L} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-L) m_{t+1+k}$$

$$P_t = \Delta^{-1}\lambda_2^{-1}(1-\lambda_1)^{-1}\{ \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1+\alpha_1\delta_1-\alpha_3 L) m_{t+1+k}$$

$$- \frac{(1+\alpha_1\delta_1-\alpha_3 L)}{1-\lambda_1 L} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-L) m_t \}.$$

Hence,

$$(C3) \qquad P_t = \lambda_1 P_{t-1} + \Delta^{-1}\lambda_2^{-1}(1-\lambda_1)^{-1}\{ (1-\lambda_1 L) \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1+\alpha_1\delta_1-\alpha_3 L)$$

$$\cdot \; m_{t+1+k} - (1+\alpha_1\delta_1-\alpha_3 L) \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-L) m_t \}.$$

As a check on (C3) it can be shown that calculation of $E_t P_{t+1}$ using (C3) yields (C2). Shifting (C3) forward yields

$$P_{t+1} = \lambda_1 P_t + \Delta^{-1}\lambda_2^{-1}(1-\lambda_1)^{-1}\{ \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_{t+1} (1+\alpha_1\delta_1-\alpha_3 L) m_{t+2+k}$$

$$- \lambda_1 \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1+\alpha_1\delta_1-\alpha_3 L) m_{t+2+k}$$

$$- (1+\alpha_1\delta_1) \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_{t+1} (1-L) m_{t+2+k}$$

$$+ \alpha_3 \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-L) m_{t+1+k} \}.$$

Applying the expectations operator,

$$E_t p_{t+1} = \lambda_1 p_t + \Delta^{-1} \lambda_2^{-1} (1-\lambda_1)^{-1} \{ \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-\lambda_1 L)(1+\alpha_1 \delta_1 - \alpha_3 L)$$

$$\cdot \, m_{t+2+k} - \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1+\alpha_1 \delta_1 - \alpha_3 L)(1-L) m_{t+2+k} \}$$

$$= \lambda_1 p_t + \Delta^{-1} \lambda_2^{-1} (1-\lambda_1)^{-1} \{ \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-\lambda_1 L - 1 + L)$$

$$\cdot \, (1+\alpha_1 \delta_1 - \alpha_3 L) m_{t+2+k} \}$$

$$= \lambda_1 p_t + \Delta^{-1} \lambda_2^{-1} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1+\alpha_1 \delta_1 - \alpha_3 L) m_{t+1+k}$$

which is (C2). To provide another check on (C3), it can be shown that transforming it leads to the f(z) that produced (C2). By the Hansen-Sargent forumla HS1,

$$\sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+1+k} = -\lambda_2 m_t + \lambda_2 \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k} .$$

Then (C3) can be written

$$(1-\lambda_1 L) p_t = \frac{1}{\Delta \lambda_2 (1-\lambda_1)} \{ (1-\lambda_1 L)[1+\alpha_1 \delta_1) \lambda_2 (-m_t + \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k})$$

$$- \alpha_3 \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k}] - (1+\alpha_1 \delta_1 - \alpha_3 L)$$

$$\cdot \, [\lambda_2 (-m_t + \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k}) - \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k}] \}$$

$$= \frac{1}{\Delta \lambda_2 (1-\lambda_1)} \{ [(1-\lambda_1 L)(1+\alpha_1 \delta_1) - (1+\alpha_1 \delta_1 - \alpha_3 L)][-\lambda_2 m_t]$$

$$+ [(1-\lambda_1 L)((1+\alpha_1 \delta_1) \lambda_2 - \alpha_3)$$

$$- (1+\alpha_1 \delta_1 - \alpha_3 L)(\lambda_2 - 1)] \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t m_{t+k} .$$

Then the appropriate transforms must satisfy

$$(1-\lambda_1 z)g(z) = \frac{1}{\Delta\lambda_2(1-\lambda_1)}\{[(1-\lambda_1 z)(1+\alpha_1\delta_1)-(1+\alpha_1\delta_1-\alpha_3 z)]$$

$$\cdot[-\lambda_2 h(z)] + [(1-\lambda_1 z)((1+\alpha_1\delta_1)\lambda_2-\alpha_3)$$

$$-(1+\alpha_1\delta_1-\alpha_3 z(\lambda_2-1)\frac{h(z)-\lambda_2^{-1}z^{-1}h(\lambda_2^{-1})}{1-\lambda_2^{-1}z^{-1}}\}$$

or

$$(C4) \qquad g(z) = \frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)}\{[(1-\lambda_1 z)(1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1})-(1+\alpha_1\delta_1-\alpha_3 z)$$

$$\cdot(1-\lambda_2^{-1})]\frac{(-\lambda_2^{-1}z^{-1}h(\lambda_2^{-1}))}{1-\lambda_2^{-1}z^{-1}}\}+\frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)}$$

$$\cdot\{[(1-\lambda_1 z)(1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1})-(1+\alpha_1\delta_1-\alpha_3 z)(1-\lambda_2^{-1})]$$

$$\cdot(\frac{1}{1-\lambda_2^{-1}z^{-1}})-[(1-\lambda_1 z)(1+\alpha_1\delta_1)$$

$$-(1+\alpha_1\delta_1-\alpha_3 z)]\}h(z).$$

The "coefficient" of $h(z)$ is

$$\frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2^{-1}z^{-1})}\{[(1-\lambda_1 z)(1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1})-(1+\alpha_1\delta_1-\alpha_3 z)$$

$$\cdot(1-\lambda_2^{-1})-(1-\lambda_2^{-1}z^{-1})[(1-\lambda_1 z)(1+\alpha_1\delta_1)-(1+\alpha_1\delta_1-\alpha_3 z)]\}$$

$$= \frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2^{-1}z^{-1})}\{1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1}-\lambda_1 z(1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1})$$

$$- (1+\alpha_1\delta_1-\alpha_3 z)+(1+\alpha_1\delta_1-\alpha_3 z)\lambda_2^{-1}-(1-\lambda_2^{-1}z^{-1})$$

$$\cdot [1+\alpha_1\delta_1-\lambda_1 z(1+\alpha_1\delta_1)-(1+\alpha_1\delta_1)+\alpha_3 z]\}$$

$$= \frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2^{-1}z^{-1})}\{-\alpha_3\lambda_2^{-1}+\alpha_3 z+(\lambda_2^{-1}-\lambda_1 z)(1+\alpha_1\delta_1)$$

$$- (1-\lambda_1)\alpha_3\lambda_2^{-1}z-(1-\lambda_2^{-1}z^{-1})(\alpha_3 z-\lambda_1 z(1+\alpha_1\delta_1))\}$$

$$= \frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2^{-1}z^{-1})}\{(\lambda_2^{-1}-\lambda_1 z)(1+\alpha_1\delta_1)-(1-\lambda_1)\alpha_3\lambda_2^{-1}z$$

$$+ (1+\alpha_1\delta_1)\lambda_1 z(1-\lambda_2^{-1}z^{-1})\}$$

$$= \frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2^{-1}z^{-1})}\{(1+\alpha_1\delta_1)[(\lambda_2^{-1}-\lambda_1 z)+\lambda_1 z-\lambda_1\lambda_2^{-1}]$$

$$- (1-\lambda_1)\alpha_3\lambda_2^{-1}z\}$$

$$= \frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2^{-1}z^{-1})}\{(1-\lambda_1)[(1+\alpha_1\delta_1)\lambda_2^{-1}-\alpha_3 z\lambda_2^{-1}\}$$

$$= \frac{1}{\Delta(1-\lambda_1 z)(\lambda_2 z-1)}\{(1+\alpha_1\delta_1-\alpha_3 z)z\}$$

$$= -\Delta^{-1}(1-\lambda_1 z)^{-1}(1-\lambda_2 z)^{-1}\{(1+\alpha_1\delta_1-\alpha_3 z)z\}.$$

The "cofficient" of h(q) in (C4) is (q = λ_2^{-1})

$$\frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2^{-1}z^{-1})}\{(1-\lambda_1 z)(1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1})-(1+\alpha_1\delta_1-\alpha_3 z)$$

$$\cdot\ (1-\lambda_2^{-1})\}(-\lambda_2^{-1}z^{-1})$$

$$=\frac{1}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2 z)}\{(1-\lambda_1 z)(1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1})$$

$$-\ (1+\alpha_1\delta_1-\alpha_3 z)(1-\lambda_2^{-1})\}$$

$$=\frac{(1-\lambda_2^{-1})}{\Delta(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2 z)}\{(1-\lambda_1 z)\frac{(1+\alpha_1\delta_1-\alpha_3\lambda_2^{-1})}{1-\lambda_2^{-1}}-(1+\alpha_1\delta_1-\alpha_3 z)\}$$

$$=\frac{(1-\lambda_2^{-1})}{(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2 z)}\{(1-\lambda_1 z)(d-\lambda_1)-\frac{(1+\alpha_1\delta_1-\alpha_3 z)}{\Delta}\}$$

by virtue of Fact A1. Then the h(q) "coefficient" becomes

$$\frac{1-\lambda_2^{-1}}{(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2 z)}\{d-\frac{1+\alpha_1\delta_1}{\Delta}-\lambda_1-(\lambda_1(d-\lambda_1)-\frac{\alpha_3}{\Delta})z\}$$

$$=\frac{1-\lambda_2^{-1}}{(1-\lambda_1)(1-\lambda_1 z)(1-\lambda_2 z)}\{1-\lambda_1-(\lambda_1(d-\lambda_1)-[\lambda_1\lambda_2-a])z\},$$

by virtue of the facts leading to Lemma A1. Now

$$\frac{\lambda_1(d-\lambda_1)[\lambda_1\lambda_2-a]}{1-\lambda_1}=a$$

by Fact A2. Then the coefficient becomes

$$\frac{1-\lambda_2^{-1}}{(1-\lambda_1 z)(1-\lambda_2 z)}\{1-az\}.$$

Hence,

$$g(z) = \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)}\{\gamma(1-az) - \frac{(1+\alpha_1\delta_1-\alpha_3 z)}{\Delta} zh(z)\}.$$

But $f(z) = z^{-1}(g(z)-\gamma)$ by assumption. So

$$f(z) = \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)} z^{-1}\{\gamma[(1-az)-(1-\lambda_1 z)(1-\lambda_2 z)]$$

$$- \frac{(1+\alpha_1\delta_1-\alpha_3 z)}{\Delta} zh(z)\}$$

$$= \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)}\{\gamma z^{-1}[1-az-1+(\lambda_1+\lambda_2)z-\lambda_1\lambda_2 z]$$

$$- \frac{(1+\alpha_1\delta_1-\alpha_3 z)}{\Delta} h(z)\}.$$

Since (Appendix A) $\lambda_1 + \lambda_2 - a = d$,

$$f(z) = \frac{1}{(1-\lambda_1 z)(1-\lambda_2 z)}\{\gamma[d-\lambda_1\lambda_2 z] - \frac{(1+\alpha_1\delta_1-\alpha_3 z)}{\Delta} h(z)\}.$$

Comparison of this expression with (C1) indicates, finally, that (C3)

is the correct solution for $\{p_t\}$.

Appendix D

The task here is to derive (5.8) and (5.9) from (5.6) and (5.7). Write (5.6) as

$$k_{t+1} = \lambda_1 k_t + \frac{\alpha_2}{\Delta\lambda_2} \sum_{k=0}^{\infty} \lambda_2^{-k} E_t (1-L) m_{t+1+k}.$$

Then

$$k_{t+1} = \lambda_1 k_t + \frac{\alpha_2}{\Delta\lambda_2} \sum_{k=0}^{\infty} \lambda_2^{-k} E_t m_{t+1+k} - \frac{\alpha_2}{\Delta\lambda_2} \sum_{k=0}^{\infty} \lambda_2^{-k} E_t m_{t+k}.$$

Assume that in (3.5'), $h(L)^{-1} = H(L) = I + H_1 L + H_2 L^2 + \ldots + H_n L^n$. Then by HS1 and HS2,

$$\sum_{k=0}^{\infty} \lambda_2^{-k} E_t m_{t+k} = h(\lambda_2^{-1}) [I + \sum_{m=1}^{n-1} \{ \sum_{k=m+1}^{n} \lambda_2^{m-k} H_k \} L^m] m_t.$$

Notice that

$$\sum_{k=0}^{\infty} \lambda_2^{-k} E_t m_{t+1+k} = E_t m_{t+1} + \lambda_2^{-1} E_t m_{t+2} + \ldots$$

$$= \lambda_2 (\lambda_2^{-1} E_t m_{t+1} + \lambda_2^{-1} E_t m_{t+2} + \ldots)$$

$$= \lambda_2 (-m_t + \sum_{k=0}^{\infty} \lambda_2^{-k} E_t m_{t+k}),$$

so that

$$\sum_{k=0}^{\infty} \lambda_2^{-k} E_t m_{t+1+k} = -\lambda_2 m_t + \lambda_2 h(\lambda_2^{-1}) [I + \sum_{m=1}^{n-1} \{ \sum_{k=m+1}^{n} \lambda_2^{m-k} H_k \} L^m] m_t.$$

Then (5.6) becomes

$$(5.8) \qquad k_{t+1} = \lambda_1 k_t + \frac{\alpha_2}{\Delta\lambda_2}[\lambda_2 + (1-\lambda_2)h(\lambda_2^{-1}) + (1-\lambda_2)h(\lambda_2^{-1})$$

$$\cdot \sum_{m=1}^{n-1}\{\sum_{k=m+1}^{n} \lambda_2^{m-k}H_k\}L^m]m_t .$$

To obtain (5.9), first separate terms in (5.7):

$$P_t = \lambda_1 P_{t-1} + \frac{1+\alpha_1\delta_1}{\Delta\lambda_2(1-\lambda_2)} \sum_{k=0}^{\infty} \lambda_2^{-k}E_t m_{t+1+k} - \frac{\alpha_3}{\Delta\lambda_2(1-\lambda_2)}$$

$$\cdot \sum_{k=0}^{\infty} \lambda_2^{-k}E_t m_{t+k}$$

$$- \frac{\lambda_1(1+\alpha_1\delta_1)}{\Delta\lambda_2(1-\lambda_2)} \sum_{k=0}^{\infty} \lambda_2^{-k}E_{t-1}m_{t+k} + \frac{\lambda_1\alpha_3}{\Delta\lambda_2(1-\lambda_2)}$$

$$\cdot \sum_{k=0}^{\infty} \lambda_2^{-k}E_{t-1}m_{t-1+k} - \frac{1+\alpha_1\delta_1}{\Delta\lambda_1(1-\lambda_2)} \sum_{k=0}^{\infty} \lambda_2^{-k}E_t m_{t+1+k}$$

$$+ \frac{1+\alpha_1\delta_1}{\Delta\lambda_2(1-\lambda_2)} \sum_{k=0}^{\infty} \lambda_2^{-k}E_t m_{t+k} + \frac{\alpha_3}{\Delta\lambda_2(1-\lambda_2)}$$

$$\cdot \sum_{k=0}^{\infty} \lambda_2^{-k}E_{t-1}m_{t+k} - \frac{\alpha_3}{\Delta\lambda_2(1-\lambda_2)} \sum_{k=0}^{\infty} \lambda_2^{-k}E_{t-1}m_{t-1+k},$$

which can be rearranged to read

$$P_t = \lambda_1 P_{t-1} + \frac{1}{\Delta\lambda_2(1-\lambda_2)}\{(1+\alpha_1\delta_1-\alpha_3) \sum_{k=0}^{\infty} \lambda_2^{-k}E_t m_{t+k} + (\alpha_3-\lambda_1$$

$$\cdot (1+\alpha_1\delta_1)) \sum_{k=0}^{\infty} \lambda_2^{-k}E_{t-1}m_{t+k} + (\lambda_1-1)\alpha_3 \sum_{k=0}^{\infty} \lambda_2^{-k}E_{t-1}m_{t-1+k}\} .$$

But,

$$\sum \lambda_2^{-k} E_{t-1} m_{t+k} = E_{t-1} m_t + \lambda_2^{-1} E_{t-1} m_{t+1} + \cdots$$

$$= \lambda_2 (\lambda_2^{-1} E_{t-1} m_t + \lambda_2^{-2} E_{t-1} m_{t+1} + \cdots)$$

$$= \lambda_2 (-m_{t-1} + \sum_{k=0}^{\infty} \lambda_2^{-k} E_{t-1} m_{t-1+k})$$

so that

(D1)
$$p_t = \lambda_1 p_{t-1} + \frac{1}{\Delta \lambda_2 (1-\lambda_2)} \{ -\lambda_2 (\alpha_3 - \lambda_1 (1+\alpha_1 \delta_1)) m_{t-1} + (1+\alpha_1 \delta_1 - \alpha_3)$$

$$\cdot \sum_{k=0}^{\infty} \lambda_2^{-k} E_t m_{t+k} + [(\lambda_1-1)\alpha_3 + \lambda_2 (\alpha_3 - \lambda_1 (1+\alpha_1 \delta_1))]$$

$$\cdot \sum_{k=0}^{\infty} \lambda_2^{-k} E_{t-1} m_{t-1+k} \},$$

an expression analogous to Fisher's (3.9). From Fact A1 and Lemma A1,

$$(\lambda_1-1)\alpha_3 + \lambda_2 (\alpha_3 - \lambda_1 (1+\alpha_1 \delta_1)) = -(1+\alpha_1 \delta_1 - \alpha_3) \frac{\alpha_2 \beta_3 + \alpha_3 \beta_2}{\Delta}$$

$$= -(1+\alpha_1 \delta_1 - \alpha_3) a.$$

Applying HS1 and HS2,

$$p_t = \lambda_1 p_{t-1} + \frac{1}{\Delta \lambda_2 (1-\lambda_2)} \{ (1+\alpha_1 \delta_1 - \alpha_3) h(\lambda_2^{-1}) [I + \sum_{m=1}^{n-1} \{ \sum_{k=m+1}^{n} \lambda_2^{m-k} H_k \}$$

$$\cdot L^m] m_t - (1+\alpha_1 \delta_1 - \alpha_3) a h(\lambda_2^{-1}) [I + \sum_{m=1}^{n-1} \{ \sum_{k=m+1}^{n} \lambda_2^{m-k} H_k \} L^m] m_{t-1}$$

$$- \lambda_2 (\alpha_3 - \lambda_1 (1+\alpha_1 \delta_1)) m_{t-1} \}.$$

Then

$$(5.9) \quad P_t = \lambda_1 P_{t-1} + \frac{(1+\alpha_1\delta_1-\alpha_3)}{\Delta\lambda_2(1-\lambda_2)} h(\lambda_2^{-1})m_t$$

$$+ \frac{1}{\Delta\lambda_2(1-\lambda_2)}\{(1+\alpha_1\delta_1-\alpha_3)h(\lambda_2^{-1})(\sum_{k=2}^{n}\lambda_2^{1-k}H_k-a)$$

$$- \lambda_2(\alpha_3-\lambda_1(1+\alpha_1\delta_1))\}m_{t-1}$$

$$+ \frac{h(\lambda_2^{-1})}{\Delta\lambda_2(1-\lambda_2)}\{\sum_{m=0}^{n-3}[(1+\alpha_1\delta_1-\alpha_3)(\sum_{k=m+3}^{n}\lambda_2^{m-k}H_k$$

$$- a\sum_{k=m+2}^{n}\lambda_2^{m-k}H_k)]L^m\}m_{t-2}$$

$$- \frac{(1+\alpha_1\delta_1-\alpha_3)a}{\Delta\lambda_2(1-\lambda_2)} h(\lambda_2^{-1})H_n m_{t-n}.$$

Appendix E

Facts A4 assure that (5.7) can be written as

$$p_t = \frac{1+\alpha_1\delta_1}{\beta_2(1+\alpha_1\delta_1)\lambda_2(1-\lambda_1)} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-aL)m_{t+1+k}$$

$$- \frac{1-aL}{1-\lambda_1 L} \sum_{k=0}^{\infty} (\frac{1}{\lambda_2})^k E_t (1-L)m_{t+1+k} \}.$$

But again, Facts A4 give $a = \lambda_1$, $\lambda_2 = (1+\beta_2)/\beta_2$. Then

$$p_t = \frac{1}{(1+\beta_2)(1-\lambda_2)} \{ \sum_{k=0}^{\infty} (\frac{\beta_2}{1+\beta_2})^k E_t (1-\lambda_1 L - 1 + L)m_{t+1+k} \}$$

$$= \frac{1}{1+\beta_2} \sum_{k=0}^{\infty} (\frac{\beta_2}{1+\beta_2})^k E_t m_{t+k},$$

which is precisely the result obtained above (4.7) for Cagan's system with $\alpha = -\beta_2$. The reader should not be alarmed at the absence of k_{t+1} in the above, though (3.3) seems to indicate that it ought to be present. But notice that if $\alpha_2 = 0$, (3.3) gives k_{t+1} as the solution to a (stable) deterministic difference equation. But the above represents only the indeterministic (regular) part of $\{p_t\}$. The price level itself is given by the above plus $(\alpha_3/(1+\alpha_1\delta_1))^t k_0$.

Footnotes

[1] These models are discussed in some detail in Sargent (1979).

[2] By predicting, in the 1970s, the possibility of sustained four percent inflation with four percent unemployment.

[3] See, for instance, Lucas and Sargent (1979).

[4] (2.4) and a similar result concerning the interest rate.

[5] See Cass-Yaari (1966), Diamond (1965), Lucas (1972), Wallace (1980), and Bryant and Wallace (1979, 1980) for examples of these models.

[6] The filter was truncated in a particular way at the end of the data record.

[7] Assume means have been extracted.

[8] (2.8) is a population relation. When applied to data, it is known as Hannan's inefficient estimator.

[9] A filter which retains low frequency power but reduces high frequency power is a "low-pass" filter.

[10] Lucas (1978) only presents the analogue to Panel e.

[11] If $h(L) = \sum_{j=-\infty}^{\infty} h_j L^j$ is a linear filter (L is the lag operator), the gain of the filter h is $\left| \sum_{j=-\infty}^{\infty} h_j e^{-iw} \right|^2$.

[12] See (2.7). Subsequently, $|\gamma(\cdot)|$ will be referred to as the "transfer modulus."

[13] In this context, "symmetric filtering" means applying the same filter to both series.

[14] The superscript w denotes "whitened." The residuals have zero mean and run from entry 25 to entry 288. A series of length 288 was used in each case, with 24 leading zeros followed by 264 residuals.

[15] In Figures 2.1-2.4, the Lucas filter, whose Fourier transform is given in 2.13, was used. The filter used in Figures 2.5-2.7 is discussed below.

[16] The quarterly averaging filter is itself a "low-pass" filter.

[17] Points plotted are $(\pi_{t_0+k}(\beta), \mu_{t_0+k}(\beta))$ for k = 1, 8, 16, ..., 192 and t_0 = 1958. Only 24 points are plotted to avoid clutter. The first and last four years were deleted to avoid "start-up" bias induced by the two-sided filter.

18/On observations from 1958 to 1973.

19/The marginal significance level is of necessity left vague: the filtering leaves substantial serial correlation in the residuals of (2.6). This point is discussed below.

20/For instance, the Durbin-Watson statistic for the residuals implicit in Panel e of Figure 2.4 is approximately zero, as is the marginal significance level of the Box-Pierce (1970) Q-statistic.

21/In subsequent pages, the articles by Lucas and Fischer will be referred to repeatedly. Lucas (1975) and Fischer (1979) will not be noted each time: it will be apparent when they are being referenced.

22/Lucas describes this setup, but Fischer is more specific about the production process.

23/Thus, expectations are rational with respect to a given information set. See Muth (1960).

24/The dating of the money stock is that used by Fischer; Lucas uses m_{t+1} in (3.4). The dating is of little consequence once an interpretation is given (in the text below), and will mitigate some of the forthcoming computational burden.

25/The standard reference on these subjects is Whittle (1963).

26/This assumption is equivalent to the assumption that h(z) is analytic in a region containing the unit disk.

27/In note 4, page 116.

28/It appears that this problem might be resolved by the addition of a lagged real balance term to (3.4), a suggestion made by Lucas in his paper.

29/A point similar to this is made by Lucas in note 15.

30/This is Lucas' "first pass." He solves a more complicated version of the system in a subsequent section of his paper. Though the models do not coincide, the solution methods do.

31/Because the system is linear.

32/Lucas demonstrates the uniqueness of his solution. In general, one cannot, of course, rely on six equations to uniquely determine six unknowns.

33/Operationally, (3.9) is not a solution for p_t, but rather, is another characterization of the price process. A solution for p_t, to be useful, must be in terms of known quantities; future values of the money stock must not appear.

34/ a, b, c, d, e, λ, and δ can be derived from information provided by Fischer.

35/ With $\{m_t\}$ generated by (3.5), the Hansen-Sargent formula (hereafter HS1) gives

$$\sum_{i=0}^{\infty} f^k E_t m_{t+k} = \sum_{j=0}^{\infty} q_j m_{t-j} = q(L)m_t$$

where

$$q(L) = \frac{1-fL^{-1}h(f)H(L)}{1-fL^{-1}}.$$

A little algebra yields a similar expression for $\sum_{i=0}^{\infty} r^i E_{t-1} m_{t+1}$.

36/ These increases are unanticipated if t_0 is the "present," anticipated if t_0 is the future.

37/ Use of processes like (3.10) and (3.11) make Fischer's world one with expectations but without uncertainty. The assumptions leading to (3.5) make this version of the model explicitly stochastic.

38/ Once a reduced form for p_t is found, a reduced form for k_{t+1} is easily derived.

39/ Recall that Lucas had to determine many, Fischer infinitely many, coefficients.

40/ For instance, the distributed lag coefficients implicit in (3.9').

41/ This set is considered by Sargent (1979). For a more thorough treatment see Naylor and Sell (1971).

42/ If $z = e^{-iw}$, $x(e^{-iw})$ is the Fourier transform of $\{x_k\}$.

43/ This property, as well as those to be stated below, is developed in many places, notably Sargent (1979), Gabel and Roberts (1973), and Liu and Liu (1975).

44/ For general two-sided sequences, Γ must be the unit circle.

45/ Covariance stationarity requires finite innovation variance.

46/ See, for instance, Whittle (1963) or Sargent (1979).

47/ See Churchill, Brown, and Verhey (1976) or Saks and Zygmund (1971).

[48]/ To generate the "certainty-equivalent" of a stochastic problem, replace random variables by their mean values.

[49]/ This problem is treated in Sargent (1978,1979).

[50]/ In fact, an intial, very intuitive "solution" was incorrect. The problem was that there were too many reasonable ways to incorporate expectations. The intuition necessary to solve the problem at the first attempt seems similar to the intuition which generates good time domain guesses. The desire to circumvent the ingenuity required to make so many seemingly arbitrary decisions led to the development of the relatively mechanical solution procedure outlined in the text above.

[51]/ Though it seems remarkable that one undetermined coefficient can be chosen so as to make two residues equal to zero, an appeal to the fact that $\{k_t\}$, $\{p_t\}$, and $\{\varepsilon_t\}$ all span the same space indicates that it is not surprising that the residues are proportional.

[52]/ A solution requires $g(z)$, a quantity easily obtained from (5.4).

[53]/ These calculations make use of the link between Fourier and z-transforms; the development follows that of Sargent (1979).

[54]/ See note 42 above.

[55]/ There is an obvious scaling problem in the Moving Average Coefficient Panels a and e for money and prices in Figures 6.4 and 6.5. For instance, the scale in Figure 6.4a should be from 0 to 1, not 39 to 40. The problem is that although there are, in essence, infinitely many nonzero moving average coefficients, only a finite number can be computed. In most cases, the problem can be circumvented by simply computing more moving average coefficients. That this is not practical in Figures 6.4 and 6.5 can be demonstrated as follows.

Consider the process

$$m_t = \frac{1}{1-\rho L}\varepsilon_t, \quad 0 < \rho < 1, \quad \varepsilon_t = m_t - E[m_t|m_{t-1},m_{t-2},\ldots].$$

The moving average coefficients for this process are $1, \rho, \rho^2, \ldots$. The method used to generate these coefficients in Figures 6.1-6.5 is to inverse Fourier transform $(1-\rho e^{-iw})$ for $w = 0, 2\pi(1/128); 2\pi(2/128), \ldots, 2\pi(127/128)$. In this manner, 128 moving average coefficients are generated. The coefficients so generated are not $1, \rho, \rho^2, \ldots$, but rather a set of coefficients which are exactly zero beyond the 128th. Notice that the process

$$m_t = \frac{1-\rho^{128}L^{128}}{1-\rho L}\varepsilon_t$$

has moving average coefficients 1, ρ, ρ^2, ..., ρ^{127}, 0, 0, 0, ... Thus, though the inverse Fourier transform of $(1-\rho e^{-iw})^{-1}$ is sought, the procedure described above recovers the inverse Fourier transform of $(1-\rho^{128}e^{-i128w})/(1-\rho e^{-iw})$. When ρ is "small," the difference between what is sought and what is recovered is inconsequential. For instance, since $(.9)^{128} < 10^{-5}$, Panel a of Figure 6.3 is approximately correct. But since $(.9999)^{128} > .98$, Panel a of Figures 6.4 and 6.5 are misscaled. There are three consolations. First, the <u>shapes</u> of the moving average coefficient distributions in Figures 6.4 and 6.5 are correct. Second, calculations done totally in the frequency domain (transfer moduli and phases) are unaffected by the above problem. Third, calculations of the distributed lag coefficients involve ratios of objects like $(1-\rho L)^{-1}$, and hence Panels f and j are approximately correct. One might think that better approximations to the moving average coefficients could be obtained by adding a correction factor to $(1-\rho e^{-iw})$ before inverse transforming. This is true, and easily done for the money coefficients in Panel a. But as equation (5.5) indicates, the correction for the price coefficients in Panel e is not so simple.

[56]/Hence the term "white noise."

[57]/The spectrum is zero because when $\alpha_2 = 0$, there is no indeterministic part in k_t . Although the pictures seem to indicate that the price spectra in Panel c of Figures 4 and 5 are zero over some frequencies, they are not; this is just resolution in the graphs.

[58]/In the figures, prices are (very weakly) damped. This is because the autoregressive parameter for money is 0.9999, not 1.0.

[59]/See note 37 above.

[60]/Recall that capital at time t + 1 can only be affected by events at time t and earlier.

[61]/When the modulus of the cross spectrum is zero, the phase is undefined. When this occurred, the phase was arbitrarily set to zero. Thus, whenever the transfer moduli in Panels g and k are zero, the phases in Panels h and l were arbitrarily set to zero.

[62]/i.e., $\pi_t = (1-L)p_t$, $\mu_t = (1-L)m_t$.

[63]/Once seasonal components have been removed from m_t.

[64]/The typical cross-equation rational expectations restrictions are embedded in (5.8) and (5.9).

References

Box, G. E. P., and D. A. Pierce. (1970) "Distribution of Residual Autocorrelations in Autoregressive-Integrated Moving Average Time Series Models," Journal of the American Statistical Association 65: 1509-26.

Bryant, John, and Neil Wallace. (1979) "The Inefficiency of Interest-Bearing National Debt," Journal of Political Economy 87: 365-81.

_____. (1980) "Open Market Operations in a Model of Regulated, Insured Intermediaries," Journal of Political Economy. Forthcoming.

Cagan, Phillip. (1956) "The Monetary Dynamics of Hyperinflation," in Studies in the Quantity Theory of Money, edited by M. Friedman. Chicago: University of Chicago Press.

Cass, David, and Menahem E. Yaari. (1966) "A Re-examination of the Pure Consumption Loans Model," Journal of Political Economy 46: 353-67.

Churchill, Ruel V., James W. Brown, and Roger F. Verhey. (1976) Complex Variables and Applications. (Third edition) New York: McGraw-Hill.

Diamond, Peter A. (1965) "National Debt in a Neoclassical Growth Model," American Economic Review 45: 1126-50.

Fischer, Stanley. (1979) "Anticipations and the Nonneutrality of Money," Journal of Political Economy 87: 225-52.

Futia, Carl A. (1979a) "Stochastic Business Cycles." Unpublished: Bell Laboratories, Inc.

_____. (1979b) "Rational Expectations in Linear Models." Unpublished: Bell Laboratories, Inc.

Gabel, Robert A., and Richard A. Roberts. (1973) Signals and Linear Systems. New York: John Wiley and Sons.

Granger, Clive W. J. (1969) "Investigating Causal Relations by Econometric Models and Cross-Spectral Methods," Econometrica 37: 424-38.

Hansen, Lars P., and Thomas J. Sargent. (1980) "Formulating and Estimating Dynamic Linear Rational Expectations Models," Journal of Economic Dynamics and Control. Forthcoming.

Liu, C. L., and Jane W. S. Liu. (1975) Linear Systems Analysis. New York: McGraw-Hill.

Lucas, Robert E., Jr. (1972) "Expectations and the Neutrality of Money," Journal of Economic Theory 4: 103-24.

_____. (1973) "Some International Evidence on Output-Inflation Tradeoffs," American Economic Review 63: 326-34.

_____. (1975) "An Equilibrium Model of the Business Cycle," Journal of Political Economy 83: 1113-44.

_____. (1976) "Econometric Policy Evaluation: a Critique," in The Phillips Curve and Labor Markets, edited by K. Brunner and A. H. Meltzer, Carnegie-Rochester Conference on Public Policy: 19-46. Amsterdam: North-Holland.

_____. (1978) "Two Illustrations of the Quantity Theory of Money." Unpublished.

_____. (1980a) "Optimal Investment with Rational Expectations," in Rational Expectations and Econometric Practice, edited by R. E. Lucas, Jr., and T. J. Sargent, Minneapolis: University of Minnesota Press. Forthcoming.

_____. (1980b) "Distributed Lags and Optimal Investment Policy," in Rational Expectations and Econometric Practice, edited by R. E. Lucas, Jr., and T. J. Sargent. Minneapolis: University of Minnesota Press. Forthcoming.

_____, and E. Prescott. (1971) "Investment Under Uncertainty," Econometrica 39: 659-81.

_____, and Thomas J. Sargent. (1978) "After Keynesian Macro-economics," in After the Phillips Curve: Persistence of High Inflation and High Unemployment, Federal Reserve Bank of Boston.

Mosca, E., and G. Zappa. (1979) "Consistency Conditions for the Asymptotic Innovations Representation and an Equivalent Inverse Regulation Problem," IEEE Transactions on Automatic Control AC-24: 501-03.

Mundell, Robert A. (1963) "Inflation and Real Interest," Journal of Political Economy 71: 280-83.

Muth, John F. (1960) "Optimal Properties of Exponentially Weighted Forecasts," Journal of the American Statistical Association 55: 299-306.

_____. (1961) "Rational Expectations and the Theory of Price Movements," Econometrica 29: 315-35.

Naylor, A. W., and G. R. Sell. (1971) Linear Operator Theory in Engineering and Science. New York: Holt.

Saks, S., and A. Zygmund. (1971) Analytic Functions, New York: Elsevier Publishing Company.

Samuelson, Paul A. (1958) "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," Journal of Political Economy 66: 467-82.

Saracoglu, Rusdu. (1977) "Frequency Domain Characterizations of Rational Expectations Equilibria with the Method of Undetermined Coefficients," Working Paper 70, Federal Reserve Bank of Minneapolis.

_____, and Thomas J. Sargent. (1978) "Seasonality and Portfolio Balance Under Rational Expectations," Journal of Monetary Economics 4: 435-58.

Sargent, Thomas J. (1978) "Estimation of Dynamic Labor Demand Schedules Under Rational Expectations," Journal of Political Economy 86: 1009-44.

_____. (1979) Macroeconomic Theory, New York: Academic Press.

Sidrauski, Miguel. (1967a) "Rational Choice and Patterns of Growth in a Monetary Economy," American Economic Review 57: 534-44.

_____. (1967b) "Inflation and Economic Growth," Econometrica 33: 671-84.

Sims, Christopher A. (1972) "The Role of Approximate Prior Restrictions in Distributed Lag Estimation," Journal of the American Statistical Association 67: 169-75.

_____. (1974) "Policy Choice in Approximate Models." Unpublished.

Taylor, John B. (1977) "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations," Econometrica 45: 1377-85.

Tobin, James. (1955) "A Dynamic Aggregative Model," The Journal of Political Economy 43: 103-15.

_____. (1965) "Money and Economic Growth," Econometrica 33: 671-84.

Vogel, Robert C. (1974) "The Dynamics of Inflation in Latin America. 1950-1969," American Economic Review 64: 102-14.

Wallace, Neil. (1979) "The Overlapping Generations Model of Fiat Money," in Models of Monetary Economies, edited by J. Kareken and N. Wallace, Federal Reserve Bank of Minneapolis. Forthcoming.

Whittle, P. (1963) Prediction and Regulation By Linear Least Squares Methods. Princeton: D. Van Nostrand.

Zellner, A., and F. Palm. (1974) "Time Series Analysis and Simultaneous Equation Econometric Models," Journal of Econometrics 2: 17-54.

Figure 2.1 BETA = 0.0

a. COHERENCE

c. TRANSFER MODULUS

b. SPECTRUM OF MONEY GROWTH (FILTERED)

d. SPECTRUM OF INFLATION (FILTERED)

Figure 2.1 (cont.)   BETA = 0.0

INFLATION (FILTERED)
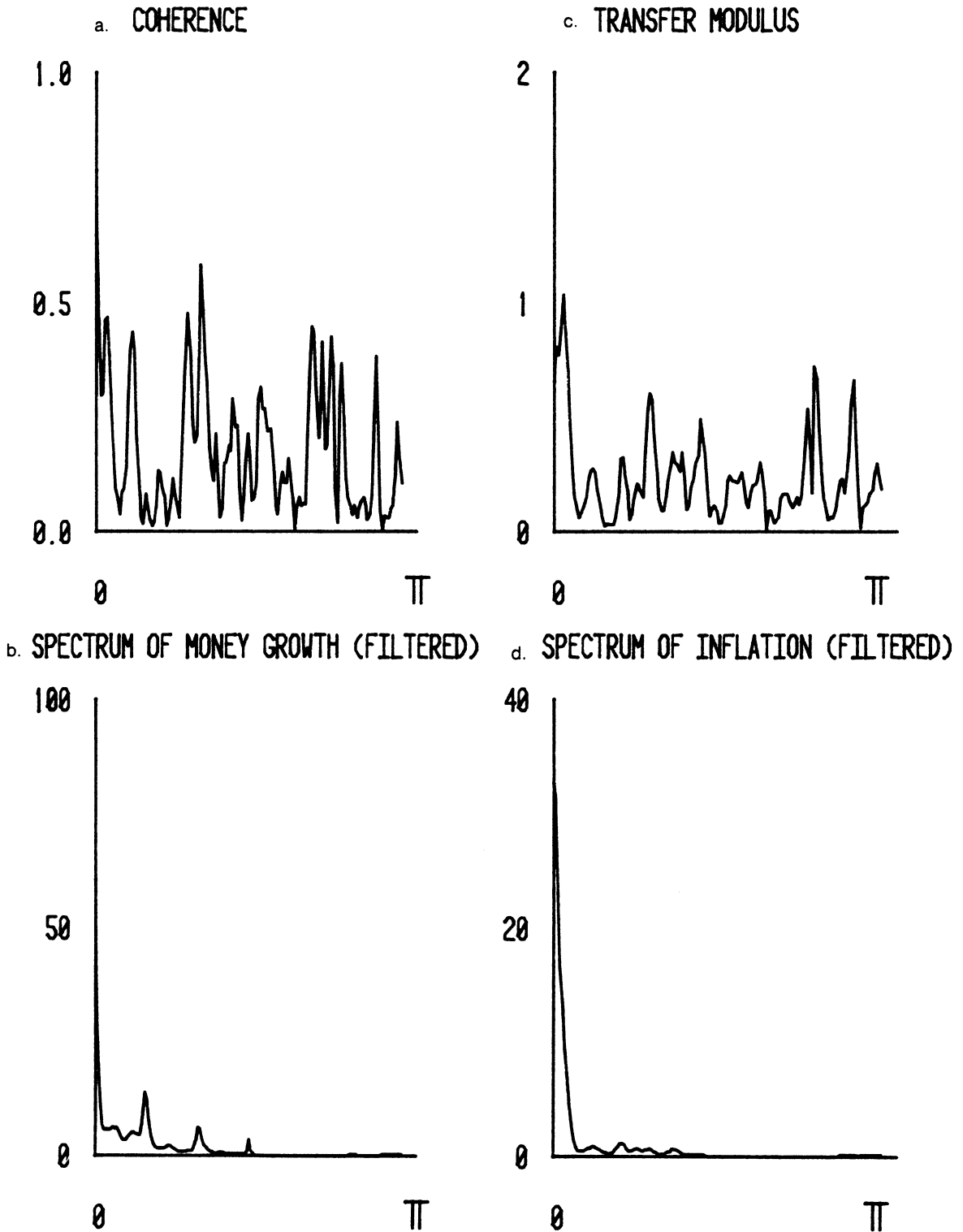
MONEY GROWTH (FILTERED)
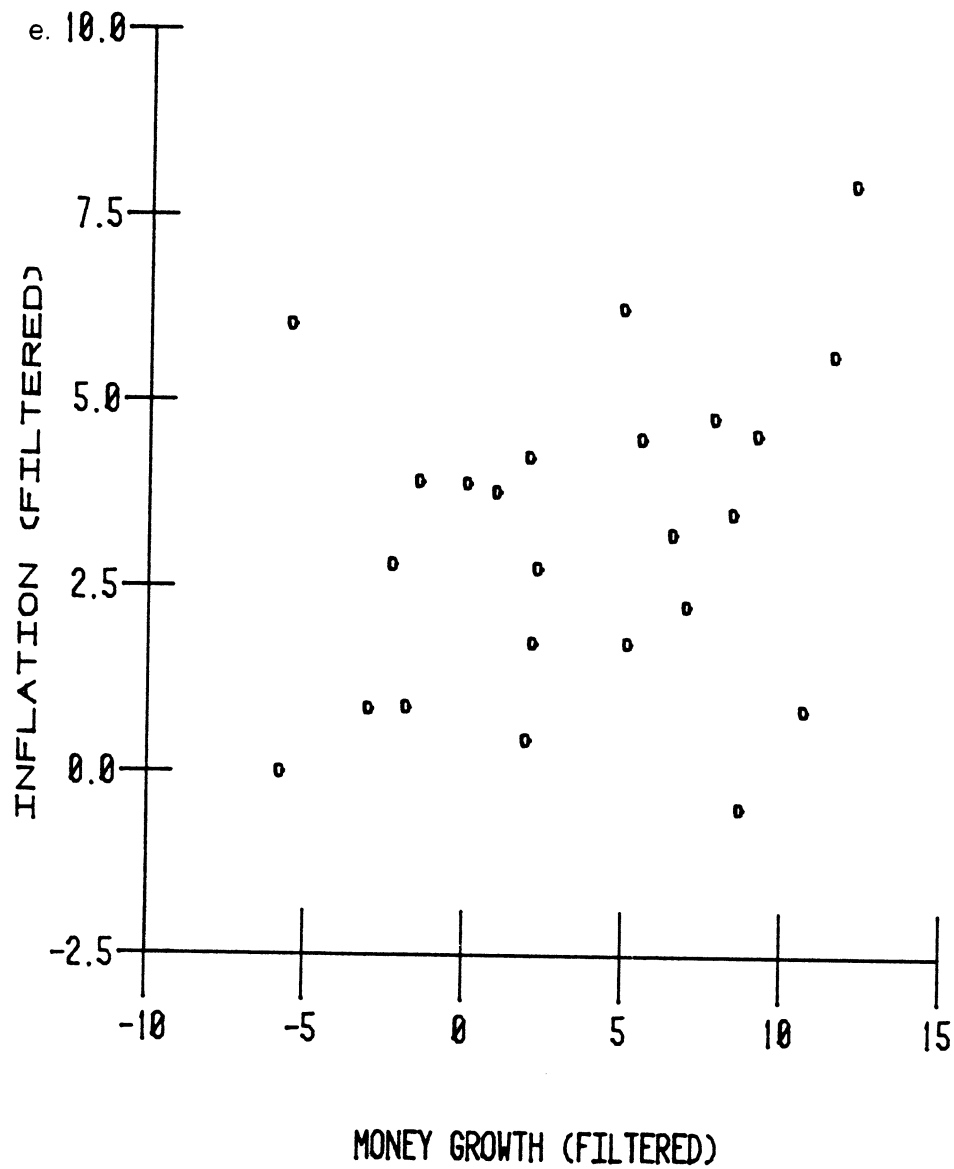
SLOPE = 0.02

$\bar{R}^2 = 0.00$

Figure 2.2   BETA = 0.5

a. GAIN OF FILTER
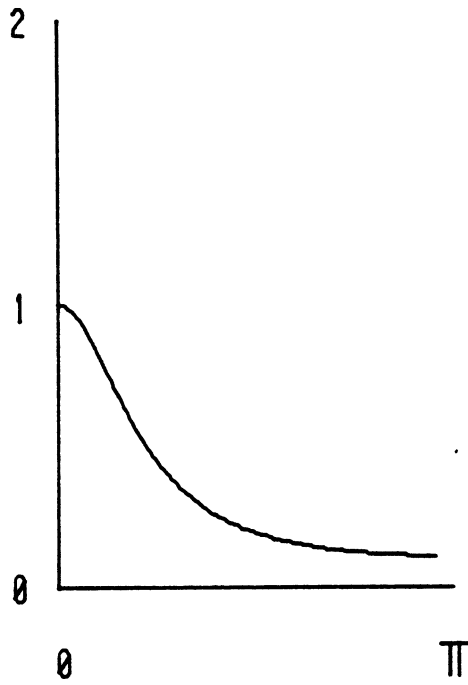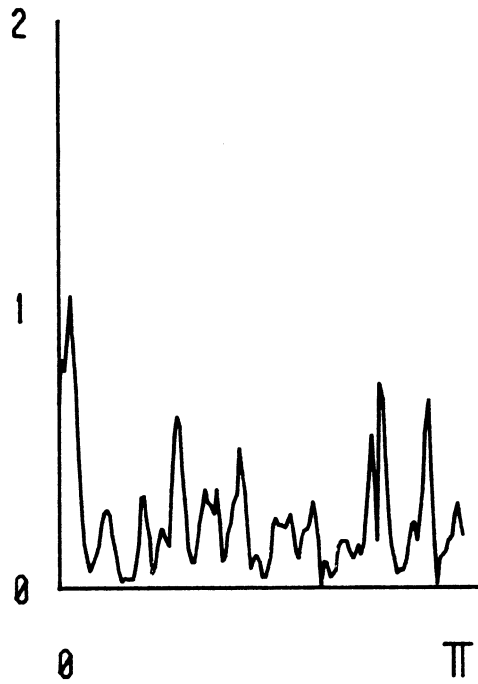
c. TRANSFER MODULUS

b. SPECTRUM OF MONEY GROWTH (FILTERED)
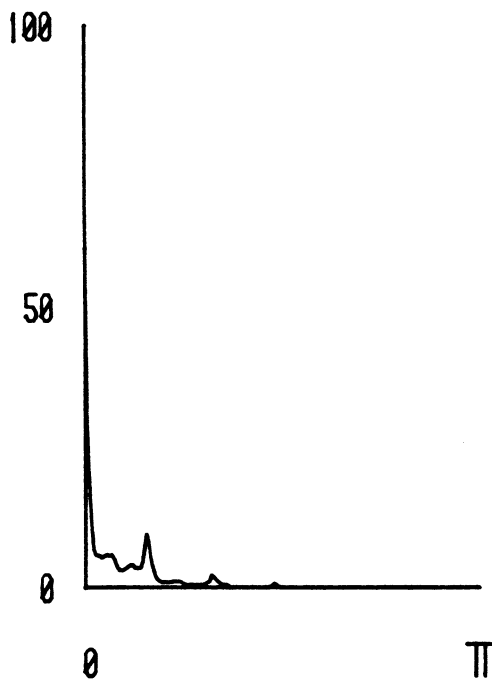
d. SPECTRUM OF INFLATION (FILTERED)
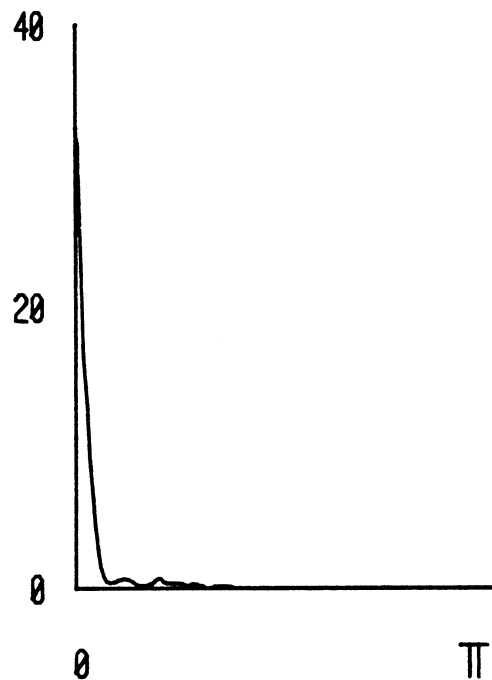
Figure 2.2 (cont.) BETA = 0.5

SLOPE = 0.09

$\bar{R}^2 = 0.03$
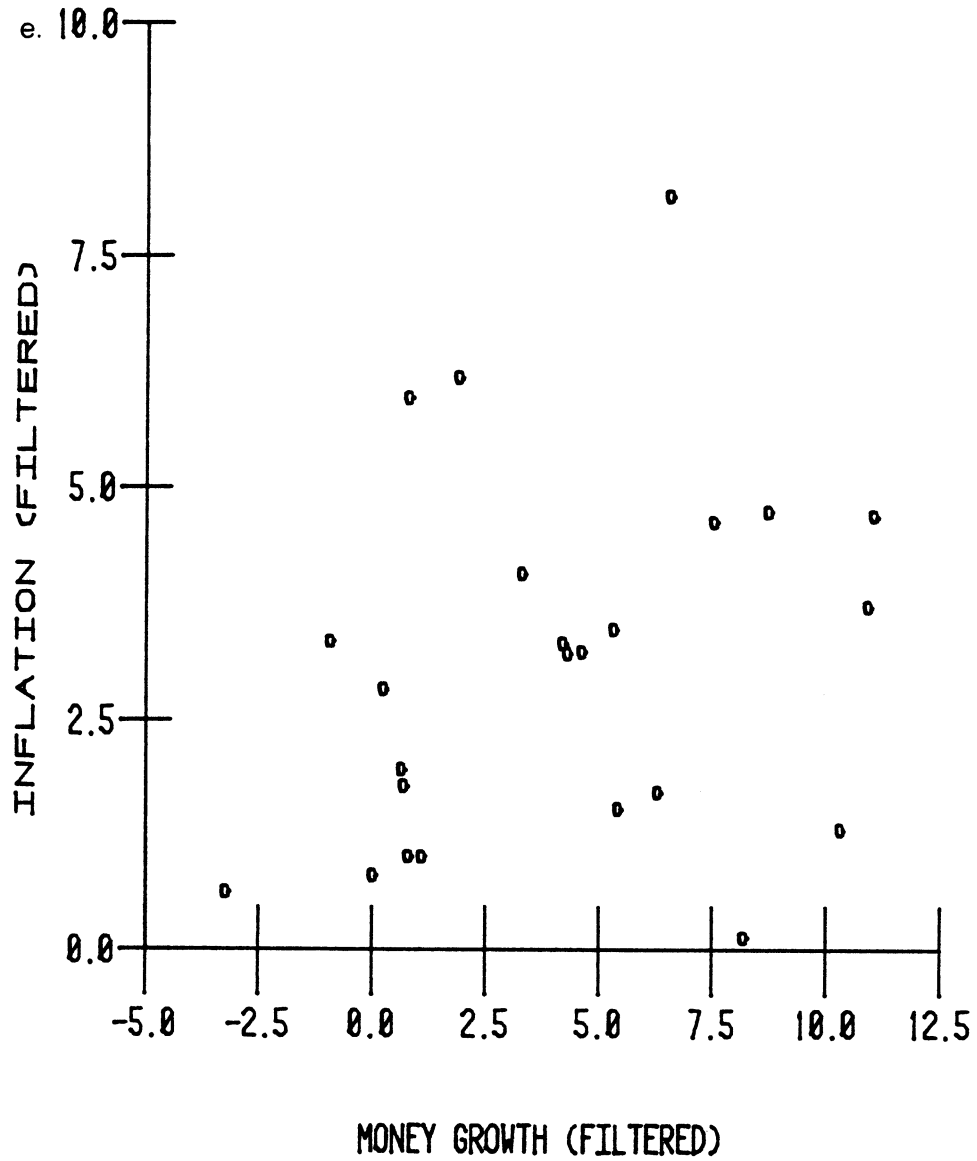
Figure 2.3   BETA = 0.9

a. GAIN OF FILTER
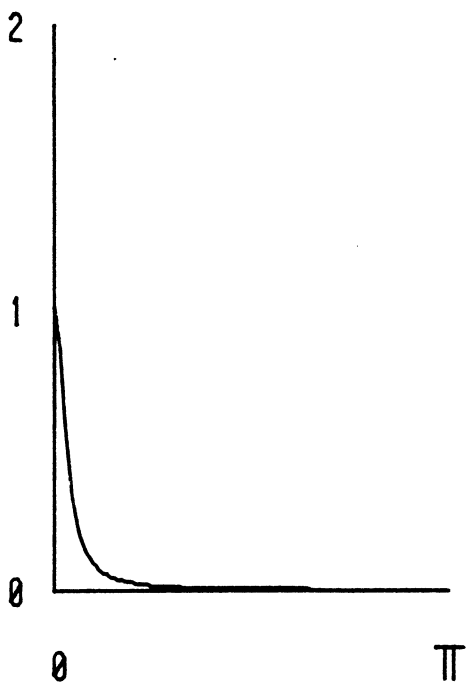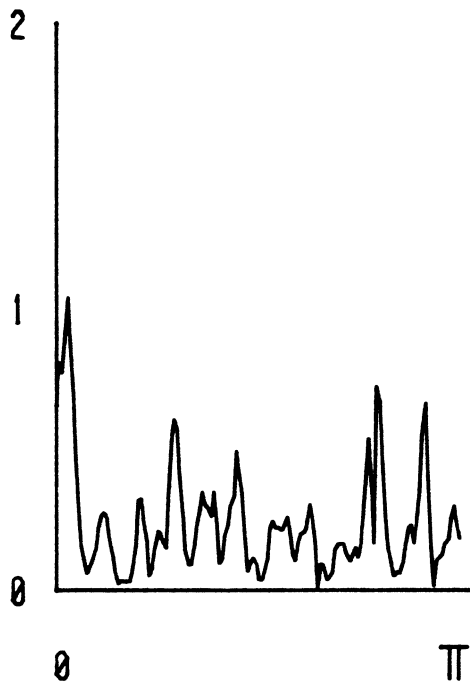
c. TRANSFER MODULUS

b. SPECTRUM OF MONEY GROWTH (FILTERED)
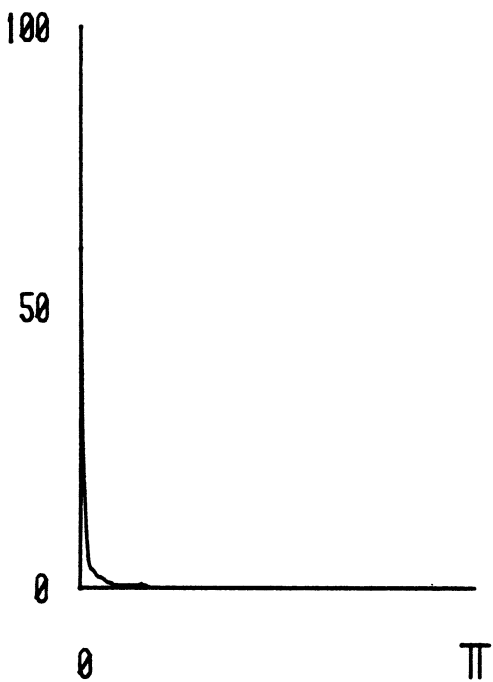
d. SPECTRUM OF INFLATION (FILTERED)
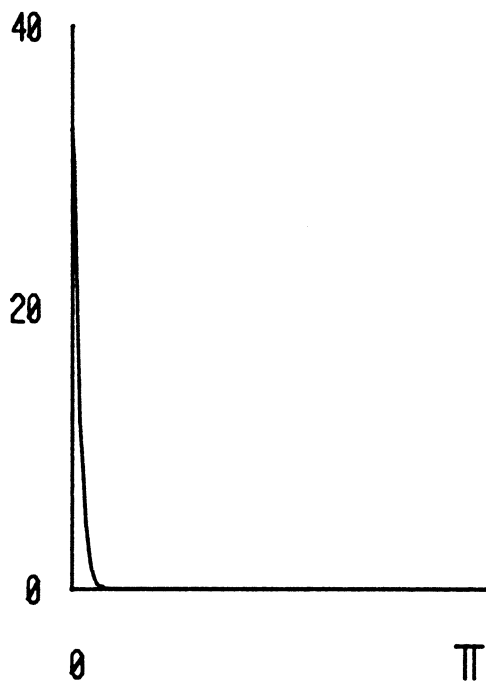
Figure 2.3 (cont.) BETA = 0.9

MONEY GROWTH (FILTERED)
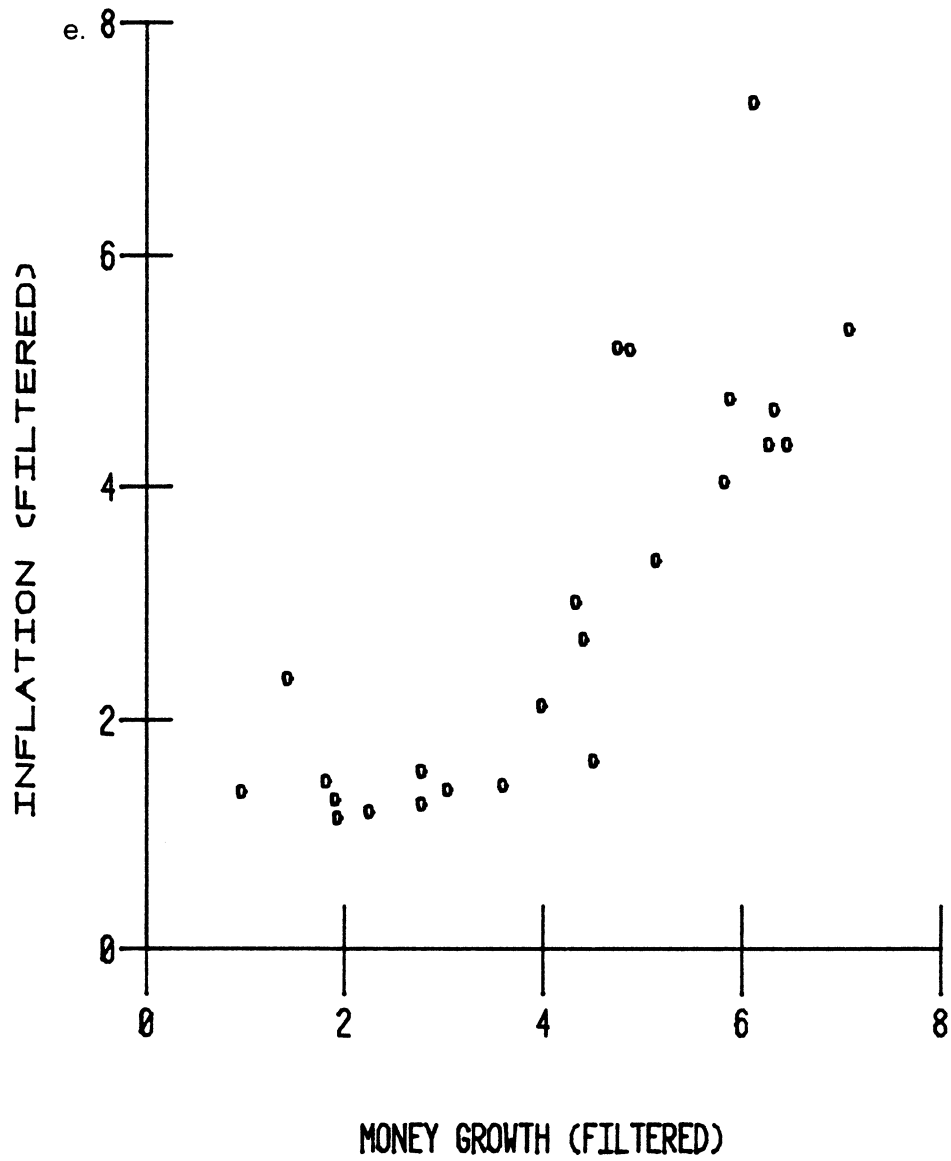
SLOPE = 0.87
$\bar{R}^2 = 0.66$

Figure 2.4 BETA = 0.95

a. GAIN OF FILTER
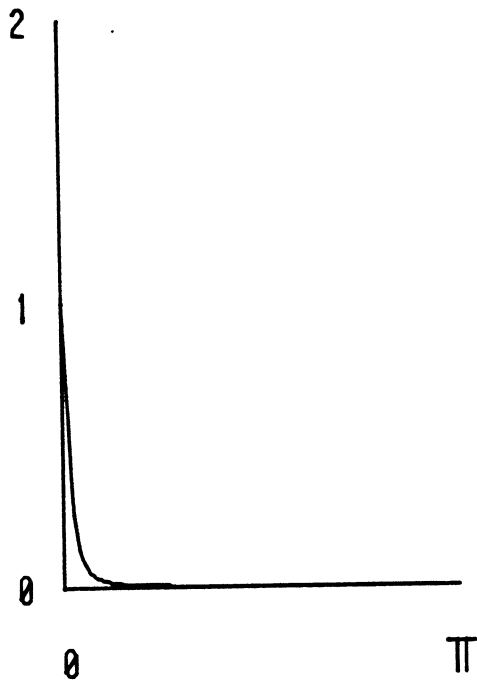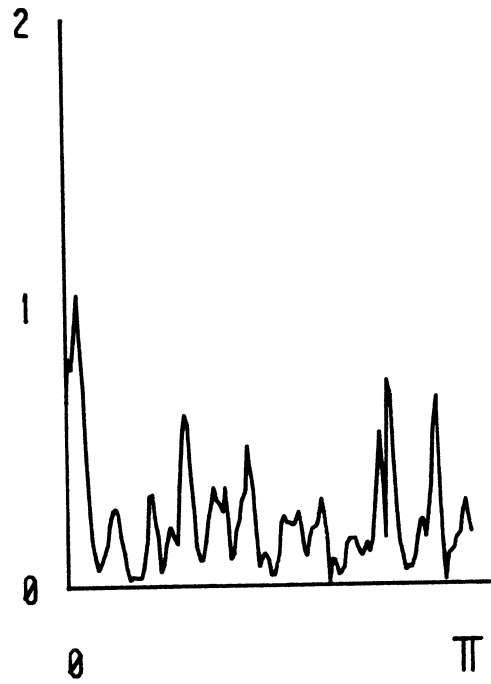
c. TRANSFER MODULUS

b. SPECTRUM OF MONEY GROWTH (FILTERED)
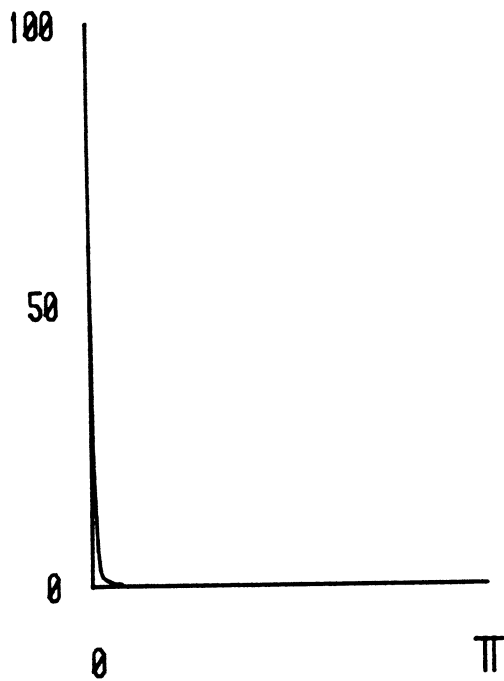
d. SPECTRUM OF INFLATION (FILTERED)
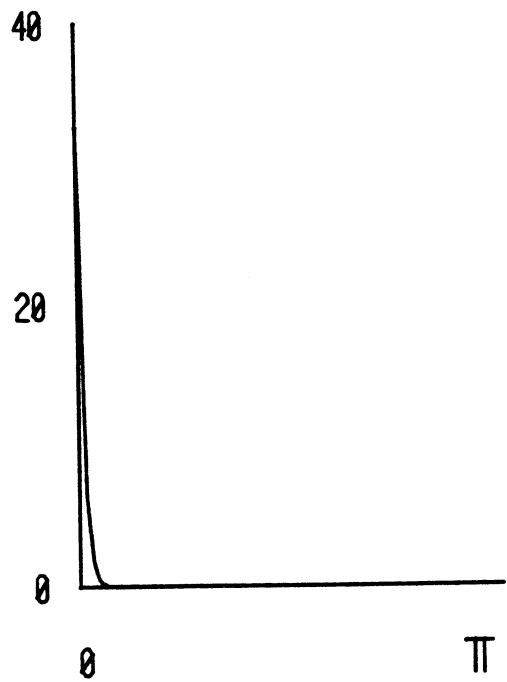
Figure 2.4 (cont.)   BETA = 0.95
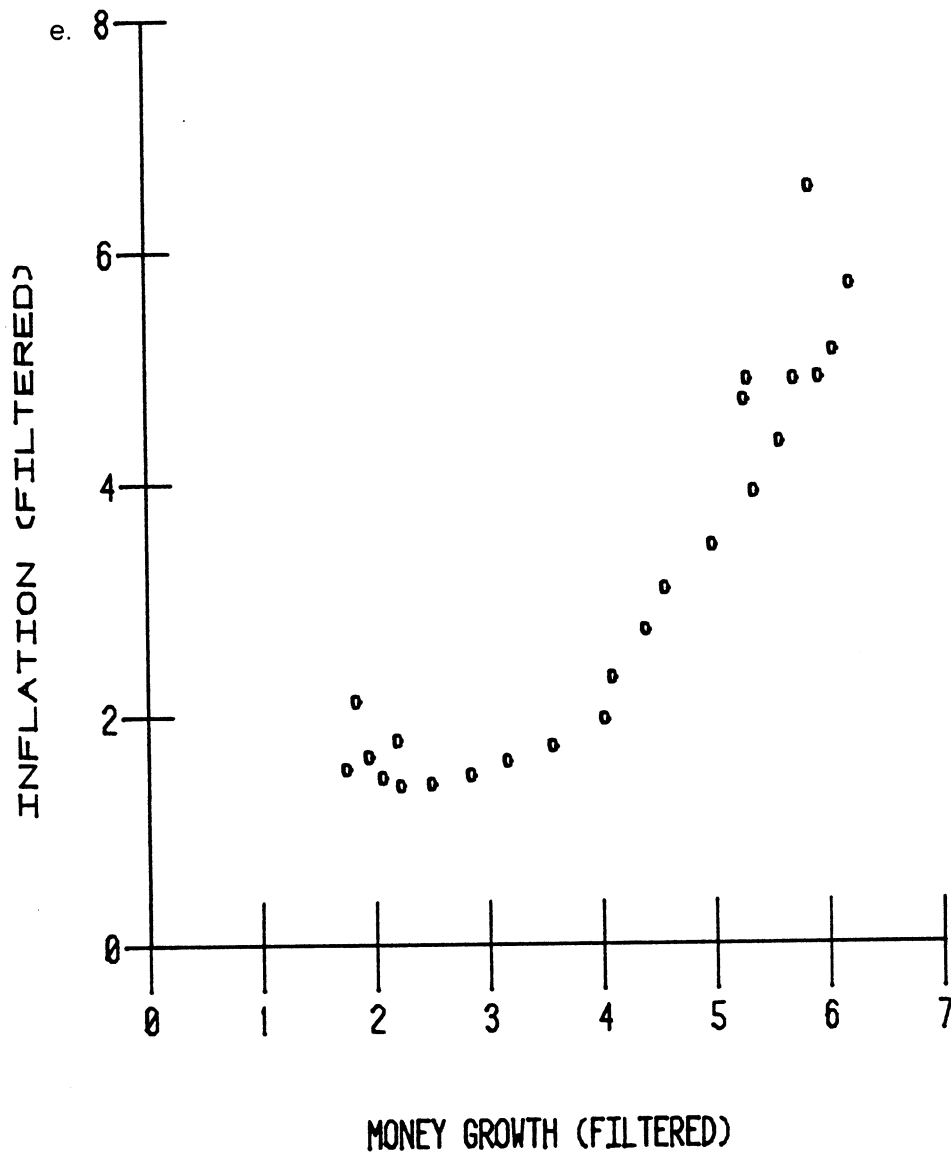
SLOPE = 0.99
$\bar{R}^2 = 0.81$

Figure 2.5   BETA = 0.5

a. GAIN OF FILTER
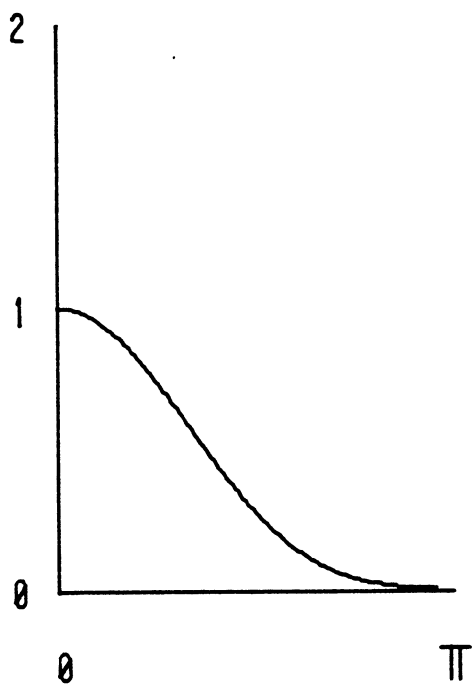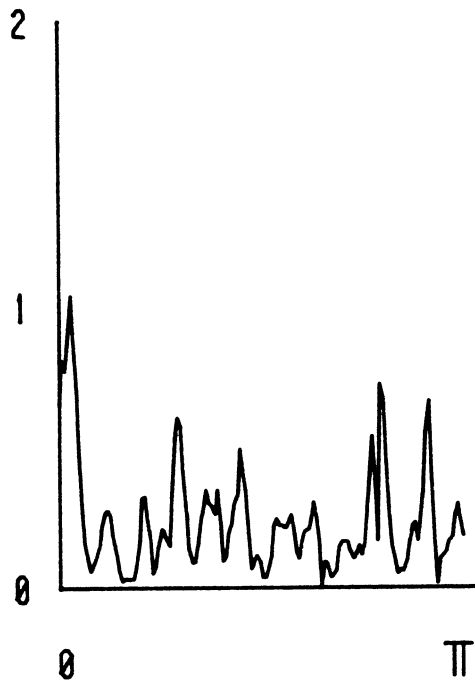
c. TRANSFER MODULUS

b. SPECTRUM OF MONEY GROWTH (FILTERED)
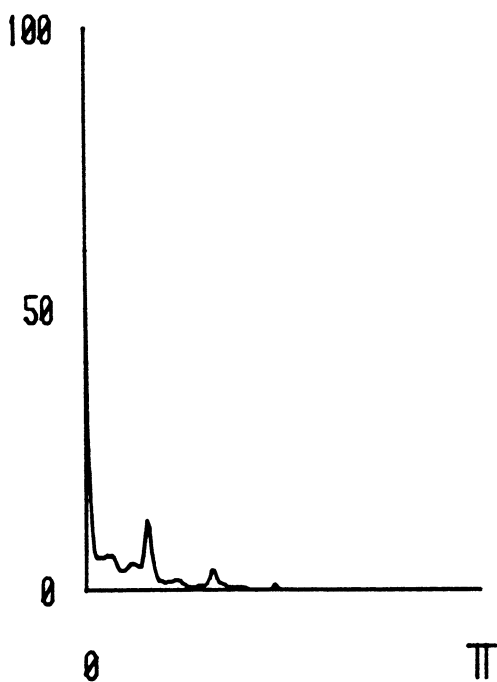
d. SPECTRUM OF INFLATION (FILTERED)

Figure 2.5 (cont.)   BETA = 0.5

e. 10.0

INFLATION (FILTERED)
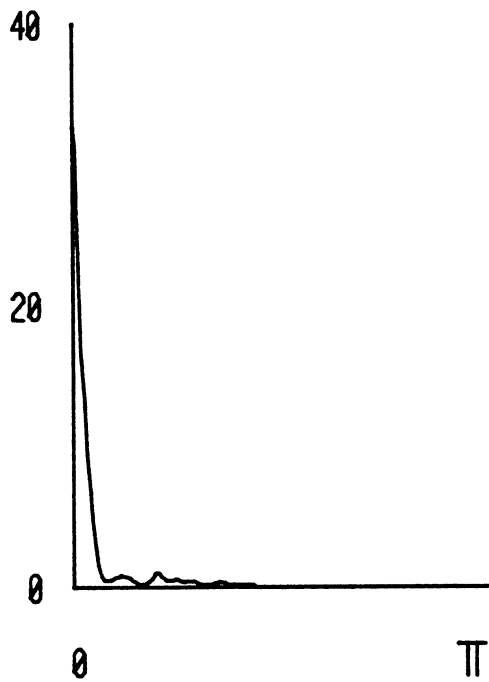
MONEY GROWTH (FILTERED)
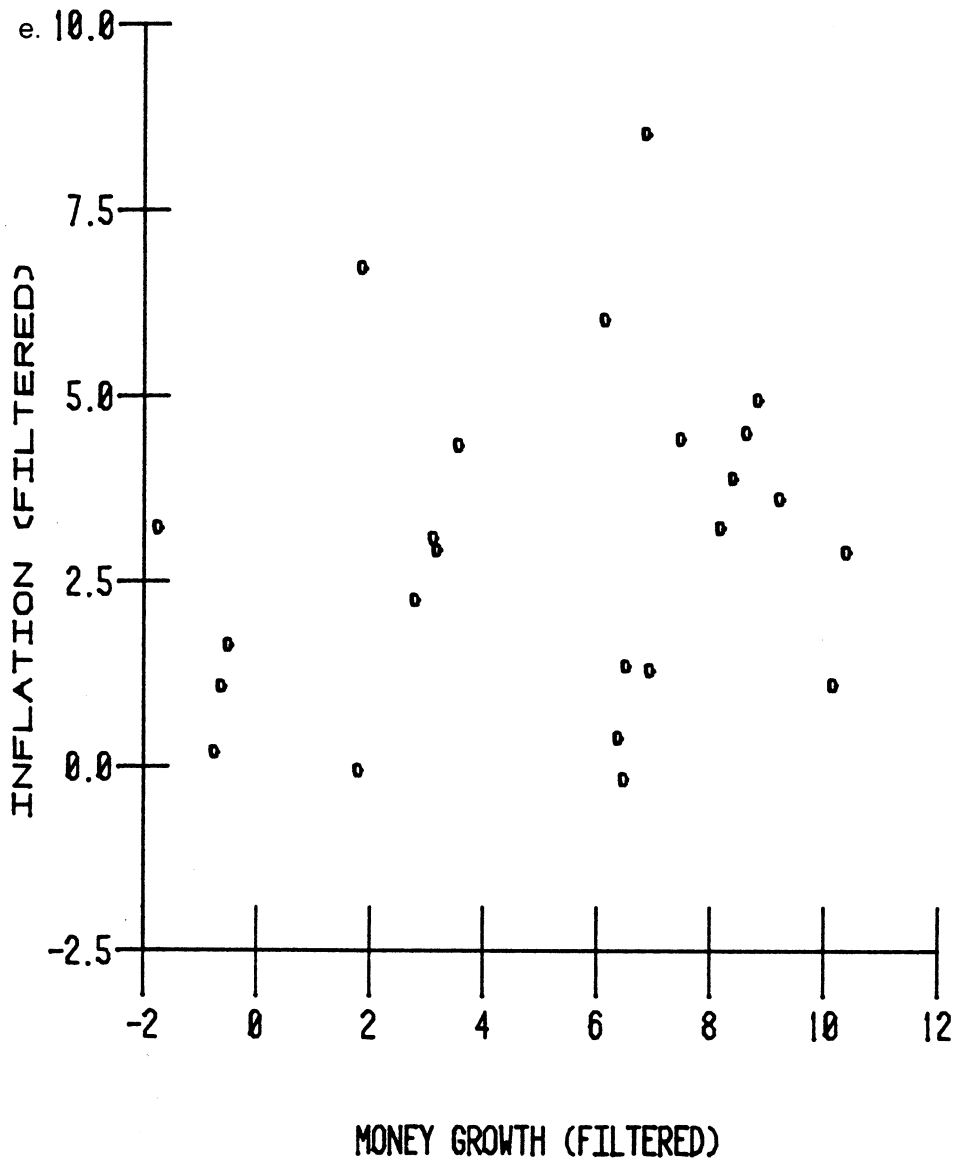
SLOPE = 0.02
$\bar{R}^2 = 0.00$

Figure 2.6 BETA = 0.9

a. GAIN OF FILTER
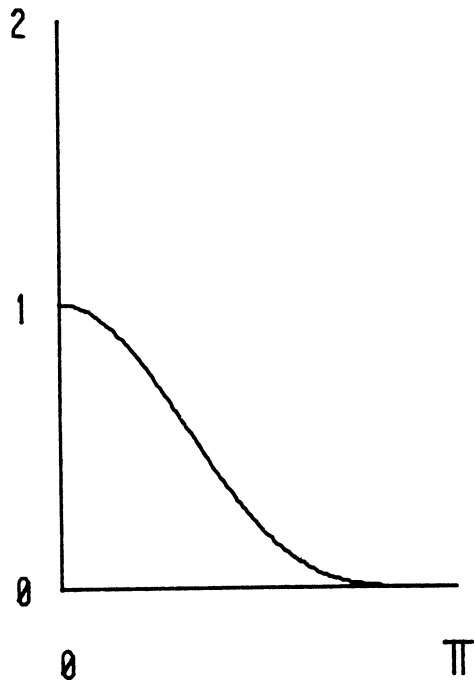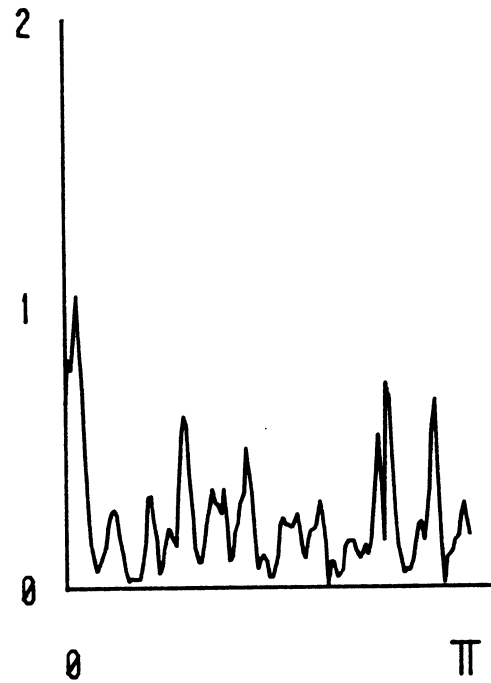
c. TRANSFER MODULUS

b. SPECTRUM OF MONEY GROWTH (FILTERED)
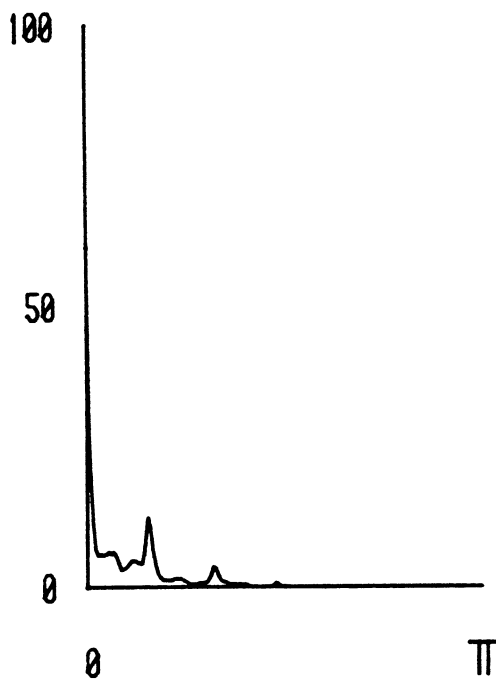
d. SPECTRUM OF INFLATION (FILTERED)
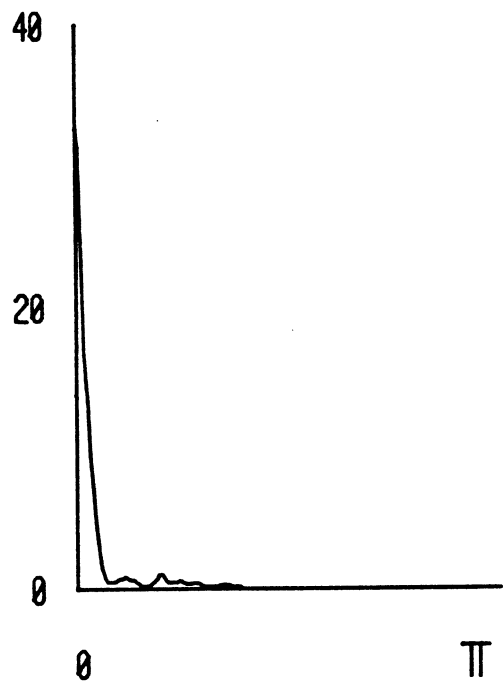
Figure 2.6 (cont.) BETA = 0.9
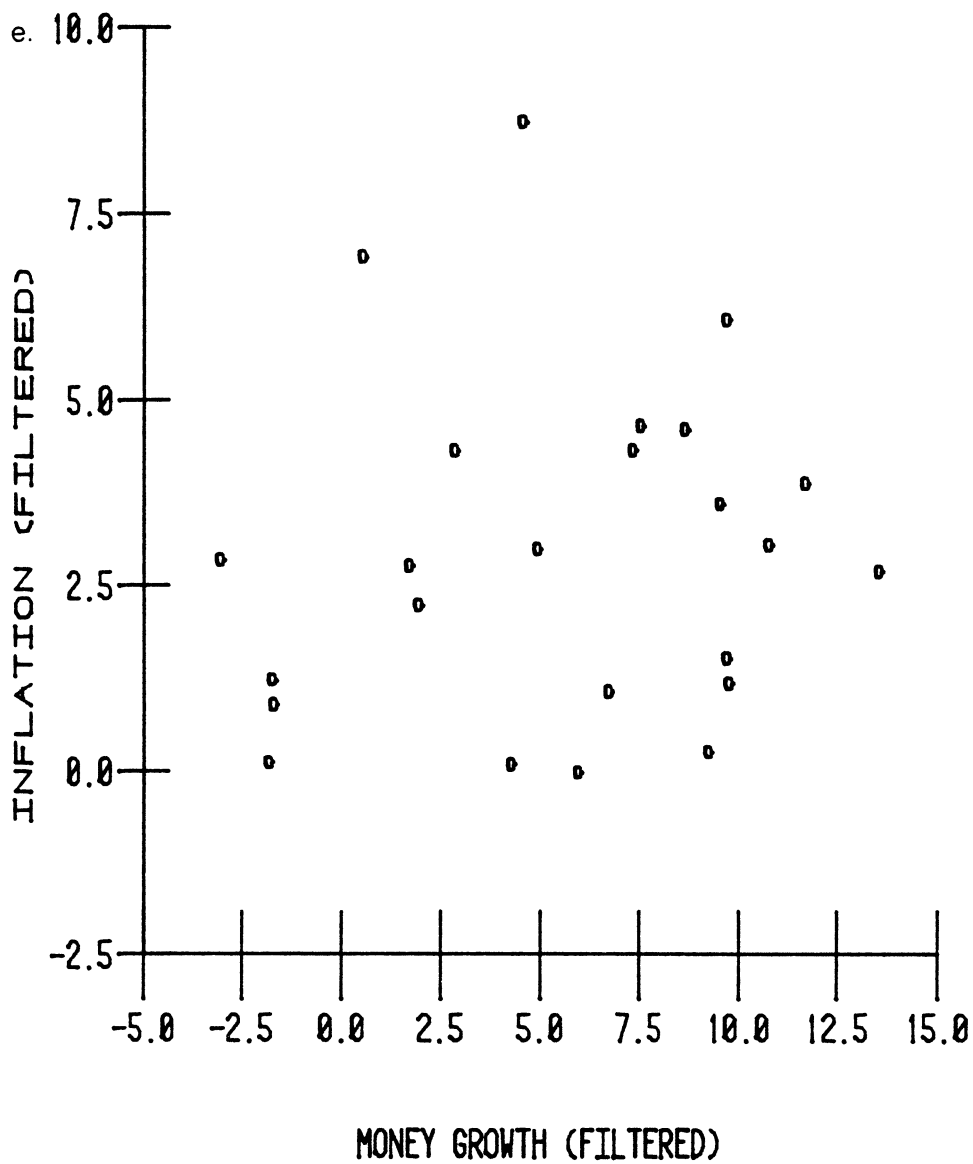
SLOPE = 0.02

$\bar{R}^2 = 0.00$

Figure 2.7 BETA = 0.95

a. GAIN OF FILTER
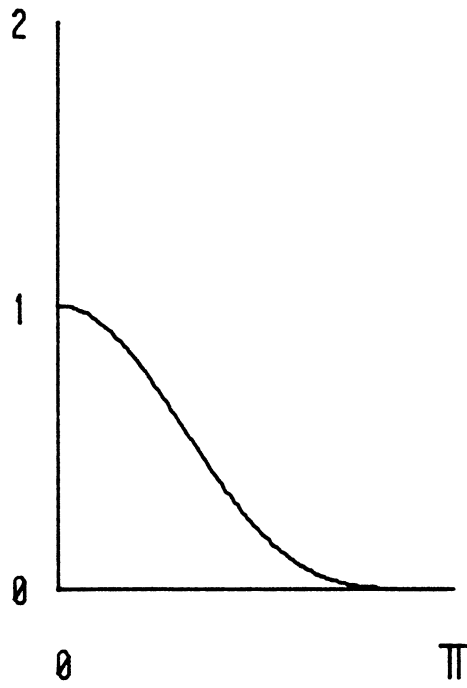
c. TRANSFER MODULUS
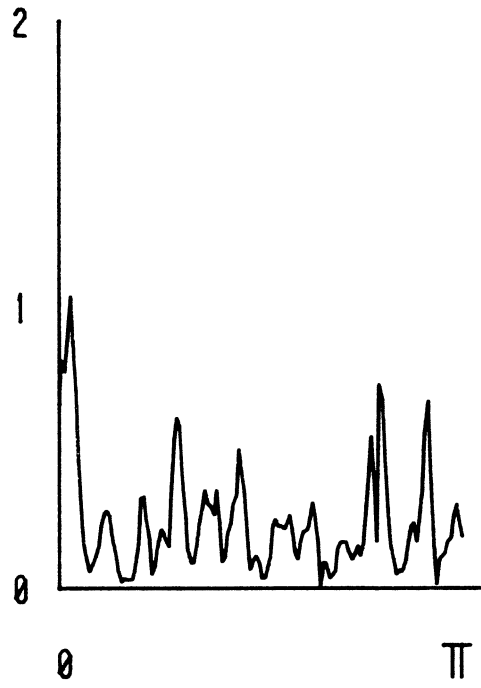
b. SPECTRUM OF MONEY GROWTH (FILTERED)
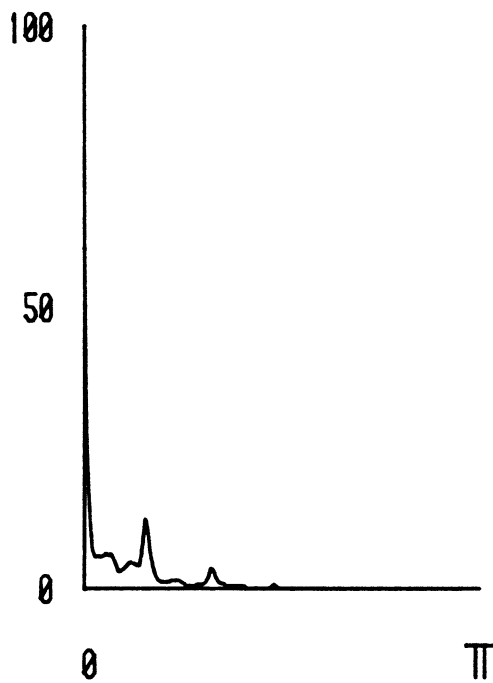
d. SPECTRUM OF INFLATION (FILTERED)
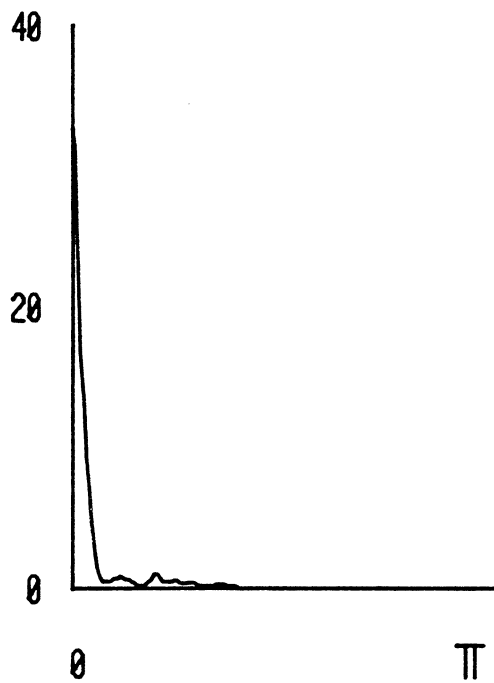
Figure 2.7 (cont.) BETA = 0.95
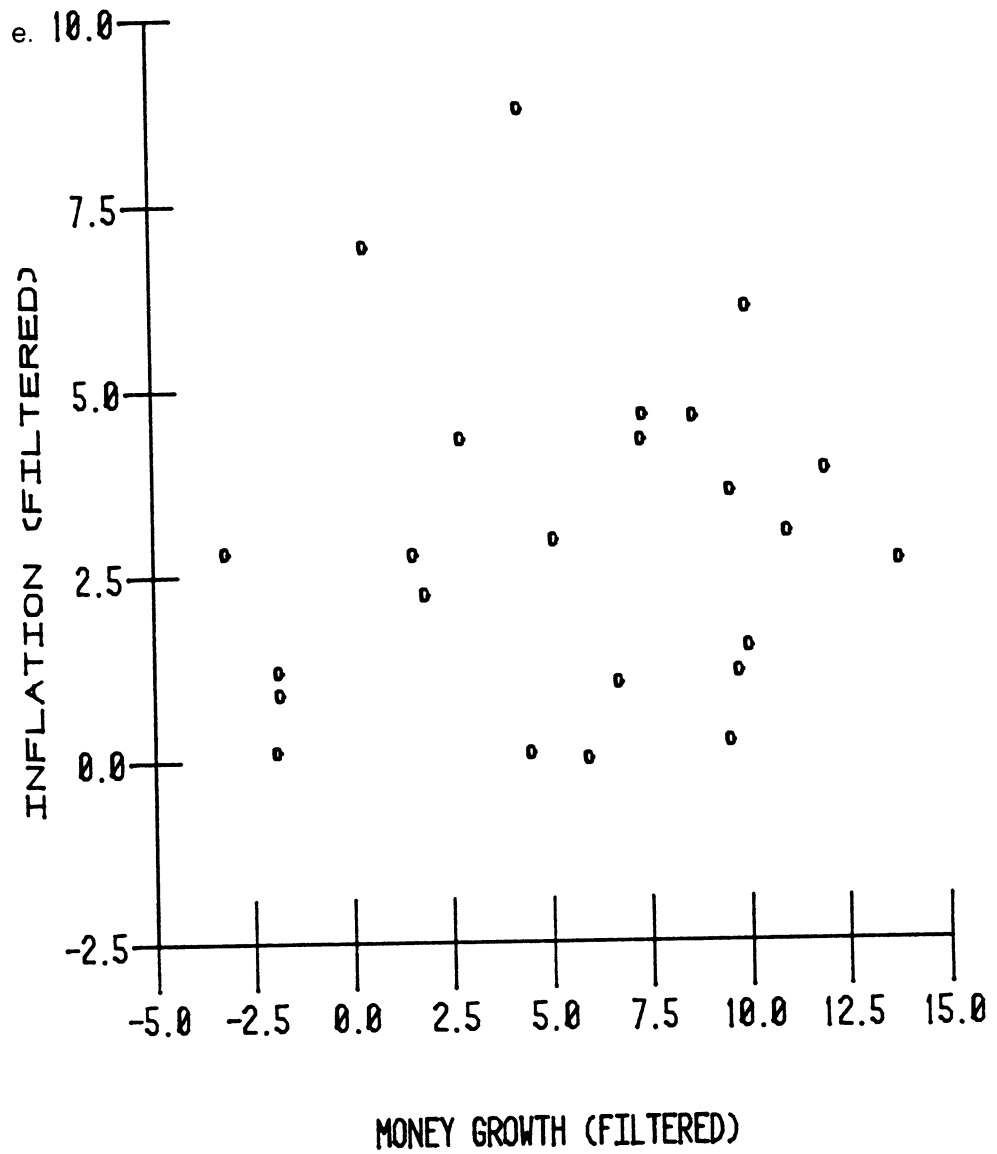
SLOPE = 0.02

$\bar{R}^2 = 0.00$

Figure 6.1

MONEY

a. MOVING AVERAGE COEFFICIENTS
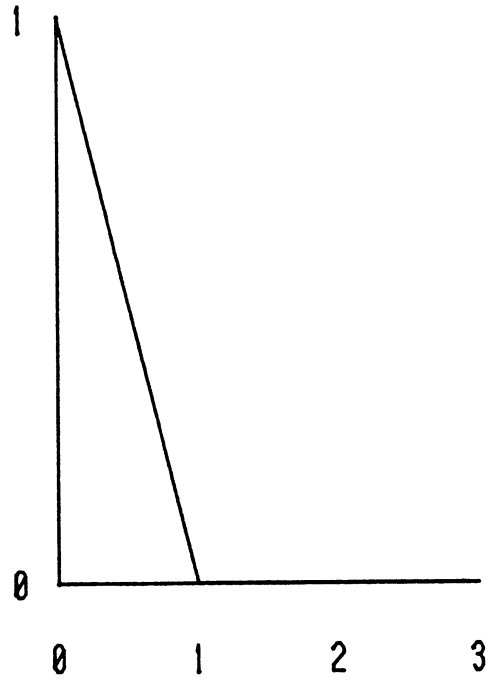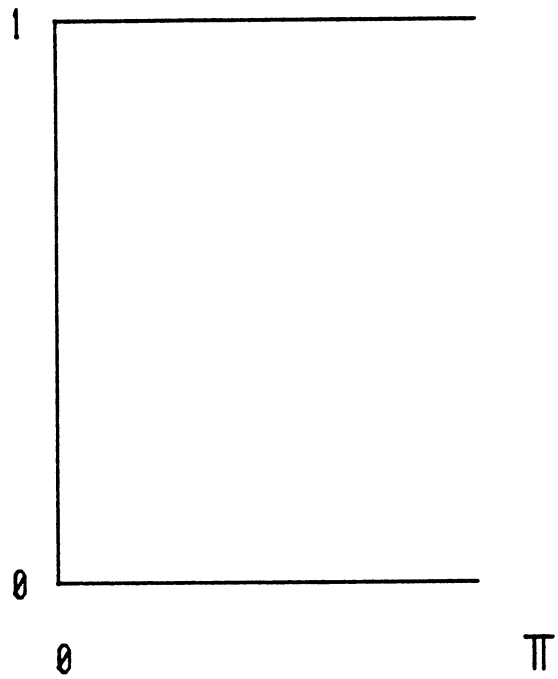
b. SPECTRUM

Figure 6.1 (cont.)

c. SPECTRUM OF PRICES
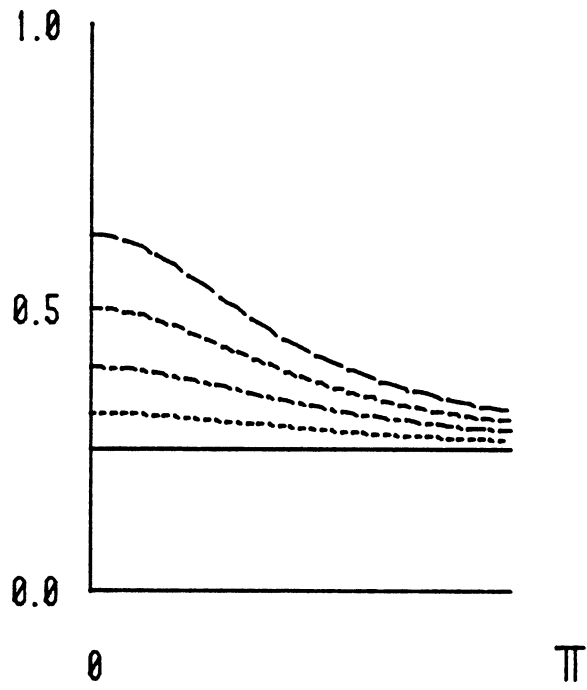
d. SPECTRUM OF CAPITAL
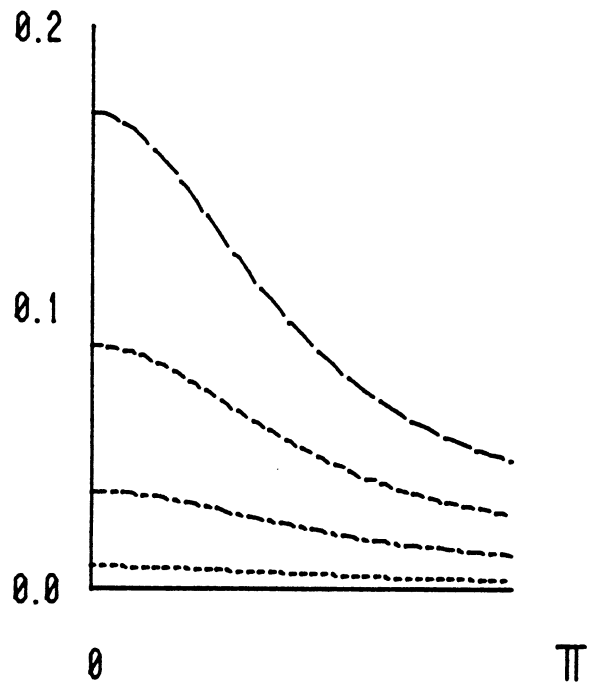
Figure 6.1 (cont.)

PRICES

e. MOVING AVERAGE COEFFICIENTS
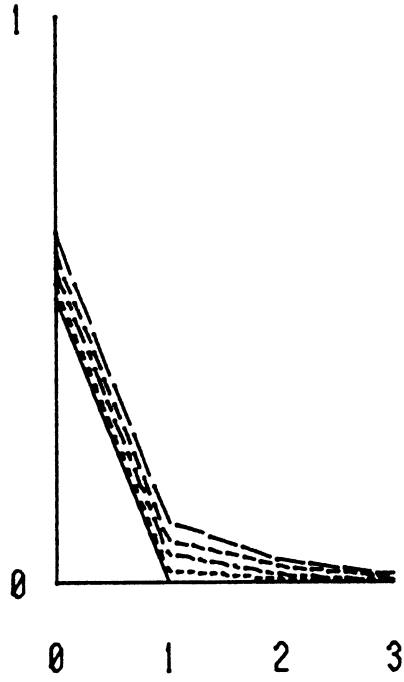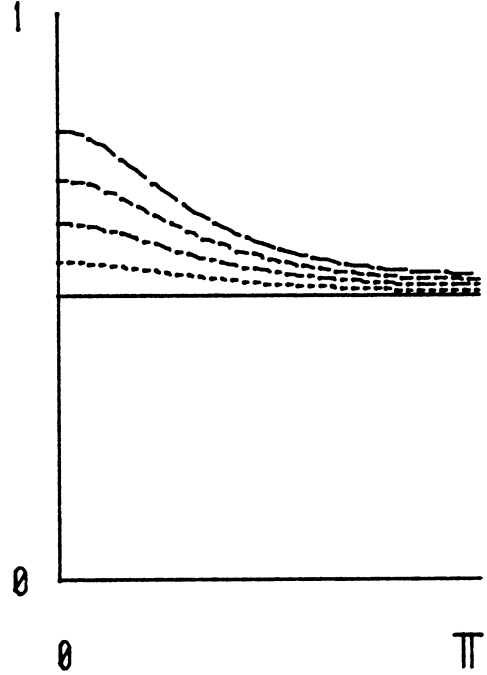
g. TRANSFER MODULUS

f. DISTRIBUTED LAG COEFFICIENTS ON MONEY
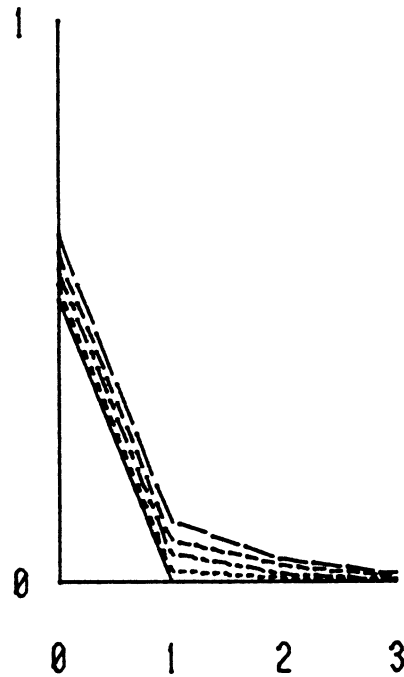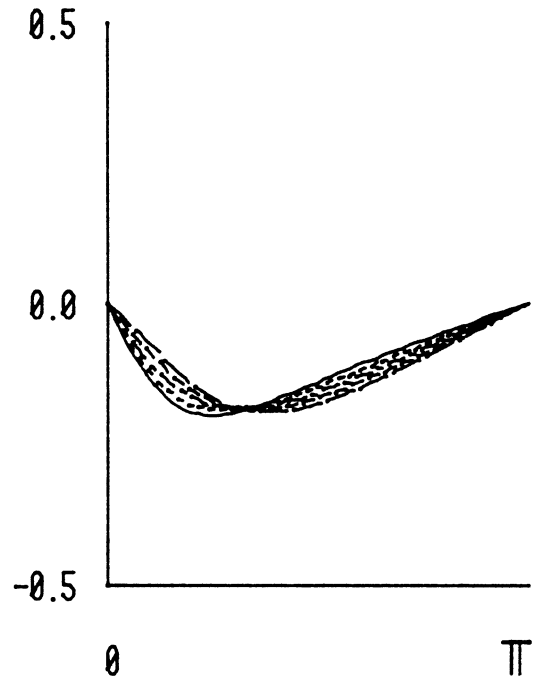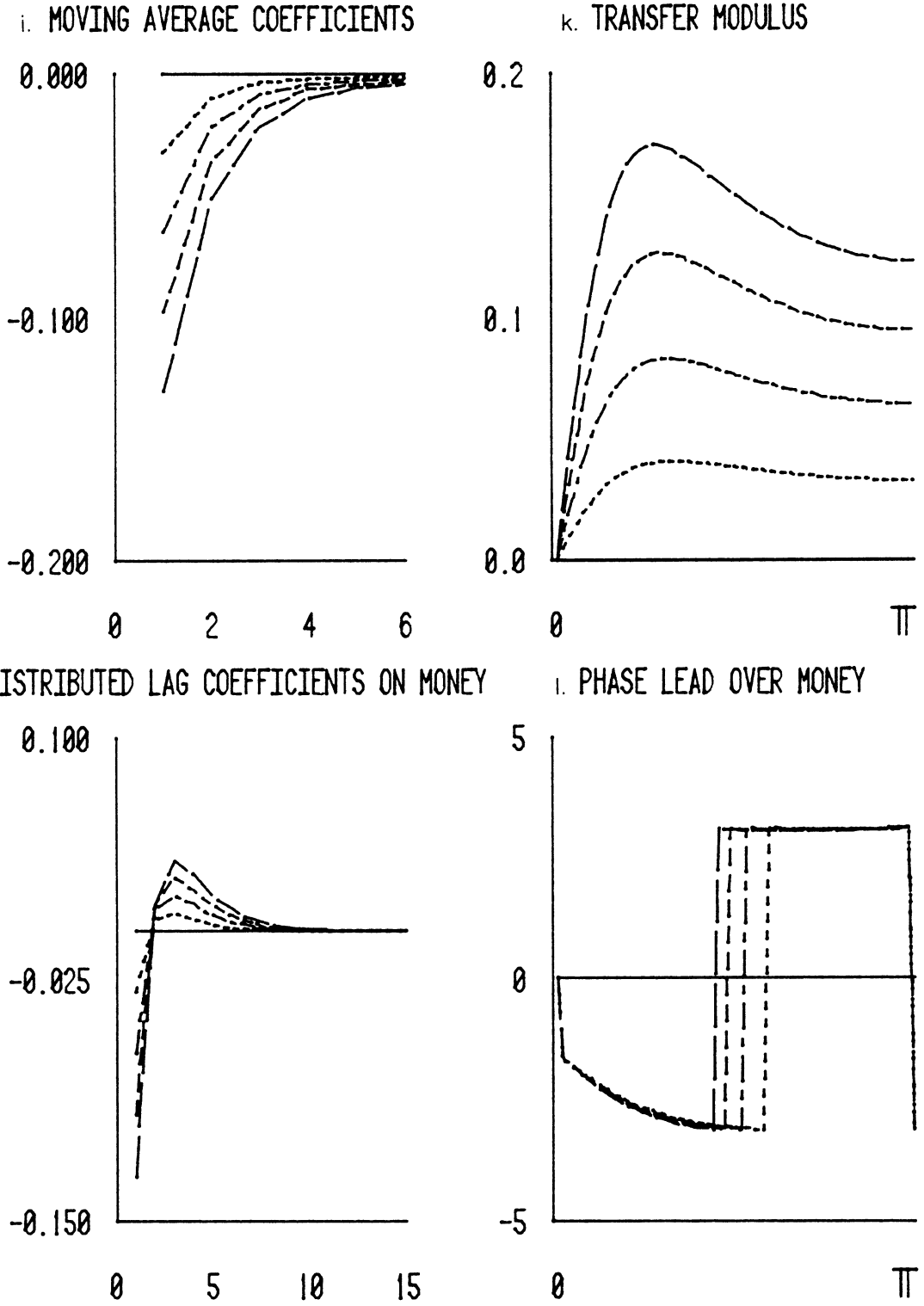

h. PHASE LEAD OVER MONEY
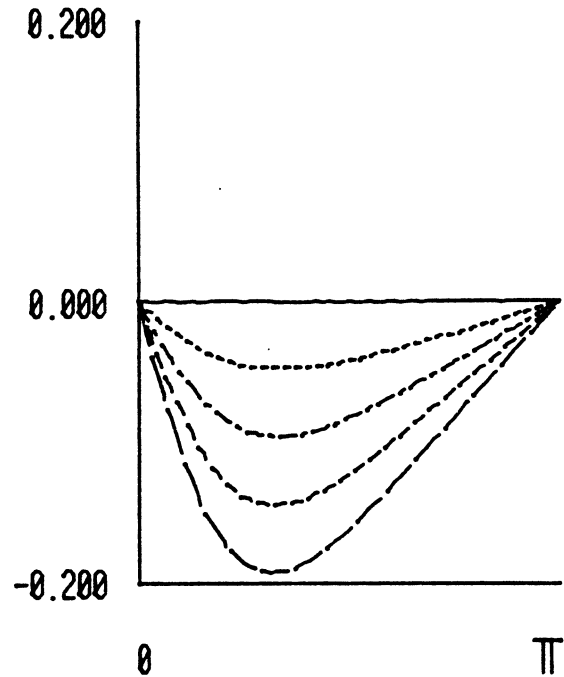
Figure 6.1 (cont.)

CAPITAL

i. MOVING AVERAGE COEFFICIENTS
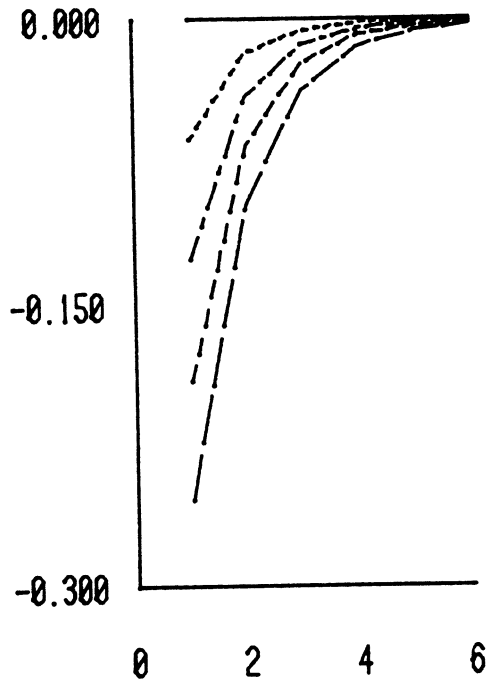
k. TRANSFER MODULUS
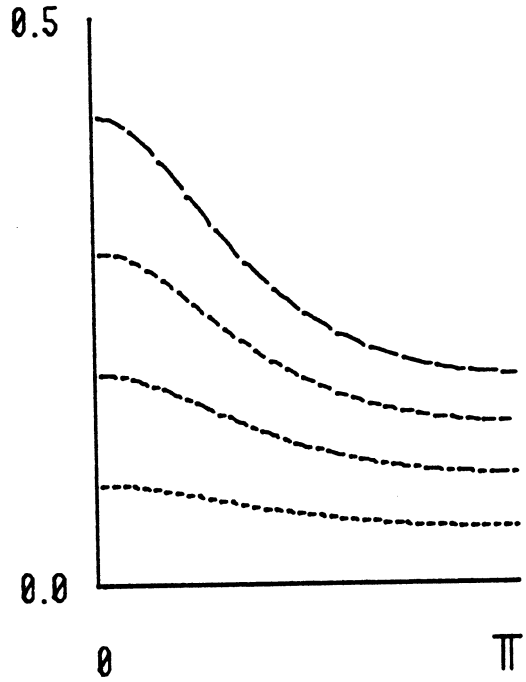
j. DISTRIBUTED LAG COEFFICIENTS ON MONEY
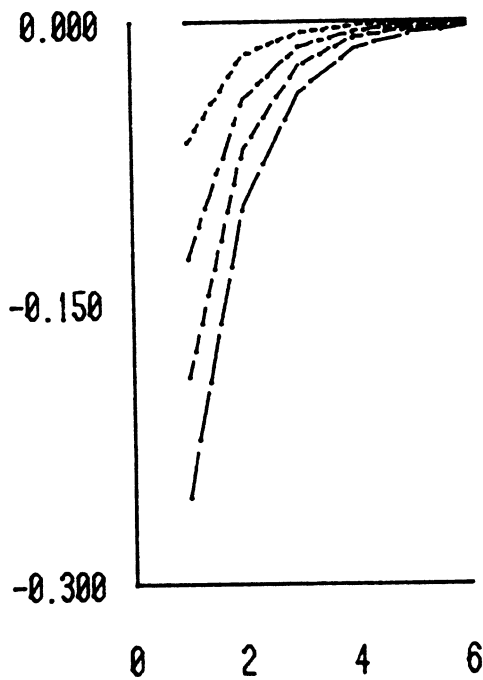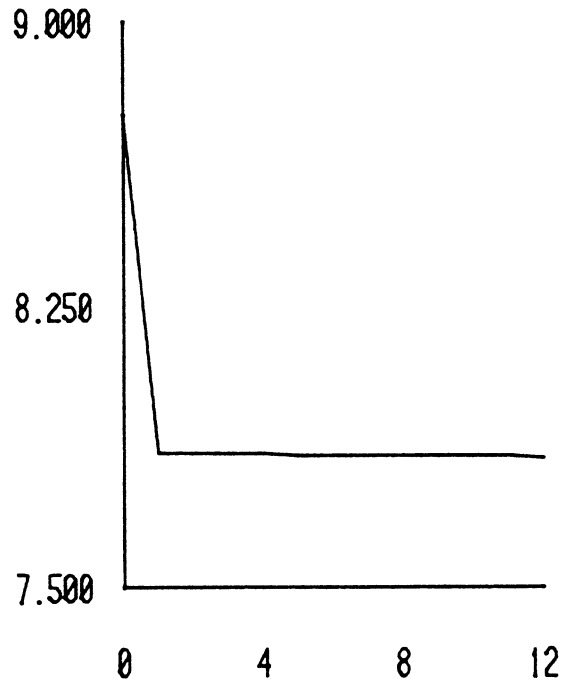
l. PHASE LEAD OVER MONEY
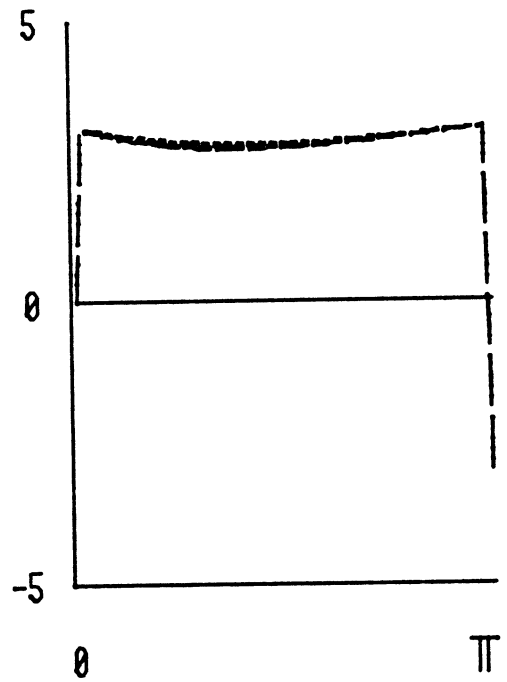
Figure 6.2

MONEY

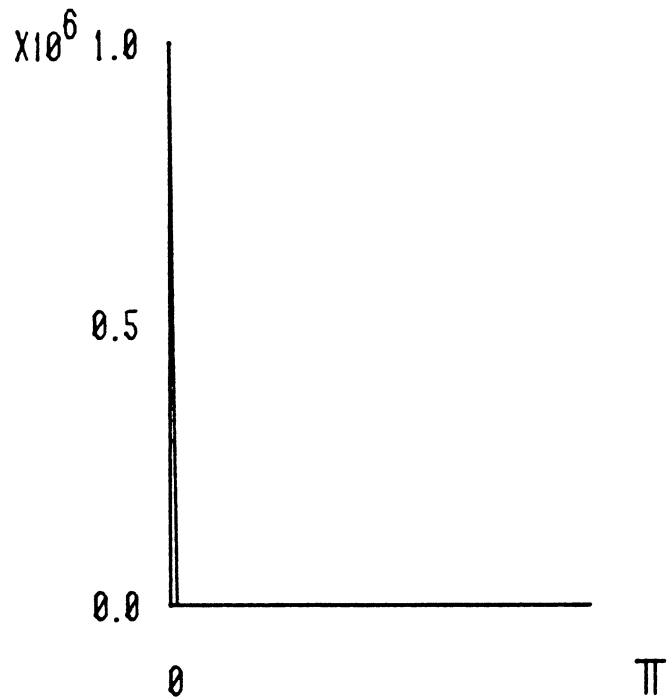a. MOVING AVERAGE COEFFICIENTS
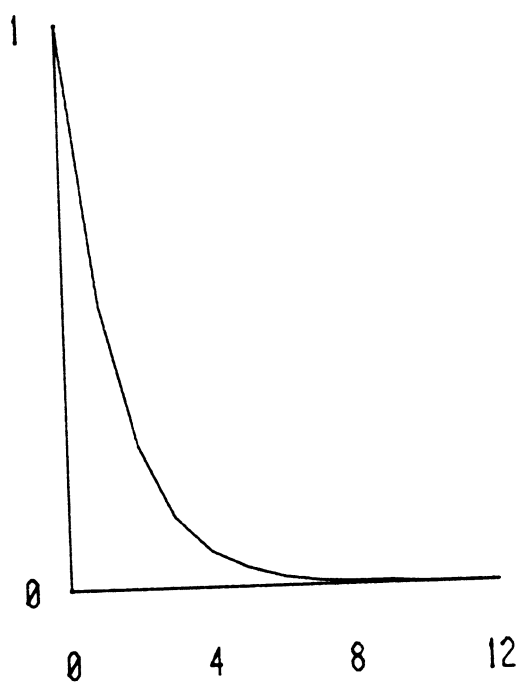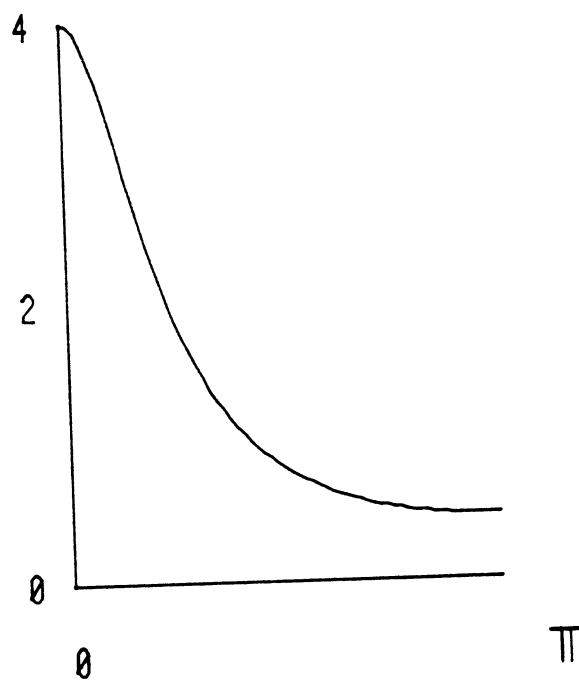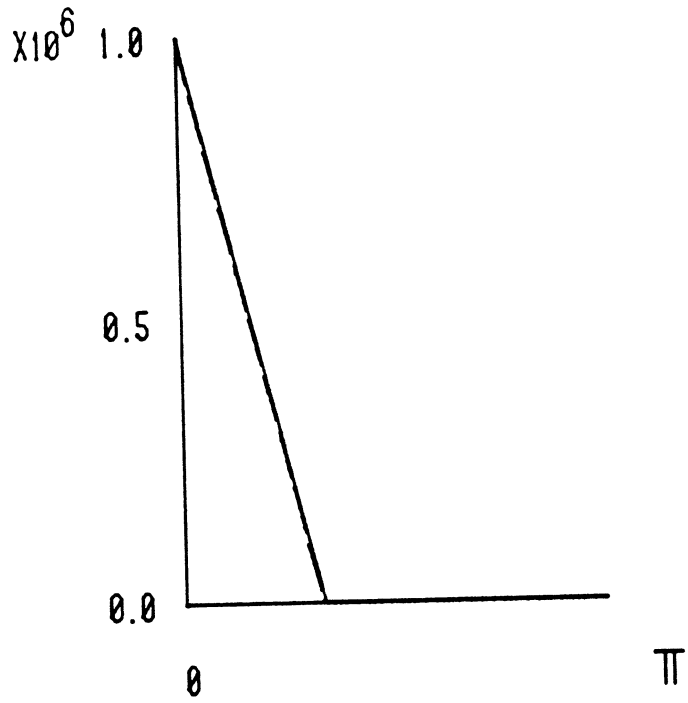

b. SPECTRUM

Figure 6.2 (cont.)

c. SPECTRUM OF PRICES
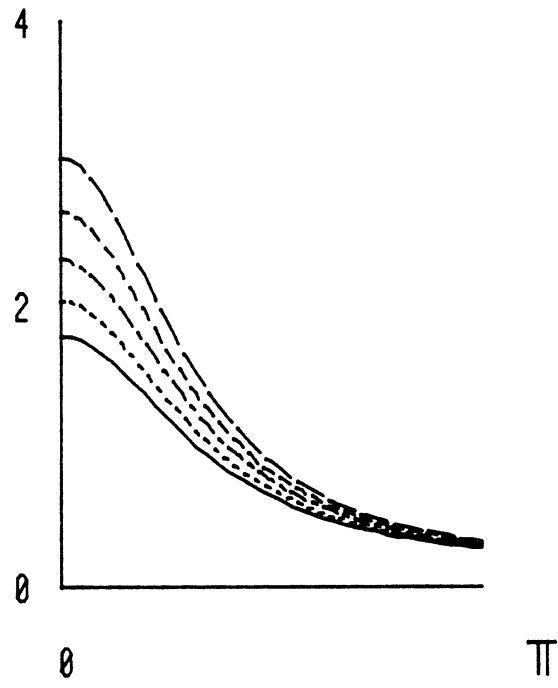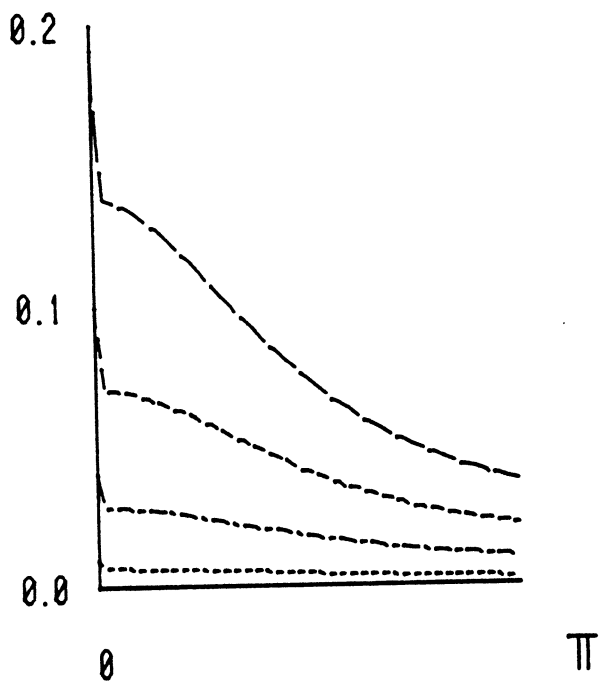

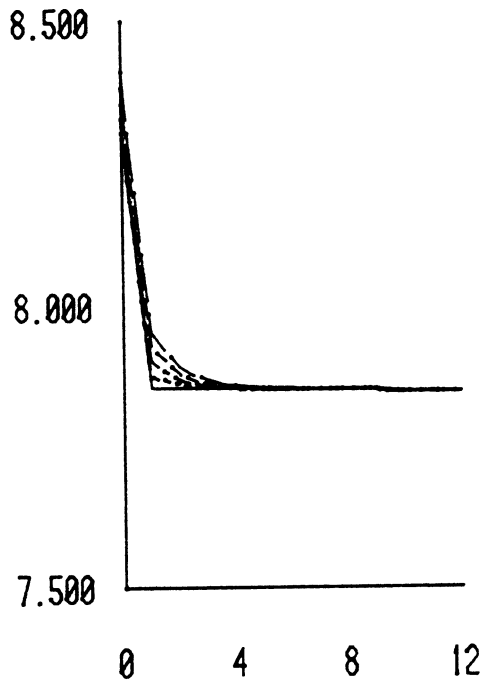d. SPECTRUM OF CAPITAL
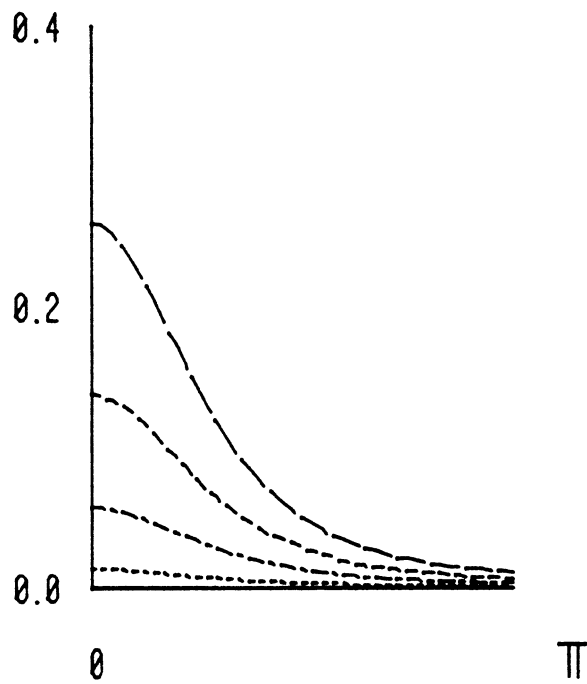
Figure 6.2 (cont.)

PRICES

e. MOVING AVERAGE COEFFICIENTS
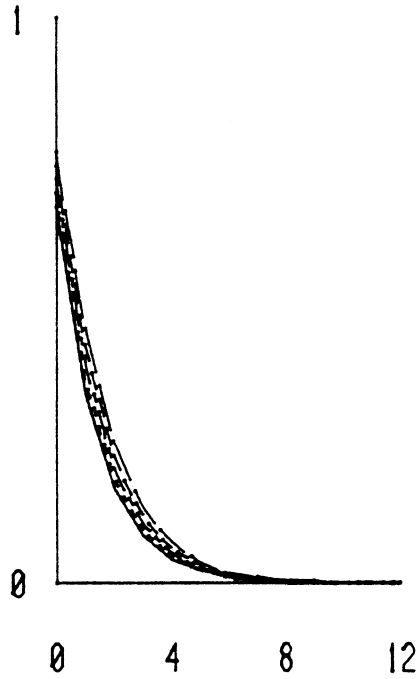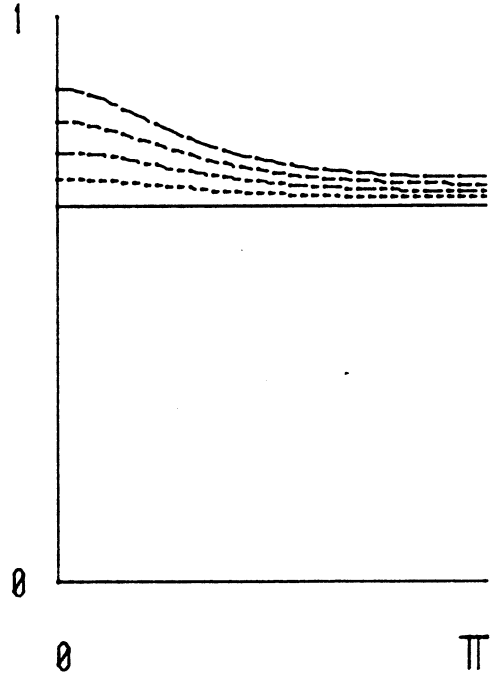
g. TRANSFER MODULUS

f. DISTRIBUTED LAG COEFFICIENTS ON MONEY
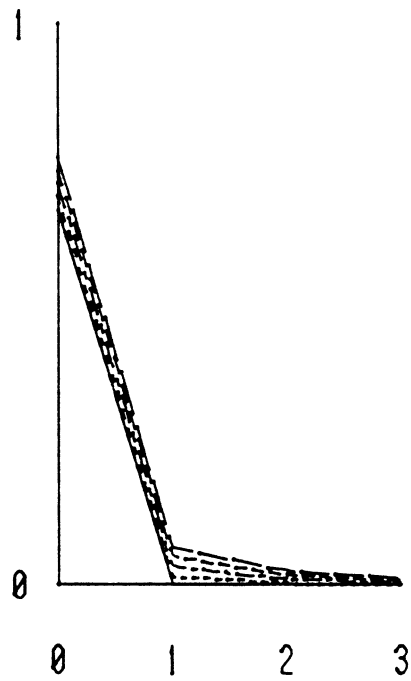
h. PHASE LEAD OVER MONEY
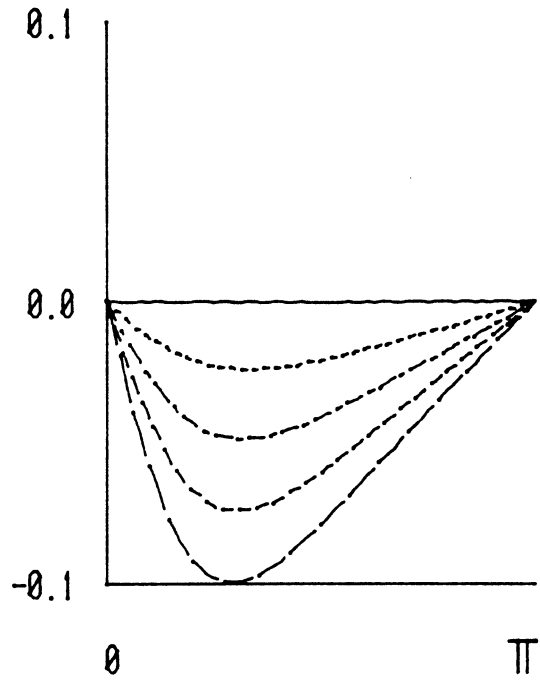
Figure 6.2 (cont.)

CAPITAL

i. MOVING AVERAGE COEFFICIENTS
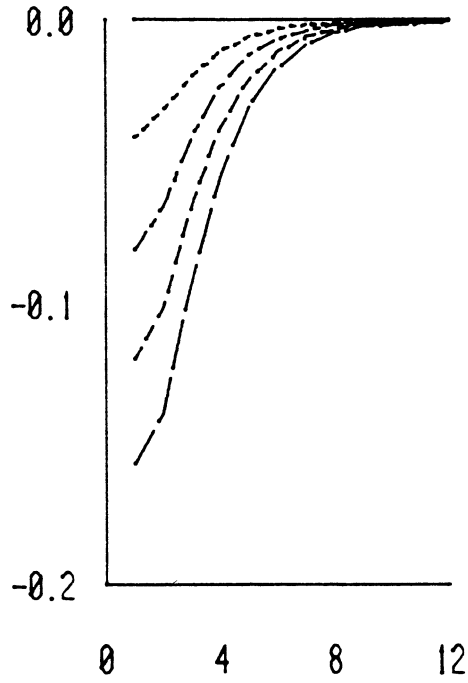
k. TRANSFER MODULUS
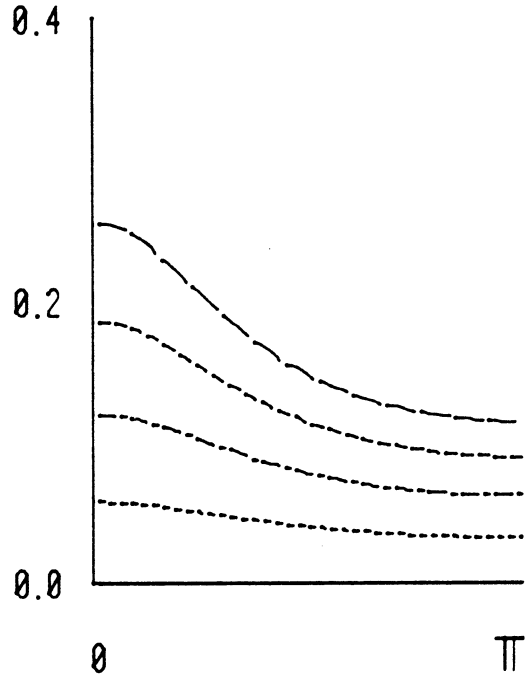
j. DISTRIBUTED LAG COEFFICIENTS ON MONEY
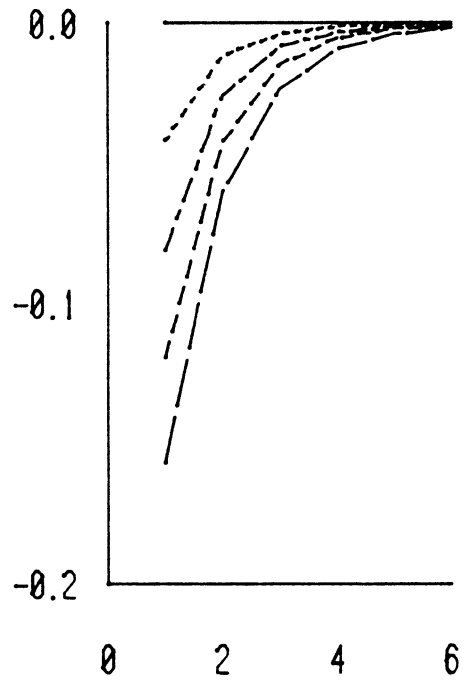
l. PHASE LEAD OVER MONEY
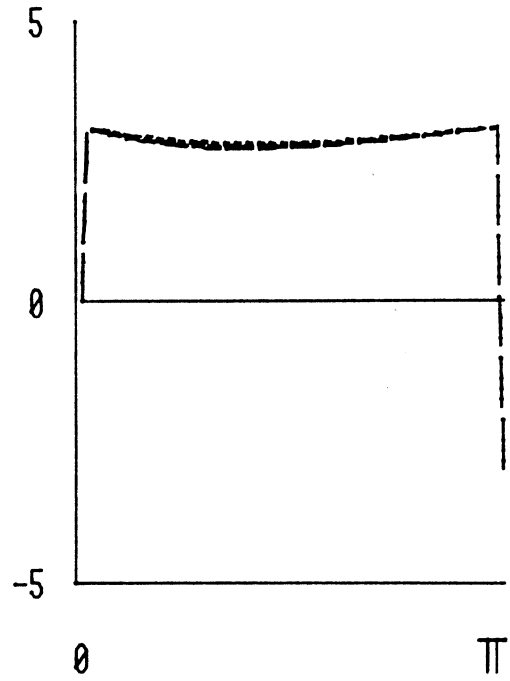
Figure 6.3

MONEY

a. MOVING AVERAGE COEFFICIENTS
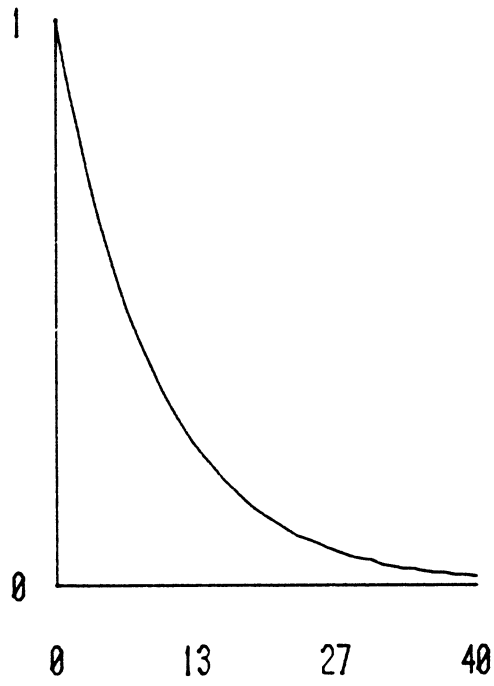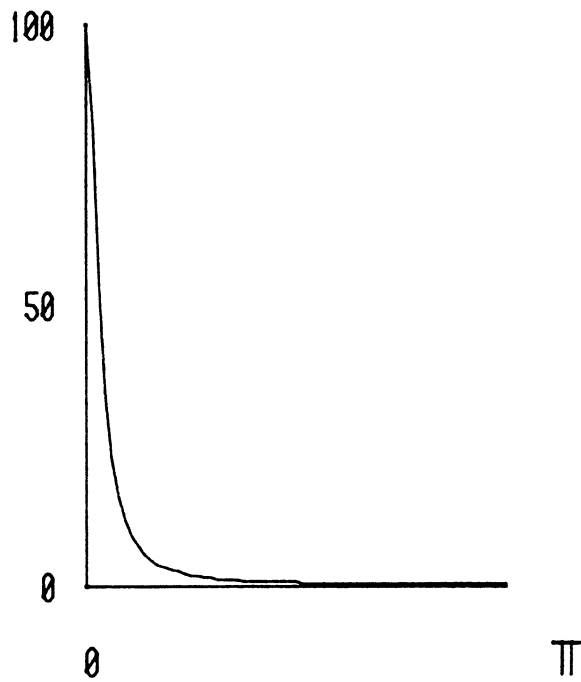
b. SPECTRUM

Figure 6.3 (cont.)

c. SPECTRUM OF PRICES
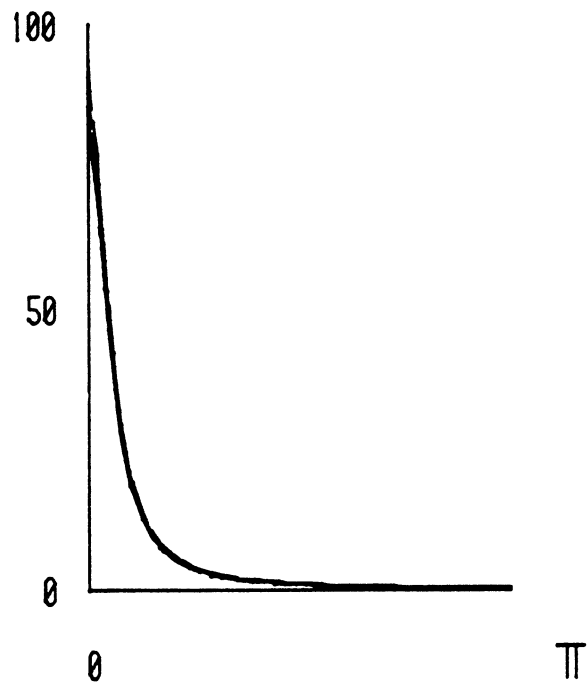
d. SPECTRUM OF CAPITAL
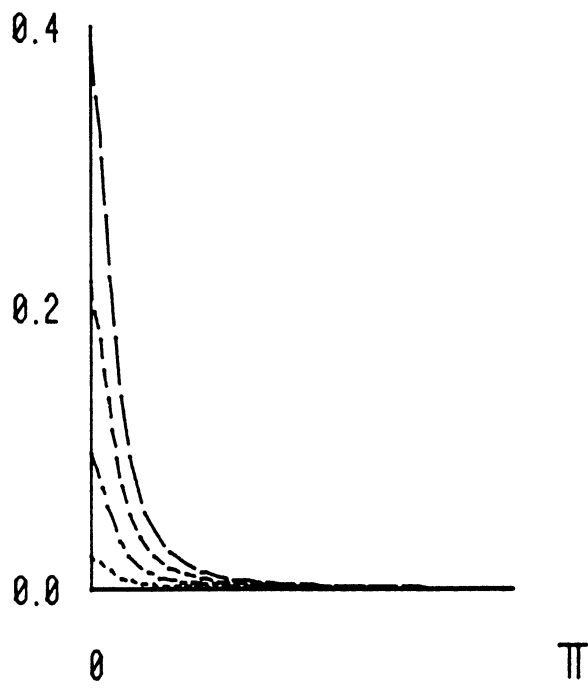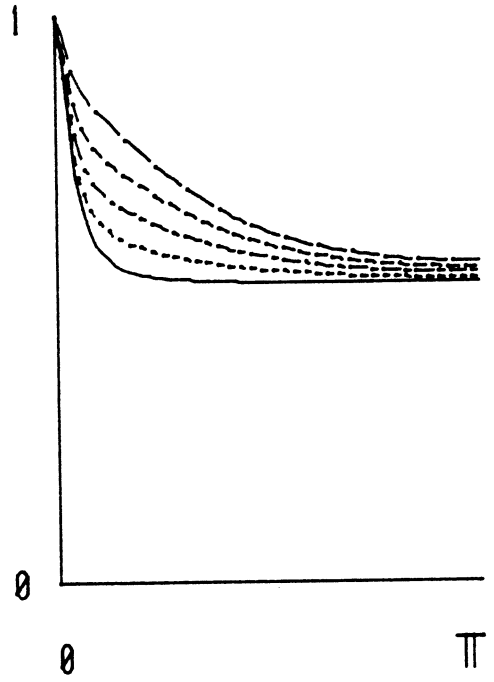
Figure 6.3 (cont.)

PRICES

e. MOVING AVERAGE COEFFICIENTS
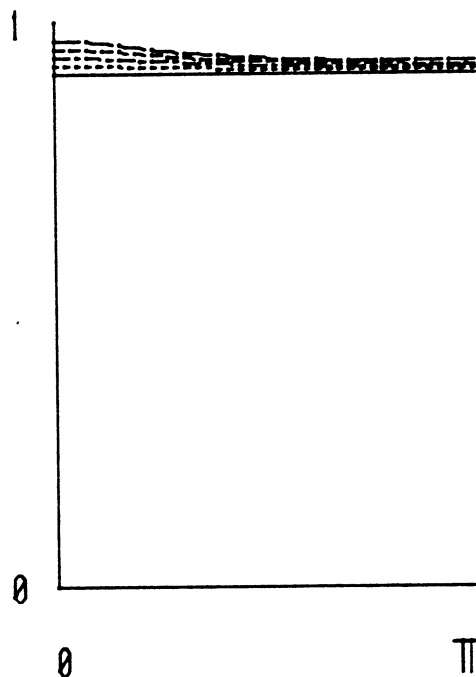
g. TRANSFER MODULUS

f. DISTRIBUTED LAG COEFFICIENTS ON MONEY
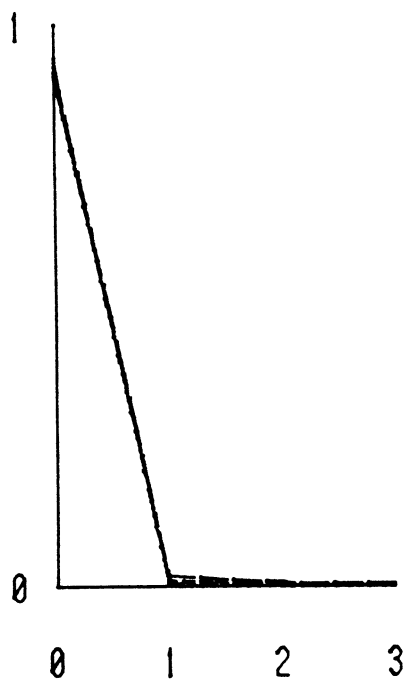
h. PHASE LEAD OVER MONEY
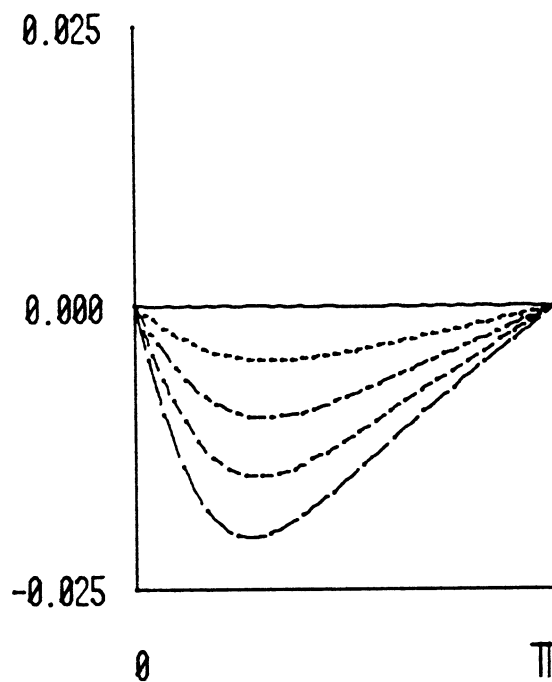
Figure 6.3 (cont.)

CAPITAL

i. MOVING AVERAGE COEFFICIENTS
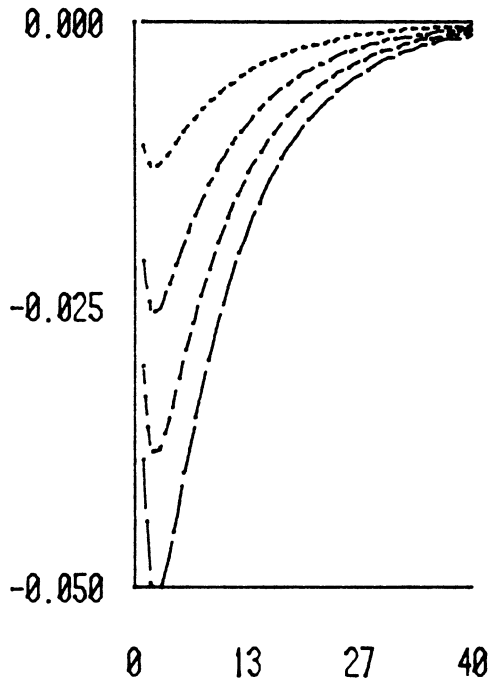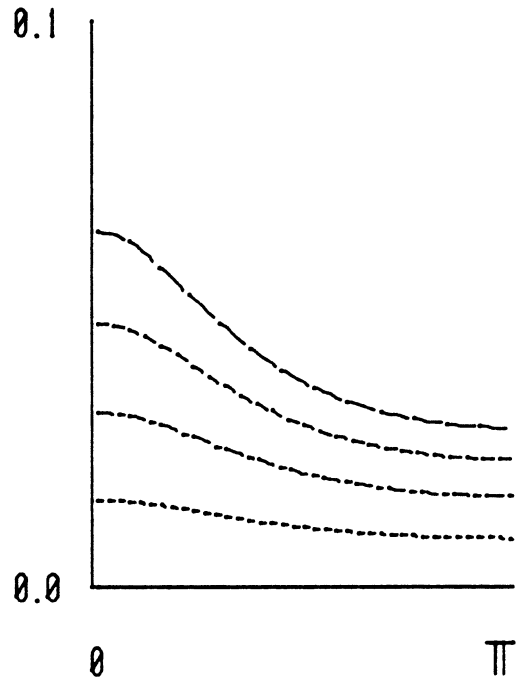
k. TRANSFER MODULUS

j. DISTRIBUTED LAG COEFFICIENTS ON MONEY
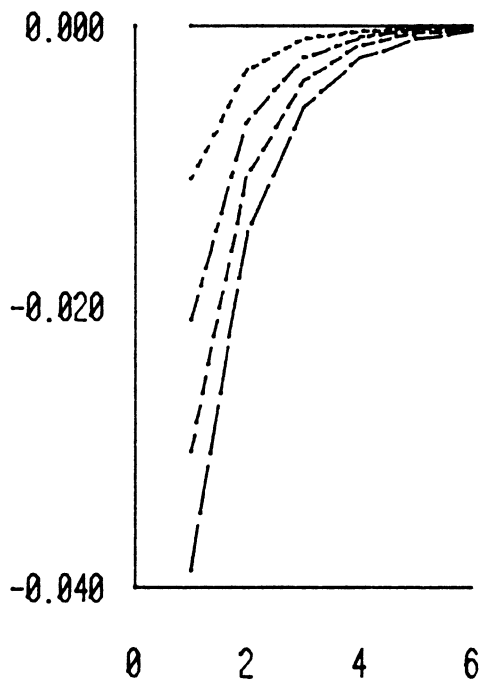
l. PHASE LEAD OVER MONEY
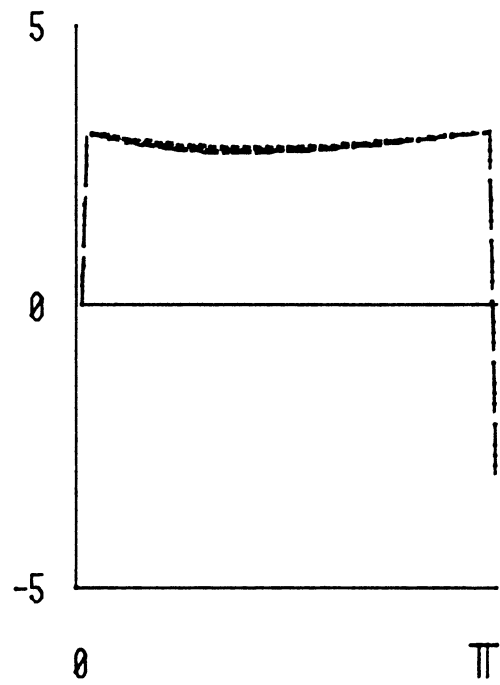
Figure 6.4

MONEY

a. MOVING AVERAGE COEFFICIENTS

b. SPECTRUM

Figure 6.4 (cont.)

c. SPECTRUM OF PRICES
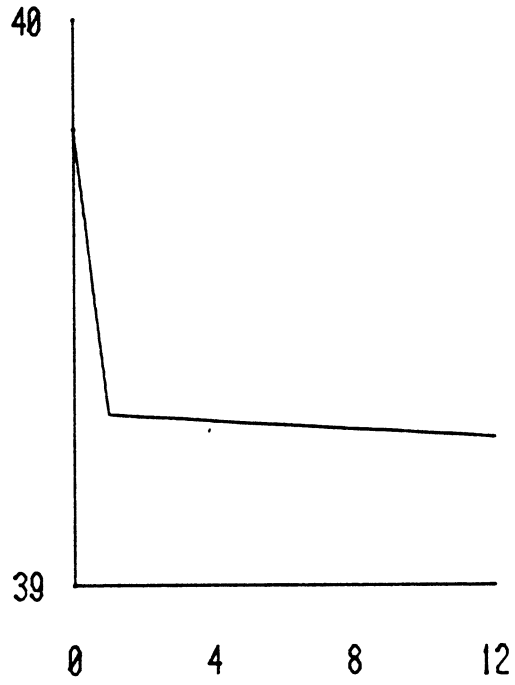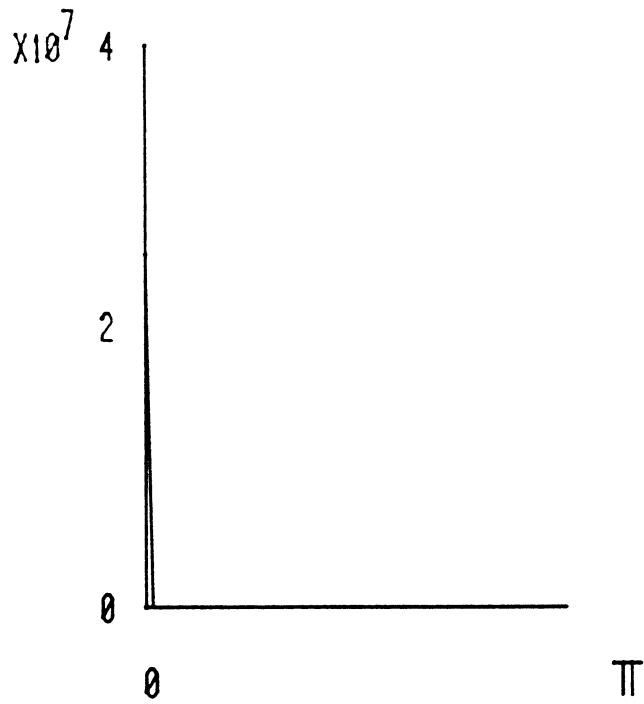
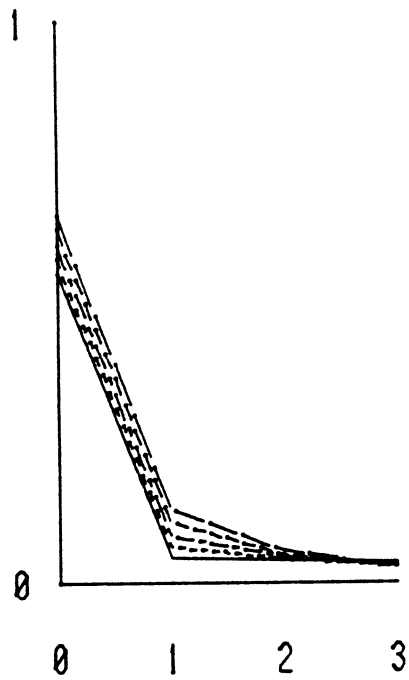d. SPECTRUM OF CAPITAL
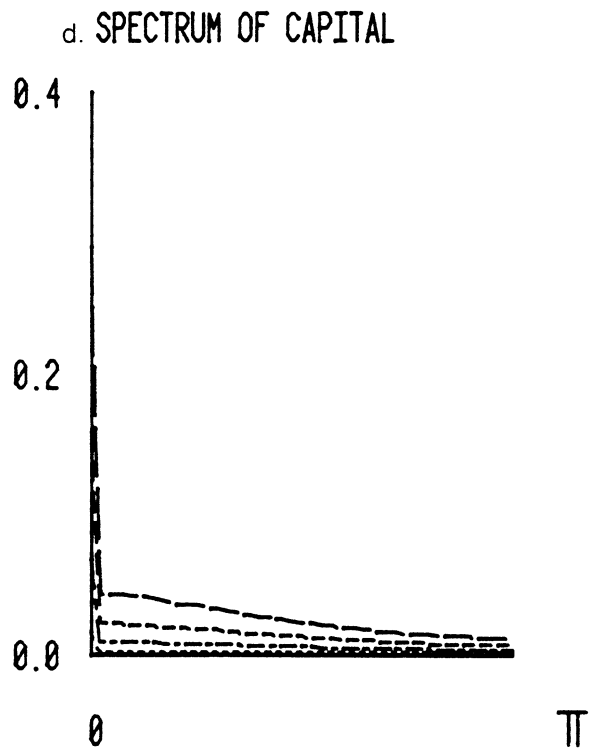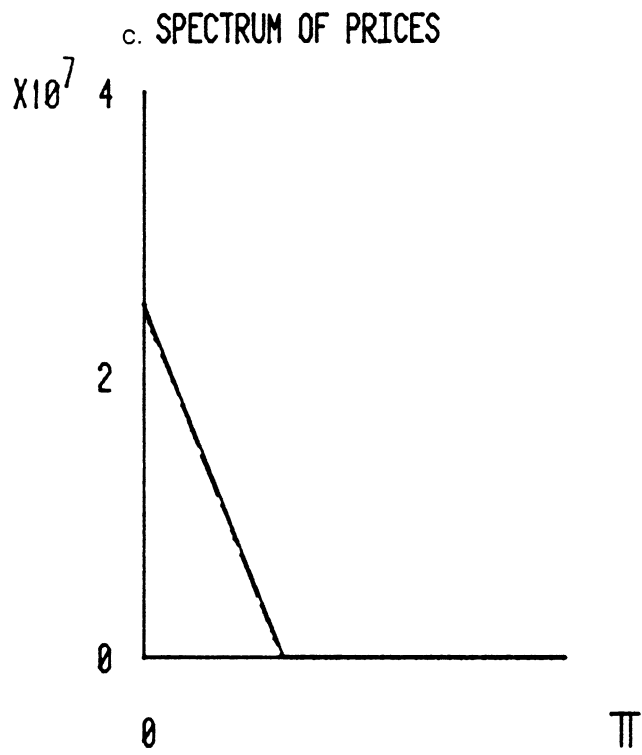
Figure 6.4 (cont.)

PRICES

e. MOVING AVERAGE COEFFICIENTS
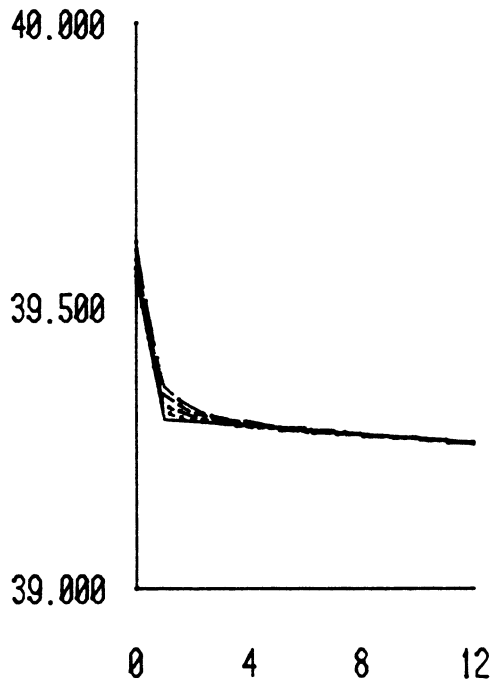
g. TRANSFER MODULUS
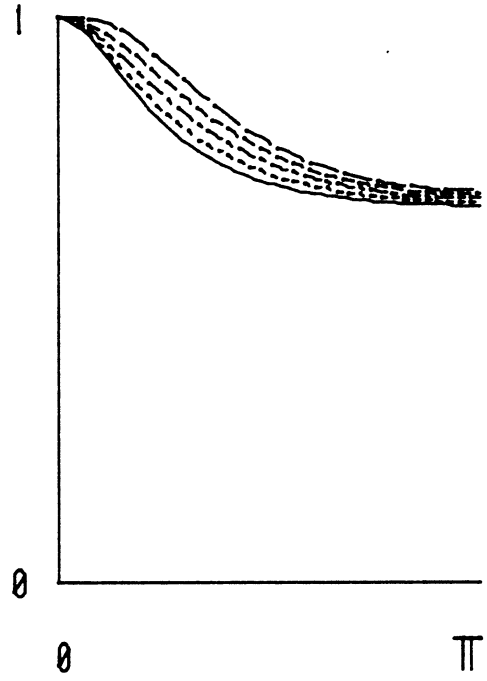
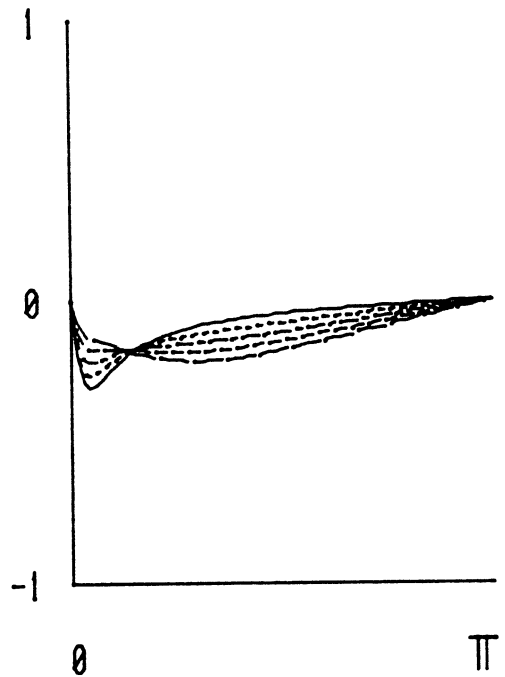f. DISTRIBUTED LAG COEFFICIENTS ON MONEY
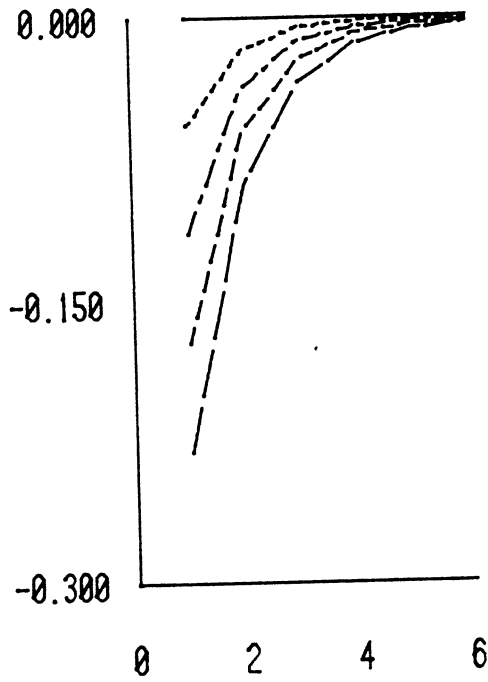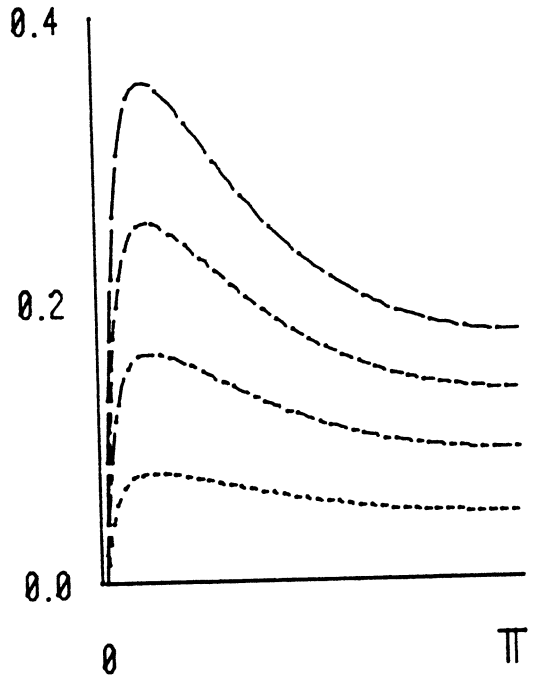
h. PHASE LEAD OVER MONEY

Figure 6.4 (cont.)

CAPITAL

i. MOVING AVERAGE COEFFICIENTS

k. TRANSFER MODULUS

j. DISTRIBUTED LAG COEFFICIENTS ON MONEY

l. PHASE LEAD OVER MONEY

Figure 6.5

MONEY

a. MOVING AVERAGE COEFFICIENTS

b. SPECTRUM

Figure 6.5 (cont.)

c. SPECTRUM OF PRICES

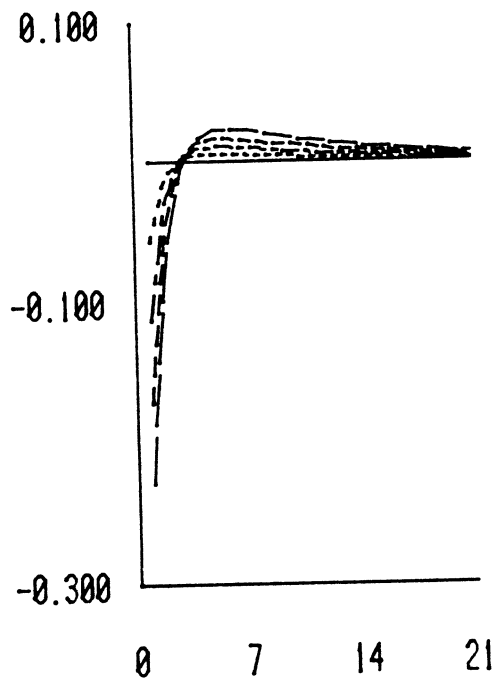d. SPECTRUM OF CAPITAL

Figure 6.5 (cont.)

PRICES

e. MOVING AVERAGE COEFFICIENTS

g. TRANSFER MODULUS

f. DISTRIBUTED LAG COEFFICIENTS ON MONEY

h. PHASE LEAD OVER MONEY

Figure 6.5 (cont.)

CAPITAL

i. MOVING AVERAGE COEFFICIENTS

k. TRANSFER MODULUS

j. DISTRIBUTED LAG COEFFICIENTS ON MONEY

l. PHASE LEAD OVER MONEY