On the Indeterminacy of
Equilibrium Exchange Rates

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Of all the possible international economic policy regimes, one has long been the favorite of a considerable majority of economists. We refer to it as the laissez-faire regime, for it is distinguished by free trade in goods, unrestricted portfolio choice (or no capital controls), and floating or market-determined exchange rates. Yet, as we argue in this paper, the laissez-faire regime is not economically feasible. The equilibrium of that regime is indeterminate. More particularly, equilibrium exchange rates are indeterminate.

That is an implication of an overlapping-generations (OLG) model. We appreciate, though, that some economists are more than a little doubtful about all OLG models, and so in the first part of this paper give a brief defense for having used one. Then in the second part we set out our assumptions, and also, among other things, state the two implied maximization problems, that of the members of generation zero, those who are old in the initial period, and that of the members of the first and all succeeding generations. In the third part of the paper we give the indeterminacy result, and thereafter, in the fourth part, indicate how a determinate world economy equilibrium might be achieved. And in the fifth or concluding part we discuss the generality of our indeterminacy result.

I. AN OLG MODEL

It is generally accepted that fiat monies, unlike gold and other commodity monies, are all intrinsically useless. No such money is ever wanted for its own sake. Thus, if an individual gives up, say, wheat for a certain number of pieces of fiat money, that can only be because he or she expects that later on another individual will take those pieces of paper in exchange for wine or something else that can be consumed.

But there is an implication. If individuals are free to choose among several assets, a fiat money being one, and if another of those assets has a greater return in all circumstances, then no one will want to hold the fiat money. Suppose to the contrary that someone does. That individual is willing to trade valuable commodities for some pieces of paper, even though he or she could enjoy more consumption subsequently by acquiring the higher-yielding asset. And why? Any explanation violates the assumption that the fiat money is intrinsically useless.

For a model of a fiat money economy to be acceptable, it must then be possible for the money (or if there are several, any one of them) to be dominated. In particular, money cannot appear as an argument of any utility or engineering production function. It is also required that money be valuable, at least over some set of parameter values. Obviously, an acceptable model cannot imply that the average of goods prices is always infinite. But only the OLG models satisfy the indicated requirements. So we really had no choice of models.

That observation is not, however, to be interpreted as a concession of anything. It has been argued that the OLG models are all of economies in which money serves only as a store of value, not as that and as a medium of exchange as well, and that their implications are therefore not to be taken seriously. We do not believe that, although fortunately it is not necessary to argue the issue here.[1] It suffices to note that if there are several fiat monies available, as there must be in a world economy with exchange rates, then to accept that exchange is costly is not to deny that one fiat money can be dominated by another. It may even be granted that the best way of making exchanges is by using paper money. That does not guarantee the Mexican peso being used as a medium of exchange in Mexico, or the Israeli pound being used in Israel.

Since one fiat money can be dominated by another even when exchange is costly, or when money is efficient in exchange, most if not all of the results to be found in the balance-of-payments literature are the fruits of implicit theorizing and therefore less general than has been appreciated. Most contributors to that literature have assumed that the residents of country k want to hold only country-k fiat money, that issued by the government of their country. A few, for example, Calvo and Rodriguez [1977] and Girton and Roper [1978], have assumed that there is limited currency substitution, or in other words, that the residents of country k want always to hold diversified portfolios of fiat monies. Neither assumption is generally consistent, though, with utility maximization, which together with the intrinsic uselessness of fiat monies is what makes dominance a possibility. We do not say that country-k residents will never want to hold diversified portfolios of fiat monies. If there is uncertainty about government policies, they may. Our point is rather that generally valid results can not be obtained using money demand functions that are consistent, if at all, with utility maximization only under particular policy regimes. That being so, using an OLG model does not appear as an altogether unfortunate strategy; there is at least the advantage of being entirely explicit.

II. PRELIMINARIES

For our purposes, it suffices that there be just two countries. They are indexed by the variable k. Each has a population that is constant over time. In period t, N_k individuals, all of whom will live for two periods, are born in country k. Since no one ever moves from one country to the other, the period-t population of country k is made up of N_k age-one individuals, the country-k members of generation t, and N_k age-two individuals, the country-k members of generation t-1. The variable h is the index of the members of generation t. And $N_k(t)$ is the set of all country-k members of generation t.

There is only one good. It is not storable. In our notation, $c_j^h(t) \geq 0$ is the consumption of that good at age $j(=1,2)$ by member h of generation t. We assume that all individuals have the same tastes. For all h and t, life-time utility is $u[c^h(t)]$, where $c^h(t) = (c_1^h(t), c_2^h(t))$ is the life-time consumption vector of member h of generation t. The function u is increasing in each of its arguments and thrice-differentiable. It is also homothetic. That is, $u_1(x_1, x_2)/u_2(x_1,x_2) = f(x)$, where $u_j = \partial u/\partial x_j$ and $x_1/x_2 = x$. And f, the marginal-rate-of-substitution function, satisfies the following conditions: $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all admissible values of x; $f(x) \rightarrow \infty$ as $x \rightarrow 0$; $f(x) \rightarrow 0$ as $x \rightarrow \infty$; and $xf'(x)/f(x) \geq -1$, which is the gross substitution condition. Thus, f has an inverse, denoted by F.

Member h of generation $t \geq 1$ gets an endowment of the one good that is available in the first period of his life, period t, but none in the second, period t+1. The period-t endowment is denoted by $y^h(t)$, and we assume that $y^h(t) = y_k$, a constant, for all h in $N_k(t)$. Consequently, for all $t \geq 1$ the world endowment is $N_1 y_1 + N_2 y_2$, which sum we denote by y.

Strictly speaking, our model is of a one-good pure-exchange economy. On certain assumptions, however, that one good can be interpreted as a composite of many produced (nonstorable) goods. Suppose (a) that utility functions are identical for all individuals, separable in the consumptions of different periods, and homothetic in the components of within-period consumption; (b) that aggregate resource endowments and production functions are constant over time and such as to yield acceptable production possibility frontiers; and (c) that there is free trade in all goods. Then equilibrium relative goods prices and factor rentals, and hence the outputs of all goods and the proportions in which they are consumed by all individuals, are independent of time and policy regime.[2/] Moreover, if it is assumed that the country-k young (for period t, the

country-k members of generation t) have command of all country-k resources, then y_k is country-k per capita GNP. The resources might be labor and land. And it might, for example, be assumed that the young of country k are endowed with all the country-k labor and land, the land having been passed on to them by members of the previous generation in accordance with some exogenous inheritance scheme.

The Competitive Choice Problems

In period t, member h of generation $t \geq 1$ consumes a portion of $y^h(t)$ and trades away what remains, getting money in exchange from one or more members of generation t-1. His period-t budget constraint is therefore

$$(1) \qquad c_1^h(t) + \sum_k P_k(t) m_k^h(t) \leq y^h(t)$$

where $P_k(t)$ is the period-t price of the money of the government of country k and $m_k^h(t)$ is his purchase of that money.

In period t+1, h receives a transfer of country-k money from the government of country k. That transfer, denoted by $x_k^h(t)$, is known in period t and is independent of $m_k^h(t)$. And in period t+1, h trades away both his period-t money purchases and his transfers, getting consumption in return from one or more members of generation t+1. Thus, his period-t+1 budget constraint is

$$(2) \qquad c_2^h(t) \leq \sum_k P_k(t+1)[m_k^h(t) + x_k^h(t)].$$

The problem for h is to maximize $u[c^h(t)]$ by the choice of a nonnegative vector $c^h(t)$ and a vector $m^h(t) = (m_1^h(t), m_2^h(t))$, subject to (1) and (2) and with $y^h(t)$, the $P_k(t)$, $P_k(t+1)$, and $x_k^h(t)$ known in period t.[3]

The members of generation zero are in the second or final period of their lives in the first period. So the problem of h of generation zero is to maximize $u[c^h(0)]$ by the choice of $c_2^h(0)$, subject to the t = 0 version of (2) and with $c_1^h(0)$ and the RHS of (2) given.[4]

Equilibrium

The economy evolves over time, starting from an arbitrary date (for us, t = 1) and arbitrary initial conditions (arbitrary initial stocks of the two fiat monies, distributed in some fashion or other over the members of generation zero). That evolution is described by a sequence in t of the endogenous variables. A sequence is an equilibrium (or equilibrium solution) if it is consistent with consumption and portfolio choices that are optimizing, with period-by-period clearing of all markets and with perfect foresight.

We restrict our attention to monetary equilibria defined as follows. For us, a monetary equilibrium is any equilibrium such that $\sum_k P_k(t+1)[m_k^h(t) + x_k^h(t)]$ is positive for all h and t and bounded away from zero. We demonstrate indeterminacy within this class of equilibria.

The Dominance Result and Money Demand

If h is subject only to (1) and (2) in his choice of $m^h(t)$, then the $P_k(t)$ of any monetary equilibrium satisfy the following conditions: if $P_k(t+1) > 0$ for all k, then $P_1(t+1)/P_1(t) = P_2(t+1)/P_2(t)$; and if $P_k(t+1) = 0$ for some k, then $P_k(t) = 0$ for that k.

As we now show, that result, referred to as the dominance result, follows from (1) and (2) and our assumption that there is no consumption bliss point.

By the definition of monetary equilibrium, $P_k(t+1) > 0$ for some k. Without loss of generality, suppose $P_1(t+1) > 0$. Now, (1) and (2) are satisfied by the equilibrium solution. Solving (2) for $m_1^h(t)$ and using the resulting expression to eliminate it from (1), one obtains

$$(3) \qquad c_1^h(t) + [P_1(t)/P_1(t+1)]c_2^h(t) + [P_2(t) - P_1(t)P_2(t+1)/P_1(t+1)]m_2^h(t)$$

$$\leq y^h(t) + P_1(t)[x_1^h(t) + P_2(t+1)x_2^h(t)/P_1(t+1)].$$

Since the aggregate endowment is finite, the equilibrium $c_j^h(t)$ must be too. It follows that the equilibrium value of the coefficient of $m_2^h(t)$ must be zero. With that observation, the proof is complete.

The period-t exchange rate, denoted by $E(t)$, is $P_2(t)/P_1(t)$. So the dominance result can be put another way. If h is subject only to (1) and (2) in his choice of $m^h(t)$, then in equilibrium $E(t+1) = E(t)$. If individuals are subject to no portfolio restrictions, then the equilibrium exchange rate is constant; there cannot be an anticipated change in the equilibrium exchange rate.

It is an easy implication of the dominance result that, in any monetary equilibrium, h of generation t faces a single (gross) rate of return on money, $\beta^h(t) = P_k(t+1)/P_k(t)$ for at least one value of k. Thus, the optimal or equilibrium $c^h(t)$ satisfies a simplified version of (3), namely,

$$(4) \qquad c_1^h(t) + c_2^h(t)/\beta^h(t) \le y^h(t) + z^h(t)$$

where $z^h(t) = P_1(t)[x_1^h(t)+E(t+1)x_2^h(t)]$ is the value of h's transfers (or taxes) in terms of period-t consumption. Given certain restrictions on the $x_k^h(t)$, spelled out below, (4) is satisfied with equality. Equivalently, (1) and (2) are. And the optimal $c^h(t)$ satisfies

$$(5) \qquad c_1^h(t)/c_2^h(t) = F[\beta^h(t)];$$

or, as is implied by (4) and (5), the optimal $c_1^h(t)$ satisfies

$$c_1^h(t) = [y^h(t)+z^h(t)]\beta^h(t)F[\beta^h(t)]/\{1+\beta^h(t)F[\beta^h(t)]\}.$$

But then, since (1) holds with equality, the optimal real money balance $q^h(t) \equiv P_1(t)m_1^h(t) + P_2(t)m_2^h(t)$, satisfies

$$(6) \qquad y^h(t) - q^h(t) = [y^h(t)+z^h(t)]\beta^h(t)F[\beta^h(t)]/\{1+\beta^h(t)F[\beta^h(t)]\}$$

which is a kind of money-demand equation. Clearly, $0 > \partial q^h(t)/\partial z^h(t) > -1$; and, by the gross substitution condition, $\partial q^h(t)/\partial \beta^h(t) \geq 0$.

Budget Policies and Money Supplies

We assume that

$$(7) \qquad x_k^h(t) = \begin{cases} (\alpha_k - 1)\alpha_k^{t-1} M_k/N_k & \text{for all } h \text{ in } N_k(t) \\ \\ 0 & \text{for all } h \text{ in } N_{k'}(t) \end{cases}$$

where $\alpha_k > 1 - N_k y_k/y$ is the policy parameter of the government of country k and M_k, an initial condition, is some positive but otherwise arbitrary amount of the fiat money of the government of country k.[5/] The country-k members of generation $t \geq 1$ receive nothing from or pay no taxes to the government of country k'. But all receive identical transfers from or pay identical taxes to the government of country k; that is, $x_k^h(t) = x_k(t)$ for all h in $N_k(t)$. Thus, since $y^h(t) = y_k$ for all h in $N_k(t)$ and $t \geq 1$, all the country-k members of generation $t \geq 1$ are alike. They all solve precisely the same maximization problem. In effect, there are only two types of members of generation $t \geq 1$, the residents of the first country and the residents of the second.

In our notation $M_k(t)$ is the period-t supply of the money of the government of country k (the amount with which the members of generation t-1 begin period $t \geq 1$ and in that period sell to the members of generation t); and

$$(8) \qquad M(t) = M_1(t) + E(t)M_2(t)$$

is the period-t world money supply.

Since there is no government intervention in the exchange market,

$$(9) \qquad M_k(t) = M_k(t-1) + N_k x_k(t-1)$$

for each k and t > 1, where $M_k(1) = M_k$. Then, by (7),

$$(10) \qquad M_k(t) = \alpha_k^{t-1} M_k.$$

It follows that with the exchange rate constant over time,

$$(11) \qquad M(t) = \alpha_1^{t-1} M_1 + \alpha_2^{t-1} E M_2.$$

More particularly, (11) holds where there is unrestricted portfolio choice.

It is immediate from (11) that

$$(12) \qquad \sigma(t) \equiv M(t+1)/M(t) = \alpha_1 \lambda(t) + \alpha_2 [1-\lambda(t)]$$

where

$$(13) \qquad \lambda(t) \equiv (M_1/M_2)(\alpha_1/\alpha_2)^{t-1} / [M_1/M_2)(\alpha_1/\alpha_2)^{t-1} + E].$$

So $\sigma(t)$, the (gross) growth rate of the world money supply is a weighted average of the α_k. For E positive and finite, the weight of the larger of the α_k increases through time and has unity as its limiting value. Thus, $\sigma(t) \geq \alpha_m \equiv \min(\alpha_1,\alpha_2)$, is monotonic increasing and has $\alpha_M \equiv \max(\alpha_1,\alpha_2)$ as its limiting value.

We note finally that with the exchange rate constant, the money transfers can be expressed as proportions of the world money supply:

$$x_1(t) = (\alpha_1-1)\lambda(t)M(t)/N_1$$

$$(14)$$

$$Ex_2(t) = (\alpha_2-1)[1-\lambda(t)]M(t)/N_2.$$

III. EQUILIBRIUM UNDER THE LAISSEZ-FAIRE (LF) REGIME

Since portfolio choice is unrestricted, we have as an equilibrium condition $E(t) = E$ for all $t \geq 1$ and $\beta^h(t) = \beta(t)$ for all h of generation t.

Then, from (14) and (9), $\sum z^h(t) = [\sigma(t)-1]P_1(t)M(t)$, where the summation is over all h of generation t. It follows upon summing (6) over all h of generation $t \geq 1$ that

(15) $y - Q(t) = \{y+[\sigma(t)-1]P_1(t)M(t)\}\beta(t)F[\beta(t)]/\{1+\beta(t)F[\beta(t)]\}$

where $Q(t) \equiv \sum q^h(t)$ is the total of desired real money holdings for period t. But it is required for equilibrium that $Q(t) = P_1(t)M(t)$ for all $t \geq 1$. And since $\beta(t) = P_1(t+1)/P_1(t)$ and/or $\beta(t) = P_2(t+1)/P_2(t)$, we have that in equilibrium

(16) $\beta(t) = Q(t+1)/\sigma(t)Q(t)$.

Using (16) to eliminate $\beta(t)$ from (15), we obtain a first-order difference equation in Q(t):

(17) $Q(t) + Q(t+1)F[Q(t+1)/\sigma(t)Q(t)] - y = 0$.

For us, then, any monetary equilibrium of the LF regime is a positive Q(t) sequence that satisfies (17) and is bounded away from zero.

In the Appendix we prove the following proposition.

Corresponding to each E in $[0,\infty]$, there exists a unique LF regime monetary equilibrium with the following properties, which hold for all $t \geq 1$:

(i) $Q(t) = \phi_A[\sigma(t)]$

where ϕ_A is a continuous function which depends on the vector $A = (\alpha_1,\alpha_2)$ but not on E;

(ii) $Q(t) \geq Q(t+1)$;

and, provided only that E is positive and finite,

(iii) $1/\sigma(t) \geq \beta(t) \geq$ limit $\beta(t) = 1/\alpha_M$.

What that proposition says, among other things, is that there is an LF regime monetary equilibrium for every possible (unchanging) value of the exchange rate. For any admissible policy choice A, the value of E determines a path for the world money supply; and there is an equilibrium for each such path.

Nor is the dependence on E trivial. If $\alpha_1 \neq \alpha_2$, then $\sigma(t)$ depends on E; and hence the equilibrium $Q(t)$ and $\beta(t)$ do too. And as is readily verified, if $\alpha_k = \alpha \neq 1$, k=1,2, then the distribution of utility over the young of the two countries depends on E, even though the equilibrium $Q(t)$ sequence does not. Further, if $\alpha = 1$, then the members of generation zero are, in general, affected by the choice of E. In sum, the value of E does not matter only if $\alpha_k = \alpha = 1$ and all members of generation zero hold the two monies in identical proportions.

Convergence and A Version of Gresham's Law

It follows from (13), (12), and (17) that if $\alpha_1 = \alpha_2$, then for any E the equilibrium $Q(t)$ sequence is constant. More specifically, if E = 0 (that is, if the money of the second country has no value), then in equilibrium $Q = y/[1+F(1/\alpha_1)]$; and if $E = \infty$ (the money of the first country has no value), then in equilibrium $Q = y/[1+F(1/\alpha_2)]$. If, on the other hand, $\alpha_1 \neq \alpha_2$, then the equilibrium $Q(t)$ sequence is nonconstant. But it converges. With E positive and finite and $\alpha_k > \alpha_{k'}$, lim $Q(t) = y/[1+F(1/\alpha_k)]$, which is precisely the equilibrium value of the $Q(t)$ when only the money of country k has value.

The limit of the real value of the relatively abundant money, lim $P_k(t)M_k(t)$ if $\alpha_k > \alpha_{k'}$, is equal to the limit of the world money supply, lim $P_1(t)M(t)$. That is our version of Gresham's Law. To see that it obtains, note that $P_1(t)M_1(t)/Q(t) = M_1(t)/M(t) = \lambda(t)$. The result follows [see (13)] from the limiting behavior of $\lambda(t)$.

The Trade Account

Trade balance is not in general a characteristic of the monetary equilibria of the LF regime. Equality between equilibrium exports and imports is a very special outcome. For almost all choices of the α_k, country k will either have a trade surplus, one which is just sufficient to offset the import of country-k' money, or a trade deficit which just offsets the export of country-k money.

By definition, the equilibrium period-t trade surplus of country k, denoted by $B_k(t)$, is the difference between the country-k endowment, $N_k y_k$, and the equilibrium consumption of the period-t residents of country k, the country-k members of generations t-1 and t. Thus, $B_k(1)$ depends on the arbitrary initial distribution of money holdings, the distributions over the first- and second-country members of generation zero. But if we can therefore say nothing about what $B_k(1)$ is, we can be quite definite about $B_k(t)$ for $t \geq 2$.

The equilibrium period-t consumption of, say, the first-country members of generation $t \geq 2$ is $N_1[y_1-q_1(t)]$, and that of the first-country members of generation t-1 is $N_1[\beta(t-1)q_1(t-1)+P_1(t)x_1(t-1)]$. Therefore, by (14),

(18) $B_1(t) = N_1 q_1(t) - [N_1 q_1(t-1)/Q(t-1)+(\alpha_1-1)\lambda(t-1)]Q(t)/\sigma(t-1)$.

It follows that if $\alpha_1 = \alpha_2 = \alpha$, then for all $t \geq 2$

(19) $B_1(t) = [N_1 q_1(t)-\lambda(t)Q(t)](\alpha-1)/\alpha$

where $q_1(t)$, $Q(t)$, and $\lambda(t)$ are all constants. Thus, if $\alpha = 1$, then $B_k(t) = 0$ for all $t \geq 2$. But if $\alpha \neq 1$, then the $B_k(t)$ are not necessarily zero. Indeed, if $\alpha \neq 1$, then $B_k(t) = 0$ for all $t \geq 2$ if and only if the period-t equilibrium desired real balances of the first-country members of generation t, $N_1 q_1(t)$, is the same as the period-t equilibrium real supply of first-country money, $P_1(t)M_1(t) = \lambda(t)Q(t)$. There is a unique value of E at which equality obtains.[6/]

It also follows from (18) that if $\alpha_1 \neq \alpha_2$, then for all $t \geq 2$

$$(20) \quad \lim B_1(t) = \begin{cases} -[N_2(\alpha_1-1)/\alpha_1]\lim q_2(t) & \text{if } \alpha_1 > \alpha_2 \\ \\ [N_1(\alpha_2-1)/\alpha_2]\lim q_1(t) & \text{if } \alpha_1 > \alpha_2 \end{cases}$$

where each of the limits on the RHS is positive, a consequence of the fact that if $\alpha_k > \alpha_{k'}$ and country-k money has value, then any country-k transfer has a limiting real value of zero. Thus, if $\alpha_1 \neq \alpha_2$, then $\lim B_k(t) = 0$ for all $t \geq 2$ if and only if $\alpha_M = 1$. Note, though, that with $\alpha_M \neq 1$, it makes all the difference whether $\alpha_M \gtrless 1$.

IV. A DETERMINATE EQUILIBRIUM

We turn now to a second international economic policy regime: as we refer to it, the portfolio autarky (PA) regime. Under that regime there is no government intervention in the exchange market. But portfolio choice is restricted. No country-k member of generation $t \geq 1$ can be either long or short country-k' money. Nor is intertemporal trade in consumption by residents of different countries permitted. A country-k member of generation $t \geq 1$ cannot, for example, get period-t consumption from a country-k' resident in exchange for period-t+1 consumption.

Under the PA regime the residents of the two countries do not in general face identical rates of return. We have $\beta^h(t) = P_k(t+1)/P_k(t) = \beta_k(t)$ for all h in $N_k(t)$ and $t \geq 1$; and the $\beta_k(t)$ may not be equal. In any event, since the PA regime budget policies of the two governments are the same as their LF regime policies, there thus are two versions of (15), one obtained by summing (6) over all h in $N_1(t)$, and the other by summing over all h in $N_2(t)$. They may be written

(21) $y_k - q_k(t) = \{y_k + (\alpha_k - 1)P_k(t)M_k(t)/N_k\}\beta_k(t)F[\beta_k(t)]/\{1 + \beta_k(t)F[\beta_k(t)]\}$

where $q_k(t)$ is the common value of $q^h(t)$ for all h in $N_k(t)$.

For the PA regime the equilibrium restrictions are $q_k(t) = P_k(t)M_k(t)/N_k$ for all k and $t \geq 1$. Consequently, since (10) holds,

$\beta_k(t) = q_k(t+1)/\alpha_k q_k(t)$.

So in place of (17) we have

(22) $q_k(t) + q_k(t+1)F[q_k(t+1)/\alpha_k q_k(t)] - y_k = 0$ for k=1,2.

And since (22) is a special case of (17), we get the following as an immediate consequence of the proposition of the preceding section.

Under the PA regime, there exists a unique monetary equilibrium with

(23) $q_k(t) = y_k/[1+F(1/\alpha_k)]$

for all k and $t \geq 1$.

Thus, under the PA regime there is (limited) autonomy. For all h in $N_k(t)$, $t \geq 1$, the equilibrium consumption allocation is independent of $\alpha_{k'}$, or of the budget policy choice of the government of country k'. More to the point, though, there is a unique equilibrium exchange rate. Since $E(t)/E(t-1) = \beta_2(t)/\beta_1(t)$, the PA regime equilibrium value of $E(t)$ is

(24) $E(t) = \alpha_1 E(t-1)/\alpha_2 = (\alpha_1/\alpha_2)^{t-1}E(1) = (\alpha_1/\alpha_2)^{t-1}(M_1/M_2)(N_2 q_2/N_1 q_1)$

an exponential function of time, where q_k is the constant value of $q_k(t)$ satisfying (23).

The PA regime is not, however, the only one under which there is a determinate equilibrium. There are such equilibria under cooperative fixed exchange rate regimes. These regimes may not be, in general, politically feasible; but they are all economically feasible, for cooperation ensures adequate

supplies of government monies.[7] And as Nickelsburg [1980] has shown, there may be a determinate equilibrium if there is the right kind of uncertainty about government policy. More particularly, with the threat of portfolio autarky being imposed sometime in the future, there is a determinate current-period equilibrium exchange rate. That is of no small importance. Some might have been tempted to argue that since there have been several instances of floating exchange rates, the OLG models, insofar as they exploit the intrinsic uselessness of fiat monies, are of no relevance. It is extremely difficult, however, to imagine individuals ever having been entirely convinced that government portfolio restrictions or intervention in exchange markets had become a thing of the past. So neither OLG models, nor our indeterminacy result, can be dismissed easily by appeal to experience.

V. CONCLUSION

Others may be tempted to argue either that individuals are bound by habit to the monies of their respective governments, or that a difference in exchange costs forces the residents of country k to use country-k money. If one or the other of those arguments justifies the restriction of macroeconomic open-economy models (residents of country k want only country-k money), that can only be for some brief period of time. Neither makes plausible that the LF regime is economically feasible.

Our indeterminacy result is an implication of a decidedly simple model, and subsequent research may well show it to have been lacking in robustness. We suspect, though, that had we admitted of, say, a real asset, whether with a deterministic or a stochastic return, we would obtain essentially the same result. There would seem to be nothing in the existence of a real asset to make one fiat money less than a perfect substitute for any other.[8] The robustness of

our indeterminacy result is, however, something to be verified. At this point, all we can do is observe that if our conjecture is established, then economists will have to go back to the issue of which among the several possible is the best of international economic policy regimes.

APPENDIX

In this Appendix we give a proof of the proposition of Section III. We begin with part (i).

Since $Q(t)$ and $\sigma(t)$ are positive for all $t \geq 1$, we can rewrite (17) as $G[\ln Q(t), \ln Q(t+1), \ln \sigma(t)] = 0$, where, with the argument of F suppressed, the partial derivatives of G, obtained from (17), are $G_1 = Q(t)\{1-\sigma(t)[\beta(t)]^2 F'\} > 0$, $G_2 = Q(t+1)[F+\beta(t)/F'] \leq 0$, $G_3 = -Q(t)[\beta(t)]^2 \sigma(t)F' > 0$. Thus, a sequence $\{Q(t)\}$ satisfies (17) if and only if it satisfies

$$(25) \qquad \ln Q(t) = \gamma[\ln Q(t+1), \ln \sigma(t)]$$

where $\gamma_1 = -G_2/G_1$ is in $[0,1)$ and $\gamma_2 = -G_3/G_1$ is in $(-1,0)$.

Equation (25) is a first-order difference equation in $\ln Q(t)$, with $\sigma(t)$, exogenously determined by $A = (\alpha_1, \alpha_2)$, as the driving variable. We will show that there is a unique equilibrium of the form

$$(26) \qquad \ln Q(t) = \phi_A[\sigma(t)] \qquad \text{for all } t \geq 1$$

where ϕ_A is a continuous (and real-valued) function defined on the interval $I(A) = [\alpha_m, \alpha_M]$.

To begin, we note that by the definition of $\lambda(t)$,

$$\lambda(t+1) = (\alpha_1/\alpha_2)\lambda(t)[1-\lambda(t)+\alpha_1\lambda(t)/\alpha_2].$$

It follows that

$$(27) \qquad \sigma(t+1) = \alpha_1 + \alpha_2 - \alpha_1\alpha_2/\sigma(t) \equiv \Psi[\sigma(t)].$$

Then, a function ϕ_A is a solution to (25) if and only if

$$(28) \qquad \phi_A(Z) = \gamma\{\phi_A[\Psi(Z)], \ln Z\}$$

identically in Z for $Z \in I(A)$.

Now, let S denote the set of all continuous (and real-valued) functions that are defined on the interval $I(A)$, bounded from above by $\ln y$ and bounded from below by $\ln L$, where L is any element of $(0, Q^*]$, and Q^* is the solution to $G[\ln Q^*, \ln Q^*, \ln \alpha_M] = 0$. (It follows from the definition of G that Q^* is unique and that $0 < Q^* < y$.)

Letting ϕ be any element of S, define the mapping Γ by

(29) $\qquad \Gamma(\phi) = \gamma\{\phi[\Psi(Z)], \ln Z\}.$

Evidently, any element ϕ_A of S that satisfies (28) is a fixed point under Γ, and vice versa.

Consider the metric space $S = (S, \rho)$ where for ϕ_1 and ϕ_2, any two elements of S, $\rho(\phi_1, \phi_2) = \max|\phi_1 - \phi_2|$. As is well known, S is complete. Therefore, all we have to show is, first, that Γ maps S into itself and, second, that Γ is a contraction. (See Kolmogorov and Fomin [1957], pp. 34, 43.)

The proof that Γ maps S into itself is as follows. Let Z be any element of $I(A)$. Then by (27) the argument of ϕ on the RHS of (29) is in $I(A)$. And therefore since γ is continuous, Γ maps continuous functions into continuous functions. So what remains is to show that $\Gamma(\phi)$ is appropriately bounded.

For any Z in $I(A)$, there is a unique number, $\bar{x}$, such that $\bar{x} = \gamma(\bar{x}, \ln Z)$. [See (17).] Moreover, since $\gamma_2 < 0$ and $Z \leq \alpha_M$, we have $\bar{x} \geq \ln Q^*$. And since $0 \leq \gamma_1 < 1$, we also have:

(a) $\qquad$ if $x \leq \bar{x}$, then $x < \gamma(x, \ln Z) \leq \bar{x}$; and

(b) $\qquad$ if $x > \bar{x}$, then $\bar{x} < \gamma(x, \ln Z) < x$.

Now, $\phi[\Psi(Z)]$ is either in $[\ln L, \bar{x}]$ or in $(\bar{x}, \ln y]$. And if it is in the first interval, then by (a)

$$\ln L \leq \phi[\Psi(Z)] < \gamma\{\phi[\Psi(Z)], \ln Z\} \leq \bar{x} \leq \ln y.$$

But if it is in the second interval, then by (b)

$$\ln L \le \ln Q^* \le \bar{x} < \gamma\{\phi[\Psi(Z)], \ln Z\} < \phi[\Psi(Z)] \le \ln y.$$

Thus, $\Gamma(\phi)$ is appropriately bounded.

Having established that Γ maps S into itself, we turn now to showing that Γ is a contraction. Suppose that for any Z, $x_1 = \phi_1[\Psi(Z)]$ and $x_2 = \phi_2[\Psi(Z)]$, where again ϕ_1 and ϕ_2 are arbitrary elements of S. Then, by the Mean Value Theorem,

$$\gamma(x_2, \ln Z) - \gamma(x_1, \ln Z) = (x_2-x_1)\gamma_1$$

where γ_1 is evaluated at $(x, \ln Z)$ for some x in $[x_1, x_2]$. And hence, since $\gamma_1 \ge 0$

$$\max_{Z} |\gamma(x_2, \ln Z) - \gamma(x_1, \ln Z)| \le (\max_{Z} \gamma_1)(\max_{Z} |x_2-x_1|)$$

$$= (\max_{Z} \gamma_1)(\max_{Z} |\phi_2-\phi_1|).$$

So we have only to show that $\max_{Z} \gamma_1 < 1$.

But $\gamma_1 < \mu/(1+\mu)$, where $\mu = -\sigma(t)[\beta(t)]^2 F' > 0$. Thus, if μ is bounded from above, Γ is a contraction. The elements ϕ_1 and ϕ_2 are, however, bounded from above and below, which as we have seen implies that γ has upper and lower bounds, and it follows that γ_1 (and hence μ) is evaluated at some point which is bounded from above and below. Consequently, μ is bounded. So we have part (i) of our proposition.

We go on now to prove parts (ii) and (iii). To prove part (ii), we first note that with Γ being a contraction, repeated application of Γ to any element ϕ of S yields a sequence, $\Gamma(\phi)$, $\Gamma(\Gamma(\phi)) = \Gamma^{(2)}(\phi)$, ..., $\Gamma^{(m)}(\phi)$, ..., that converges to ϕ_A. If $\phi^{(0)} = \ln Q^*$ is taken as the starting point, then the resulting sequence is (with the argument of σ suppressed) $\phi^{(1)}(\sigma) = \gamma[\ln Q^*,$

$\ln \sigma]$, $\phi^{(2)}(\sigma) = \gamma[\phi^{(1)}(\sigma), \ln \sigma]$, ..., $\phi^{(n)}(\sigma) = \gamma[\phi^{(n-1)}(\sigma), \ln \sigma]$, ..., $\lim \phi^{(n)}(\sigma) = \phi_A(\sigma)$.

Since $\gamma_2 < 0$, we have $\phi^{(1)}(\sigma_0) > \phi^{(1)}(\sigma_1)$, where σ_0 and σ_1 are any two values of σ satisfying the restriction $\sigma_0 < \sigma_1$. And since $\gamma_1 \geq 0$, $\phi^{(n)}(\sigma_0) \geq \phi^{(n)}(\sigma_1)$ for all $n \geq 1$. Thus, $\phi_A(\sigma_0) \geq \phi_A(\sigma_1)$. And part (ii) of the proposition follows from the observation that $\{\sigma(t)\}$ is nondecreasing for any admissible choice of A.

Having shown that $Q(t) \geq Q(t+1)$ for all $t \geq 1$, we also have that $P_1(t)M(t) \geq P_1(t+1)M(t)\sigma(t)$, from which the first of the inequalities of part (iii) is immediate. And by the definition of β we have that

$$\ln \beta(t) - \ln \beta^* = \ln Q(t+1) - \ln Q(t) - \ln \sigma(t) + \ln \sigma^*$$

$$= \ln Q(t+1) - \gamma[\ln Q(t+1), \ln \sigma(t)] - \ln \sigma(t) + \ln \sigma^*$$

where $\beta^* = \lim \beta(t)$ and $\sigma^* \equiv \lim \sigma(t)$. Also, by the Mean Value Theorem,

$$\gamma[\ln Q(t+1), \ln \sigma(t)] = \gamma(\ln Q^*, \ln \sigma^*)$$

$$+ \gamma_1[\ln Q(t+1)-\ln Q^*] + \gamma_2[\ln \sigma(t)-\ln \sigma^*]$$

$$= \gamma_1 \ln Q(t+1) + (1-\gamma_1)\ln Q^* + \gamma_2[\ln \sigma(t)-\ln \sigma^*]$$

where $Q^* = \lim Q(t)$. Therefore,

$$\ln \beta(t) - \ln \beta^* = (1-\gamma_1)[\ln Q(t+1)-\ln Q^*] - (1+\gamma_2)[\ln \sigma(t)-\ln \sigma^*].$$

But then with the γ_i bounded in the ways indicated above, it must be that $\ln \beta(t) - \ln \beta^* \geq 0$ for all $t \geq 1$. For the sequences $\{Q(t)\}$ and $\{\sigma(t)\}$ are respectively monotonic decreasing and increasing.

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FOOTNOTES

[1] A recent publication of the Federal Reserve Bank of Minneapolis, Models of Monetary Economies, [Kareken and Wallace (1980)] contains several papers on the usefulness of OLG models. See, in particular, the introduction, Tobin's discussion, and the paper by Cass and Shell.

[2] See Kareken and Wallace [1977], wherein the recursiveness is actually proved. That paper also contains a welfare analysis that applies to the model of this paper.

[3] Note that $m^h(t)$ is not restricted to being nonnegative. A country-k member of generation $t \geq 1$ may be short country-k money, and also, unless prevented by government regulation from holding any country-k' money, that money as well.

[4] The initial or first-period stocks of the two monies are distributed arbitrarily over the first- and second-country members of generation zero. That is, we require no particular distribution. Nor, therefore, does it matter which regime was in force prior to the first period.

[5] The restrictions on the α_k guarantee that the equilibrium $c_j^h(t)$ are strictly positive for all h, t, and j.

[6] At $E = 0$, $\lambda = 1$ and $q_2 > 0$; and at $E = \infty$, $\lambda = 0$ and $q_1 > 0$. And uniqueness follows from $\partial(N_1 q_1 - \lambda Q)/\partial E > 0$, where the inequality is implied by the way q^h depends on the transfer [see (6)] and the further fact that with the $\alpha_k = \alpha$, Q and β are independent of E.

[7] The cooperative and certain noncooperative fixed rate regimes are considered in Kareken and Wallace [1978a], as is a welfare appraisal of the different regimes. A principal virtue of the framework used in this paper is that it allows for the analysis of a wide range of policy rules and for their appraisal in terms of allocative efficiency and distribution.

[8/]Under the LF regime, the presence of a real asset affects the values of α_m and α_M for which there is indeterminacy. For example, suppose the consumption good is storable with a constant gross rate of return of r. If $\alpha_M \leq 1/r$, then the proposition of Section III holds; if $\alpha_m \leq 1/r < \alpha_M$, then the only monetary equilibrium is one in which the more rapidly expanding money is worthless; if $1/r < \alpha_m$, then no monetary equilibrium exists.

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