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Notes for Another Paper on the  
Dynamics of Hyperinflation\*

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\*Note: This paper breaks the record for the number of papers written on a single topic by one author or set of authors. The previous record was held by Sargent for his papers on the Gibson paradox.

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## Introduction

During six of the seven twentieth-century hyperinflations in Europe, real balances tended to fall over time while inflation rates tended to rise. Cagan's model of hyperinflation under rational expectations implies that such a systematic pattern could not occur, except by chance. (See Sargent and Wallace [ ] and Sargent [ ]). According to that model, real balances and inflation may drift, but are predicted not to move systematically upward or downward. There is also a long-standing claim that the European hyperinflations appeared to proceed at rates that exceeded the rates that would maximize the revenue from seignorage. (See Cagan [ ] or Sargent [ ]). This paper describes a model that aims to account for these possibilities.

We study a linear stochastic dynamic system in the price level and per capita base money, designed to reflect aspects of hyperinflations. The dynamic system is a solution of two difference equations, one each that describe the behavior of the public and the government. The public's behavior is described by a linear version of Cagan's demand function for real balances, which we express as

$$(a) \quad p(t) = \lambda E_t p(t+1) + \gamma h(t) + u(t), \quad \gamma > 0, \quad 1 > \lambda > 0$$

where  $p(t)$  is the price level at  $t$ ,  $h(t)$  is per capita base money at  $t$ , and  $E_t(\cdot)$  is the linear least squares projection of  $(\cdot)$ , conditional on information available at  $t$ , which is assumed to include at least  $(h(t), h(t-1), \dots, p(t), p(t-1), \dots)$ . In (a),  $u(t)$  is a zero mean random process, possibly nonstationary, that re-

flects disturbances to portfolio balance decisions. The government's behavior is described by the budget constraint

$$(b) \quad h(t) = \frac{1}{1+n} h(t-1) + \xi p(t) + \epsilon(t)$$

where  $h(t)$  is per capita base money,  $n$  is the growth rate of the population,  $\xi p(t) + \epsilon(t)$  is the nominal government deficit per capita. In (b),  $\epsilon(t)$  is a zero mean, possibly nonstationary random disturbance to the per capita nominal government deficit, while  $\xi > 0$  is a constant that measures the average level of the per capita real deficit. The force of (b) is that the government prints base money to finance a real per capita deficit that is on average constant at the rate  $\xi$ . From the interaction of (a) and (b) there results a closed system that determines the evolution of  $p(t)$  and  $h(t)$  as stochastic processes.

Sargent and Wallace [ ] analyzed the deterministic version of this system that emerges when  $u(t)$  and  $\epsilon(t)$  are each set identically to zero. They showed that the evolution of that version of the system is represented by the difference equation

$$(c) \quad \pi(t+1) = \phi - (1/(1+n)\lambda) \cdot 1/\pi(t)$$

where  $\pi(t+1) \equiv p(t+1)/p(t)$ , and  $\phi \equiv (\lambda^{-1} + (1+n)^{-1} - \xi\gamma/\lambda)$ . Equation (c) is graphed in figure 1,

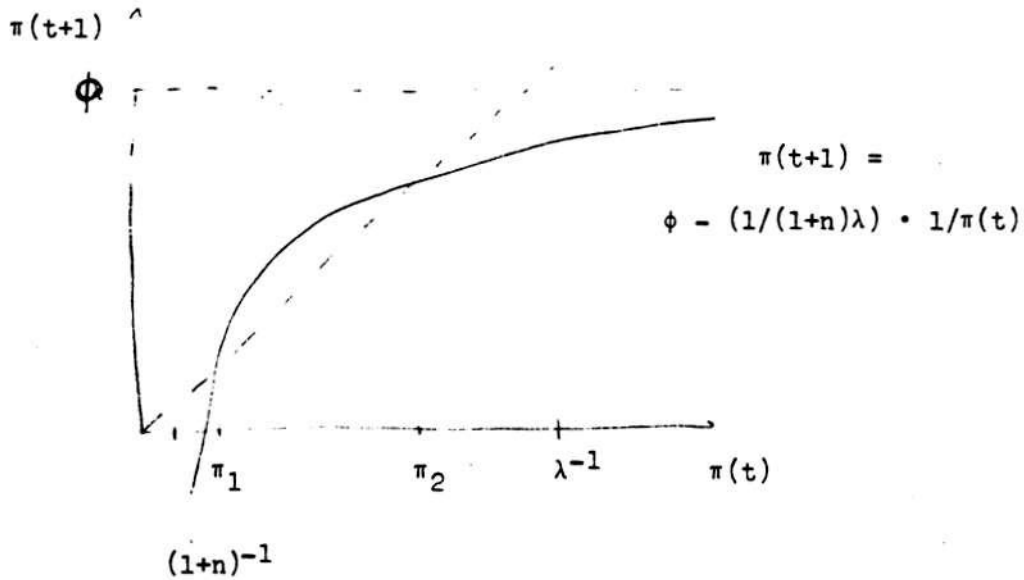


Figure 1

which indicates that there are two stationary points  $\pi_1$  and  $\pi_2$  with  $(1+n)^{-1} < \pi_1 < \pi_2 < \lambda^{-1}$ . These stationary points correspond to two alternative stationary levels of the gross inflation rate  $p(t+1)/p(t)$  that satisfy portfolio balance and that finance a constant real per capita deficit  $\xi$ . For an initial  $\pi(0)$  in the interval  $(\pi_1, \infty)$ , the system converges to  $\pi_2$ . The lower stationary point  $\pi_1$  is unstable in this sense. Figure 1 reflects that fact that the deterministic version of the system formed by (a), (b) possesses a continuum of equilibria within the class of equilibria for which  $p(t)$  and  $h(t)$  are of exponential order strictly less than  $\lambda^{-1}$ . Here equilibria are defined as elements of the space of sequences of  $(p(t), h(t), t > 0)$ . Evidently, this multiplicity of equilibria is logically distinct from the multiplicity of "speculative bubble" equilibria, which are constructed by using the freedom of adding transient terms of exponential order  $\lambda^{-1}$  to the solution for the price process. (See Sargent and Wallace for more details.)

The present paper studies a stochastic version of the system for several reasons in order to try to account for the time series observations on real balances and inflation cited above. To accomplish this task, we first have to characterize whether and how the multiplicity of equilibria of exponential order less than  $\lambda^{-1}$  will surface in a stochastic system. A stochastic version of the system is convenient for studying time series observations, and for making the model econometrically operational. Some of the forces captured in this model were alluded to by us in an informal way in earlier work that attempted to rationalize the pattern of Granger causality between (logarithms) of base money and prices that appears throughout a number of hyperinflations. There is a marked tendency for prices to Granger cause money, but much weaker evidence that money Granger causes prices. In our earlier work, we posited a system with extensive feedback from prices to money, which we argued informally might reflect dynamics coming from the government budget constraint. The current paper returns to this issue and presents a formal analysis that is permitted by our adopting a related but distinct parameterization to the one used in Sargent and Wallace [ ] and Sargent [ ]. A general reason for studying the current system is that it is a laboratory for exhibiting what can be learned about the demand for money from observations on money and prices drawn from a system in which there are extensive dynamic interactions between money and prices that reflect the behavior both of private agents and agents for the government.

## 2. Equilibrium

We consider the system

$$(1) \quad p(t) = \lambda E_t p(t+1) + \gamma h(t) + u(t)$$

$$h(t) = \frac{1}{1+n} h(t-1) + \xi p(t) + \varepsilon(t)$$

where  $1 > \lambda > 0$ ,  $\gamma > 0$ ,  $\xi > 0$ ,  $n \geq 0$ .

We assume that the system starts out at time  $t = 0$ , and that  $(u(t), \varepsilon(t)) = (0, 0)$  for  $t < 0$ . We assume that for  $t \geq 0$ ,  $(u(t), \varepsilon(t))$  is a vector stochastic process with diagonal noncontemporaneous cross-covariance matrices. In particular, we assume that

$$(2) \quad u(t) = a_1(L)w(t),$$

$$\varepsilon(t) = a_2(L)w(t)$$

where

$$w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix},$$

$Ew(t) = 0$  for  $t \geq 0$ .

$$Ew(t)w(t-s)^T = \begin{cases} 0 & \text{for } s \neq 0 \text{ and } t \geq 0 \\ \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{12}(t) & \sigma_{22}(t) \end{pmatrix} & \text{for } s = 0 \text{ and } t \geq 0. \end{cases}$$

In (2),  $a_1(L)$  and  $a_2(L)$  are each  $(1 \times 2)$  vectors in the lag operator that are one-sided and square summable in nonnegative powers of the lag operator  $L$ . In this section, we study the serially uncorrelated case in which

$$a_1(L) = (1 \ 0)$$

$$a_2(L) = (0 \ 1).$$

We seek solutions of the system of expectational stochastic difference equations (1) of the form

$$(3) \quad p(t) = d(L)w(t) + \pi_1 {}^t F_1 + \pi_2 {}^t F_2$$

$$h(t) = g(L)w(t) + \pi_1 {}^t J_1 + \pi_2 {}^t J_2.$$

Here  $\pi_1$  and  $\pi_2$  are the two zeroes of the same characteristic polynomial analyzed by Sargent and Wallace [ ], and which will reappear below. The parameters  $F_1$ ,  $F_2$  and  $J_1$ ,  $J_2$  are constants that can be regarded as representing the initial position of the system at time  $t = 0$ . In (3),  $d(L)$  and  $g(L)$  are each  $(1 \times 2)$  vector square summable polynomials in the lag operator that are one-sided in nonnegative powers of  $L$ . We seek solutions of the form (2) in which  $p(t)$  and  $h(t)$  are each of mean exponential order less than  $\lambda^{-1}$ , i.e., solutions for which

$$\lim_{j \rightarrow \infty} E_t \lambda^j p(t+j) = \lim_{j \rightarrow \infty} E_t \lambda^j h(t+j) = 0.$$

For the base money process  $h(t)$ , this condition is imposed to guarantee that the geometric sum

$$E_t \sum_{j=0}^{\infty} \lambda^j h(t+j)$$

converges for each  $t$ . The price level at  $t$  can be represented as a linear function of this geometric sum of expected future  $h(t+j)$ 's, and an analogous sum of expected future  $u(t+j)$ 's, so that convergence of this sum is a necessary condition for the existence of a solution of the difference equation system (1).

As we note more fully in section 6, the entire class of solutions of mean exponential order less than  $\lambda^{-1}$  cannot be represented in the form of (3). There are additional solutions depending on spurious indicators.

The exponential terms in  $\pi_1$  and  $\pi_2$  represent the deterministic part of the solution, which was solely focused upon by Sargent and Wallace, and which continues to play a role in the solution of the stochastic version of the system.

We shall proceed by deducing the restrictions that the model imposes on  $d(L)$  and  $g(L)$ , and across the  $F_i$ 's and  $J_i$ 's. The restrictions on these stochastic and deterministic parts of the solution can be deduced separately. Turning first to the deterministic parts, we have that (1), (2), and (3), imply that

$$(\pi_1 {}^t F_1 + \pi_2 {}^t F_2) = \lambda [\pi_1 {}^{t+1} F_1 + \pi_2 {}^{t+1} F_2] + \gamma [\pi_1 {}^t J_1 + \pi_2 {}^t J_2]$$

$$(\pi_1 {}^t J_1 + \pi_2 {}^t J_2) = \frac{1}{1+n} [\pi_1 {}^{t-1} J_1 + \pi_2 {}^{t-1} J_2] + \xi [\pi_1 {}^t F_1 + \pi_2 {}^t F_2].$$

Recall from Sargent and Wallace that  $\pi_1$  and  $\pi_2$  satisfy the characteristic equation

$$(1 - (\frac{1}{\lambda} + \frac{1}{1+n} - \frac{\gamma\xi}{\lambda})L + \frac{1}{(1+n)\lambda}L^2) = (1 - \pi_1 L)(1 - \pi_2 L).$$

It then follows that the above pair of equations is solved by  $(F_i, J_i)$  pairs satisfying

$$(4) \quad F_1 = \frac{\gamma}{1 - \lambda\pi_1} J_1$$
$$F_2 = \frac{\gamma}{1 - \lambda\pi_2} J_2.$$



These equations express the coefficient on  $\pi_j$  in the price level solution as a weighted geometric sum of future values of the coefficients on  $\pi_j^t$  in the solution for the money supply (see (1)). The restrictions that the model imposes across  $F_j$ 's and  $J_j$ 's thus have a natural interpretation. We shall regard  $J_1$  and  $J_2$  as free parameters and use (5) to determine  $F_1$  and  $F_2$ .

We now deduce the restrictions that the model imposes on  $d(L)$  and  $g(L)$ . Using (1), (2), (3), and the Wiener-Kolmogorov prediction formula, we find the following restrictions on  $d(L)$ ,  $g(L)$ :

$$d(L) = \lambda \left[ \frac{d(L)}{L} - \frac{d_0}{L} \right] + \gamma g(L) + a_1(L)$$

$$g(L) = \frac{1}{1+n} g(L)L + \xi d(L) + a_2(L).$$

Rearranging these in matrix form gives

$$\begin{bmatrix} (1-\lambda L^{-1}) & -\gamma \\ -\xi & (1 - \frac{1}{1+n} L) \end{bmatrix} \begin{bmatrix} d(L) \\ g(L) \end{bmatrix} = \begin{bmatrix} a_1(L) - \lambda d_0 L^{-1} \\ a_2(L) \end{bmatrix}$$

Premultiplying by the inverse of the matrix on the left and substituting  $a_1(L) = (1 \ 0)$ ,  $a_2(L) = (0 \ 1)$  gives

$$(5) \quad \begin{pmatrix} d(L) \\ g(L) \end{pmatrix} = \frac{-\lambda^{-1}}{(1-\pi_1 L)(1-\pi_2 L)} \begin{bmatrix} (1 - \frac{1}{1+n} L)(L-\lambda d_{01}), -\lambda d_{02}(1 - \frac{1}{1+n} L) + \gamma L \\ \xi(L-\lambda d_{01}), -\xi \lambda d_{02} + (L-\lambda) \end{bmatrix}$$

where the determinant of the matrix on the left side of (4) satisfies

$$\begin{aligned}
 & (1-\lambda L^{-1})(1 - \frac{1}{1+n} L) - \gamma \xi \\
 & = -L^{-1}\lambda(1 - (\frac{1}{\lambda} + \frac{1}{1+n} - \frac{\gamma \xi}{\lambda})L + \frac{1}{(1+n)\lambda} L^2) \\
 & = -\lambda L^{-1}(1-\pi_1 L)(1-\pi_2 L)
 \end{aligned}$$

where  $\pi_1, \pi_2 = [(\lambda^{-1} + (1+n)^{-1} - \xi\gamma/\lambda) \pm$

$\sqrt{(\lambda^{-1} + (1+n)^{-1} - \xi\gamma/\lambda)^2 - 4/(1+n)\lambda}] / 2$ , and where  $(1+n)^{-1} < \pi_1 < \pi_2 <$

$\lambda^{-1}$ . (This characteristic polynomial is identical with the one analyzed by Sargent and Wallace [ ], who establish the inequalities.) We assume that  $(\lambda^{-1} + (1+n)^{-1} - \xi\gamma/\lambda)^2 - 4/(1+n)\lambda > 0$ , which is a necessary condition for an equilibrium to exist. This condition places an upper limit on  $\xi$ , the average per capita real deficit. The upper limit is given by  $\frac{1}{\lambda}$

$$\xi \max = \frac{\lambda}{\gamma} \left[ \frac{1}{1+n} + \frac{1}{\lambda} - 2\sqrt{\frac{1}{\lambda(1+n)}} \right].$$

In (5), we have adopted the notation

$$d_0 = (d_{01} \quad d_{02})$$

where  $d_0$  is the coefficient on  $L^0$  in  $d(L)$ .

Equations (1), (4) and (5) represent the class of form (3) of solutions of mean exponential order less than  $\lambda^{-1}$  and reveal that there is a continuum of solutions within this class.<sup>2/</sup> The continuum is multidimensional, and surfaces both in the deterministic and the moving average part of the solution. First, given any level of initial per capita nominal balances  $h(-1)$ , it is possible to choose the free parameters  $J_1$  and  $J_2$  in a continuum of ways. This is the dimension of the continuum that

was focused on by Sargent and Wallace [ ]. Second, there is another dimension of the continuum that is conveniently indexed by the two underdetermined parameters  $d_{01}$ ,  $d_{02}$ . For any values of  $(d_{01}, d_{02})$ , (5) gives a solution to our system within the admissible class. Since  $(1+n)^{-1} < \pi_1 < \pi_2$ , the form of the representation (5) reveals that for almost all of the solutions, both  $p(t)$  and  $h(t)$  will eventually become processes of mean exponential order  $\pi_2$ .

Within the preceding class of solutions, there is a unique solution which is of mean exponential order  $\pi_1$ .<sup>3/</sup> This solution requires, first, that  $J_2 = 0$ . Second, this solution also requires that  $d_{01}$  and  $d_{02}$  be set at those values that cause each of the moving average polynomials in the matrix in brackets on the right side of (5) to have zeroes at  $\pi_2^{-1}$ , thereby cancelling the denominator polynomial  $(1-\pi_2 L)$ . Since there are four moving average polynomials in the matrix on the right side of (5), and only two free parameters  $d_{01}$  and  $d_{02}$ , it needs to be shown that choices of  $d_{01}$  and  $d_{02}$  exist that achieve the desired cancellation. Such values do exist, namely,

$$(6) \quad d_{01} = \frac{1}{\lambda \pi_2}$$

$$d_{02} = \frac{-\gamma(1+n)}{\lambda(1-\pi_2(1+n))}.$$

With these values of  $d_{01}$  and  $d_{02}$ , the solution (5) for  $(d(L), g(L))$  assumes the special form

$$(7) \quad \begin{pmatrix} d(L) \\ g(L) \end{pmatrix} = \frac{\lambda^{-1}}{(1-\pi_1 L)} \begin{pmatrix} (1 - \frac{1}{1+n}L)\frac{1}{\pi_2}, & -\gamma(1+n)/(1-\pi_2(1+n)) \\ \xi/\pi_2, & \lambda-(\xi\gamma(1+n))/(1-\pi_2(1+n)) \end{pmatrix}$$

This is the unique special case in which at time  $t$ , the price level and the per capita nominal balances are each expected eventually to rise at a gross rate  $\pi_1$ . For all other settings of  $d_{01}$ ,  $d_{02}$ , and  $J_2$ , price and per capita nominal balances are each expected eventually to rise at gross rates of  $\pi_2 > \pi_1$ . In particular, even if  $J_2 = 0$ , unless  $d_{01}$  and  $d_{02}$  are set to satisfy (6), the price level and per capita nominal balances will eventually rise at mean exponential orders approaching  $\pi_2$ .

We now offer some observations about interpreting the multiplicity of equilibria in this model. One way to interpret the multiplicity of equilibria is as reflecting the incompleteness of the "feedback law" (b) as a description of the evolution of  $h(t)$ . According to (b), namely

$$(b) \quad h(t) = \frac{1}{1+n} h(t-1) + \xi p(t) + \xi(t),$$

the monetary authority simply prints enough new money to buy  $(\xi + \varepsilon(t)/p(t))$  goods per capita, at the ruling price level  $p(t)$ . With this specification of policy, there are many equilibria.

An alternative description of policy is that the authority chooses  $J_1$ ,  $J_2$  and a kernel  $g(L)$  in the moving average representation for  $h(t)$ ,

$$h(t) = g(L)w_t + F_1\pi_1^t + F_2\pi_2^t,$$

subject to the condition that the deficit is financed and that portfolio balance prevails. This is equivalent with the authority choosing  $F_1$ ,  $F_2$  and  $d_0$ . Notice that to formulate policy in this way, the authority has to know the parameters of the model, and to

see current and lagged  $w(t)$ 's. To execute (b), the authority needs much less information. It simply prints enough new money to purchase  $\xi + \xi(t)/p(t)$  goods in period  $t$ .

From this viewpoint, it is understandable that the equilibrium is not unique with a less complete specification of policy, and that uniqueness can be obtained by specifying the evolution of policy actions in a more restrictive way. In a way, this example illustrates a general point, namely, the importance of the specification of strategy spaces in influencing matters of existence and uniqueness of equilibria.

Table 1 shows values of  $\xi \max$ ,  $\pi_1$  and  $\pi_2$  for various values of the free parameters. Given  $\lambda$ ,  $\gamma$  has been selected so that  $h/p$  evaluated at an expected gross inflation rate of unity and a zero disturbance to portfolio balance, which we denote  $(h/p)(1)$ , assumes the indicated value. The value of  $\gamma$  that achieves this is  $\gamma = (1-\lambda)/(h/p)(1)$ . Under this specification, the maximum sustainable per capita deficit  $\xi \max$  is given by

$$\xi \max = \frac{h}{p}(1) \cdot \frac{\lambda}{1-\lambda} \left[ \frac{1}{1+n} + \frac{1}{\lambda} - 2\sqrt{\frac{1}{\lambda(1+n)}} \right].$$

Notice that  $\xi \max$  varies proportionally with  $h/p(1)$ , which is the base for the inflation tax, directly with  $n$ , and inversely with  $\lambda$ . When  $n = 0$ ,  $\lim_{\lambda \rightarrow 1} \xi \max = 0$ , as application of l'hospital's rule to the above equation shows.

Table 1

$\lambda$	$h/p(1)$	$n$	$\xi_{max}$	$\xi$	$\pi_1$	$\pi_2$
.005	100	0	86.7918	$\xi_{max}$	14.14	$\pi_1$
.1	100	0	51.9494	$\xi_{max}$	3.1624	$\pi_1$
.3	100	0	29.2221	$\xi_{max}$	1.82579	$\pi_1$
.4	100	0	22.5148	$\xi_{max}$	1.58	$\pi_1$
.4	100	0	22.5148	15	1.2116	2.063
.5	100	0	17.157	$\xi_{max}$	1.414	$\pi_1$
.5	100	.015	17.777	$\xi_{max}$	1.4037	$\pi_1$
.5	100	0	17.157	15	1.25	1.6
.7	100	0	8.8933	$\xi_{max}$	1.195	$\pi_1$
.7	100	0	8.8933	8	1.129	1.265
.8	100	0	5.5728	$\xi_{max}$	1.118	$\pi_1$
.9	100	0	2.6334	$\xi_{max}$	1.0541	$\pi_1$
.9	200	0	5.2668	$\xi_{max}$	1.0541	
.95	100	0	1.2823	$\xi_{max}$	1.026	$\pi_1$
.98	100	0	.50506	$\xi_{max}$	1.010	$\pi_1$
.98	100	.005	.783265	$\xi_{max}$	1.008	$\pi_1$
.995	100	0	.1253	$\xi_{max}$	1.0025	$\pi_1$

### 3. Simulations and Evidence

Figures 1-2 show simulations of the system with  $n =$  \_\_\_\_\_,  $\lambda =$  \_\_\_\_\_,  $\gamma =$  \_\_\_\_\_, and  $\sigma_{11}(t) =$  \_\_\_\_\_, and  $\sigma_{22}(t) =$  \_\_\_\_\_,  $\sigma_{12}(t) = 0$ . In the simulation summarized in figure 1,  $d_0$  has been set to satisfy (6) and  $J_2 = 0$ . Here gross inflation oscillates around  $\pi_1 =$  \_\_\_\_\_, while real balances oscillate around the average that obtains under a gross inflation rate averaging  $\pi_1$ . Figure 2 simulates a system with  $J_2 = 0$ , but with  $d_0$  violating (6). Here inflation starts out near  $\pi_1$ , then grows toward  $\pi_2$ . Real balances decrease over time, converging toward the average behavior appropriate for an average gross inflation rate of  $\pi_2$ .

Figure 3 plots the time series of logs of inflation and real balances during the German hyperinflations. The last two months of October and November 1923, which are often excluded in empirical work, are included in these graphs.

Figures 4, 5, 6, 7, 8, and 9 plot inflation and real balances during the hyperinflations in Greece, Hungary after World War II, Austria, Hungary after World War I, Poland and Russia, respectively. The data are those used by Cagan. Of these seven countries, only Russia for long seems to have been close to a case in which the effects of  $\pi_2$  have been zeroed out. (See Keynes's remarks about the ingenuity of the Soviets in extracting seignorage. Also, notice Russia's position in the table of Cagan, reproduced by Sargent [     ].) The remaining countries more closely resemble equilibria of the kind depicted theoretically in figure 2, in which the  $\pi_2$  components gradually assume increasing importance.

#### 4. Identification

We first study identification in the singular case in which  $J_2 = 0$  and  $d_0$  satisfies (6). In this case, the bivariate  $(p(t), h(t))$  process evolves according to

$$(8) \quad \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \frac{\lambda^{-1}}{1-\pi_1 L} \begin{bmatrix} (1 - \frac{1}{1+n} L) \frac{1}{\pi_2}, & -\gamma(1+n)/(1-\pi_2(1+n)) \\ \xi/\pi_2, & \lambda - (\xi\gamma(1+n))/(1-\pi_2(1+n)) \end{bmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \\ + \begin{pmatrix} \gamma J_1 / (1-\lambda\pi_1) \\ J_1 \end{pmatrix} \pi_1^t .$$

We assume that the covariance matrix of  $w(t)$  grows geometrically at the rate  $\pi_1^{t/2}$  i.e.,

$$Ew(t)w(t)^T = \pi_1^t V$$

where  $V$  is a positive definite matrix. Let  $G^{-1}$  be a lower triangular matrix that normalizes and diagonalizes  $V$ , i.e.,  $I = G^{-1}VG^{-1T}$ . Define the transformed disturbance vector

$$\eta(t) = \pi_1^{-t/2} G^{-1} w(t).$$

Thus,  $E\eta(t)\eta(t)^T = I$ , while  $E\tilde{\eta}(t)\tilde{\eta}(t)^T = \pi_1^t I$ , where  $\tilde{\eta}(t) = \pi_1^{t/2} \eta(t)$ .

Now using  $w(t) = G\tilde{\eta}(t)$ , we can express (8) as



$$(1-\pi_1 L)I \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \lambda^{-1} \begin{bmatrix} (1 - \frac{1-L}{1+n})\frac{1}{\pi_2}, & \frac{-\gamma(1+n)}{1-\pi_2(1+n)} \\ \xi/\pi_2 & , \lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)} \end{bmatrix}$$

$$\begin{bmatrix} \xi_{11} & 0 \\ \xi_{21} & \xi_{22} \end{bmatrix} \begin{bmatrix} \tilde{n}_1(t) \\ \tilde{n}_2(t) \end{bmatrix} + \begin{bmatrix} \gamma J_1 / (1-\lambda\pi_1) \\ J_1 \end{bmatrix} \pi_1^t$$

or

$$(9) \quad (1-\pi_1 L) \begin{pmatrix} p(t) \\ h(t) \end{pmatrix}$$

$$= \lambda^{-1} \begin{bmatrix} (1 - \frac{1-L}{1+n})\frac{\xi_{11}}{\pi_2} - \frac{\gamma\xi_{21}(1+n)}{1-\pi_2(1+n)}, & \frac{-\xi_{22}\gamma(1+n)}{1-\pi_2(1+n)} \\ \xi_{11}/\pi_2 + \xi_{21}(\lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)}), & \xi_{22}(\lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)}) \end{bmatrix} \begin{bmatrix} \tilde{n}_1(t) \\ \tilde{n}_2(t) \end{bmatrix}$$

$$+ \begin{pmatrix} \gamma J_1 / (1-\lambda\pi_1) \\ J_1 \end{pmatrix} \pi_1^t.$$

In this representation, the three parameters,  $\xi_{11}$ ,  $\xi_{21}$ ,  $\xi_{22}$  represent the covariance matrix of the original  $w(t)$  process. The  $\tilde{n}(t)$  process was constructed by orthonormalizing the  $w(t)$  process with the matrix  $G$ . The matrix  $G$  thus summarizes all of the information in  $V$ .

Equation (9) is a vector autoregressive moving average representation whose identifiable parameters can be displayed as follows. Represent (9) as

$$(1-\pi_1 L) \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \begin{pmatrix} c_{11}^0 + c_{12}^1 L, & c_{12} \\ c_{21}, & c_{22} \end{pmatrix} \begin{pmatrix} \tilde{n}_1(t) \\ \tilde{n}_2(t) \end{pmatrix} + \begin{pmatrix} c_{13} \\ c_{23} \end{pmatrix} \pi_1^t.$$

The identifiable parameters are  $\pi_1$  and the  $c_{ij}^k$ 's. From (9) and the formula for  $\pi_1, \pi_2$ , these parameters are linked to the deep parameters of the model by the following equations:

$$(10) \quad c_{11}^0 = \lambda^{-1} \left( \frac{\xi_{11}}{\pi_2} - \frac{\gamma \xi_{21} (1+n)}{1-\pi_2 (1+n)} \right)$$

$$(11) \quad c_{11}^1 = \frac{-\lambda^{-1} \xi_{11}}{(1+n)\pi_2}$$

$$(12) \quad c_{12} = \frac{-\lambda^{-1} \xi_{22} \gamma (1+n)}{1 - \pi_2 (1+n)}$$

$$(13) \quad c_{21} = -\lambda^{-1} \left( \frac{\xi_{11}}{\pi_2} + \xi_{21} \left( \lambda - \frac{\xi \gamma (1+n)}{1-\pi_2 (1+n)} \right) \right)$$

$$(14) \quad c_{22} = \lambda^{-1} \xi_{22} \left( \lambda - \frac{\xi \gamma (1+n)}{1-\pi_2 (1+n)} \right)$$

$$(15) \quad \pi_1, \pi_2 = \left\{ (\lambda^{-1} + (1+n)^{-1} - \xi \gamma / \lambda) \right. \\ \left. \pm [(\lambda^{-1} + (1+n)^{-1} - \xi \gamma / \lambda)^2 - 4 / ((1+n)\lambda)]^{1/2} \right\} / 2$$

$$(16) \quad c_{13} = \gamma J_1 / (1 - \lambda \pi_1)$$

$$(17) \quad c_{23} = J_1.$$

The known variables in the nine equations (10)-(17) are  $\pi_1, c_{11}^0, c_{11}^1, c_{12}, c_{21}, c_{22}, c_{13}, c_{23}$ . The unknown variables to be determined are  $\lambda, \xi, \gamma, n, \pi_2, \xi_{11}, \xi_{21}, \xi_{22}, J_1$ . We thus have nine

equations to be solved for nine unknowns, which is promising from the viewpoint of identification.

To highlight the role of equations (16) and (17) in helping to achieve identification, we shall begin by ignoring them. This amounts to considering identification in the system from which the deterministic components  $\pi_1^t \begin{pmatrix} J_1 \\ F_1 \end{pmatrix}$  have been removed prior to estimation. In this case, equations (10)-(15) form seven equations in the eight unknowns  $\lambda$ ,  $\xi$ ,  $\gamma$ ,  $n$ ,  $\pi_2$ ,  $g_{11}$ ,  $g_{21}$ , and  $g_{22}$  so that these parameters are in general underidentified. However, under the special assumption that  $g_{21} = 0$ , local identification obtains. The assumption that  $g_{21} = 0$  is equivalent with the hypothesis that  $w_1(t)$  and  $w_2(t)$ , the disturbances to portfolio balance and to the government budget, respectively, are orthogonal.

In the case that  $g_{21} = 0$ , identification can be thought to proceed as follows. Equations (10) and (11) imply that

$$(1+n) = -c_{11}^0/c_{11},$$

while equations (13) and (10) imply that

$$\xi = -c_{21}/c_{11}^0.$$

So  $n$  and  $\xi$  are identified.

After some algebra, (14), (12) and (15) imply that

$$(18) \quad c_{22}/c_{12} = (1-\pi_1\lambda)/-\gamma$$

Given knowledge of  $n$  and  $\xi$ , equation (18) together with (15) for  $\pi_1$ , namely,

$$\pi_1 = \left\{ (\lambda^{-1} + (1+n)^{-1} - \frac{\xi\gamma}{\lambda}) - \sqrt{(\lambda^{-1} + (1+n)^{-1} - \frac{\xi\gamma}{\lambda})^2 - 4/(1+n)\lambda} \right\},$$

form two equations in  $\gamma$  and  $\lambda$  which possess a locally unique solution. Given  $\pi_1$ ,  $\lambda$ ,  $\xi$ , and  $n$ ,  $\pi_2$  can be obtained from (15). Then  $g_{11}$  can be obtained from (10), and  $g_{22}$  from (12). This completes the discussion of identification in the special case in which  $g_{21} = 0$  and  $J_2 = 0$ .

With  $J_1 = 0$  and  $g_{21}$  an unknown to be identified, the parameters of the model become underidentified. In this case, we are one restriction short of having an identified system. When  $J_1 \neq 0$ , equations (16) and (17) add two equations but only one unknown to the system. This leaves us with a system of nine equations in the nine unknown parameters to be identified.

The preceding analysis shows that in the singular case in which the root  $\pi_2$  has been eliminated from the system, identification is delicate. Identification hinges either on including the deterministic component  $(\pi_1^t)$  explicitly in the estimation process, or by a priori imposing orthogonality between  $w_1(t)$  and  $w_2(t)$ .

We now briefly discuss identification in the more general case in which  $\pi_2$  has not been zeroed out. In this case, we would assume

$$Ew(t)w(t)^T = \pi_2^t v.$$

We then proceed as above, defining  $G^{-1}$  to be the lower triangular matrix such that  $I = G^{-1}VG^{-1T}$ .

In this general case, the solution has representation

$$(20) \quad (1-\pi_1 L)(1-\pi_2 L) \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = -\lambda^{-1} \begin{pmatrix} (1 - \frac{1}{1+n}L)(L-\lambda d_{01}), & -\lambda d_{02}(1 - \frac{1}{1+n}L) + \gamma L \\ \xi(L-\lambda d_{01}) & , & -\xi \lambda d_{02} + (L-\lambda) \end{pmatrix}$$

$$\begin{pmatrix} \xi_{11} & 0 \\ \xi_{21} & \xi_{22} \end{pmatrix} \begin{pmatrix} \tilde{n}_1(t) \\ \tilde{n}_2(t) \end{pmatrix} + \begin{pmatrix} \gamma J_1 \\ 1-\lambda \pi_1 \\ J_1 \end{pmatrix} \pi_1^t + \begin{pmatrix} \gamma J_2 \\ 1-\lambda \pi_2 \\ J_2 \end{pmatrix} \pi_2^t$$

This can be represented in terms of identifiable parameters as

$$(21) \quad (1-\pi_1 L)(1-\pi_2 L) \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} =$$

$$\begin{pmatrix} c_{11}^0 + c_{11}^1 L + c_{11}^2 L, & c_{12}^0 + c_{12}^1 L \\ c_{21}^0 + c_{21}^1 L, & c_{22}^0 + c_{22}^1 L \end{pmatrix} \begin{pmatrix} \tilde{n}_1(t) \\ \tilde{n}_2(t) \end{pmatrix} + \begin{pmatrix} c_{31} \\ c_{32} \end{pmatrix} \pi_1^t + \begin{pmatrix} c_{41} \\ c_{42} \end{pmatrix} \pi_2^t.$$

The identifiable parameters are  $\pi_1$ ,  $\pi_2$ , and the  $c_{ij}^k$ 's. These are linked to the 11 deep free parameters ( $\lambda$ ,  $\gamma$ ,  $\xi$ ,  $n$ ,  $d_{01}$ ,  $d_{02}$ ,  $\xi_{11}$ ,  $\xi_{12}$ ,  $\xi_{22}$ ,  $J_1$ ,  $J_2$ ) by the 15 equations created by (15) and by matching coefficients in (20) and (21). According to these "order" conditions, the model is overidentified.

### 5. Equilibria Depending on Spurious Indicators

We now generalize the preceding solution. We assume that  $u(t) = (a_{11}(L) \ 0 \ 0)w(t)$  and  $\varepsilon(t) = (0 \ a_{22}(L) \ 0)w(t)$ , where  $w(t)$  is now a  $(3 \times 1)$  vector white noise, and  $a_{11}(L)$  and  $a_{22}(L)$  are each square summable and invertible polynomials in nonnegative powers of  $L$ . The third component  $w_3(t)$  of  $w(t)$  is

included to permit a "nonfundamental" or "self-fulfilling" stochastic component to the solution for  $(p(t), h(t))$ . We now let  $d(L)$  and  $g(L)$  each be  $(1 \times 3)$  row vector polynomials in  $L$ .

The model restricts the polynomials  $(d(L), g(L))$  to satisfy

$$\begin{pmatrix} d(L) \\ g(L) \end{pmatrix} = \frac{\lambda^{-1}L}{(1-\pi_1 L)(1-\pi_2 L)} \begin{bmatrix} (1 - \frac{1}{1+n}L) & \gamma \\ \xi & (1-\lambda L^{-1}) \end{bmatrix} + \begin{bmatrix} a_{11}(L) - \lambda d_{01} L^{-1}, & -\lambda d_{02} L^{-1}, & -\lambda d_{03} L^{-1} \\ 0 & a_{22}(L), & 0 \end{bmatrix}$$

where  $d_0 = (d_{01}, d_{02}, d_{03})$ . Carrying out the multiplication on the right side, we have

$$(22) \quad \begin{pmatrix} d(L) \\ g(L) \end{pmatrix} = \frac{-\lambda^{-1}L}{(1-\pi_1 L)(1-\pi_2 L)} \begin{bmatrix} (1 - \frac{1}{1+n}L)(a_{11}(L) - \lambda d_{01} L^{-1}), & -(1 - \frac{1}{1+n}L)\lambda d_{02} L^{-1} + \gamma a_{22}(L), \\ \xi(a_{11}(L) - \lambda d_{01} L^{-1}), & -\xi \lambda d_{02} L^{-1} + a_{22}(L)(1-\lambda L^{-1}) \\ -\lambda d_{03} L^{-1}(1 - \frac{1}{1+n}L) \\ -\xi \lambda d_{03} L^{-1} \end{bmatrix}$$

The first two columns agree with our earlier solution for the special case in which  $a_{11}(L) = 1$ ,  $a_{22}(L) = 1$ . Equation (\*) represents a solution for any values of  $(d_{01}, d_{02}, d_{03}) \in R^3$ .

Equation (22) indicates that there is an additional dimension to the multiplicity of solutions described earlier.

This additional dimension is conveniently indexed by the undetermined parameter  $d_{03}$ . Solutions with  $d_{03}$  are "self-fulfilling" or "nonfundamental" solutions in which a white noise random process  $w_3(t)$  that has no influence directly upon the "fundamental" disturbances  $u(t)$  and  $\epsilon(t)$  nevertheless plays a role in the rational expectations solution.

The third columns of  $d(L)$ ,  $g(L)$ , namely,

$$\begin{pmatrix} d_3(L) \\ g_3(L) \end{pmatrix} = \frac{1}{(1-\pi_1 L)(1-\pi_2 L)} \begin{bmatrix} d_{03} \left(1 - \frac{1}{1+n} L\right) \\ -\xi d_{03} \end{bmatrix}$$

have the property that there is no nontrivial choice of  $d_{03}$  that "zeroes out" the polynomial  $(1-\pi_2 L)$ . That is, if  $d_{03} \neq 0$ , then the solution must eventually become of mean exponential order  $\pi_2$ .

The reader can convince himself that moving averages in any number of self-fulfilling white noises can be added to the solution, provided that they bear polynomials in  $L$  that are proportional to those of  $(d_3(L), g_3(L))$ .

The preceding findings are consistent with slightly reinterpreted versions of the results of Charles Whiteman. Working in a stationary context, Whiteman pointed out quite generally that in cases in which there exist multiple linear rational expectations equilibria in a Hilbert space of lagged fundamental processes, there exist many additional equilibria that can be formed by adding to those solutions moving averages of a "spurious" white noise.<sup>5/</sup>

## 6. Granger Causality

First consider the special case of the model in which  $a_{11}(L) - a_{22}(L) = 0$ ,  $w_{3t} = 0$ . Also assume that (6) and  $J_2 = 0$  hold, so that the  $\pi_2$  mode of the system has been deactivated. In this special case, the solution for  $p(t)$ ,  $h(t)$  has the representation

$$\begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \frac{\lambda^{-1}}{1-\pi_1 L} \begin{bmatrix} (1 - \frac{1}{1+n}L) \frac{1}{\pi_2}, & -\gamma(1+n)/(1-\pi_2(1+n)) \\ \xi t_2, & \lambda - (\xi\gamma(1+n))/(1-\pi_2(1+n)) \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \\ + \begin{pmatrix} \gamma J_1 / (1-\lambda\pi_1) \\ J_1 \end{pmatrix} \pi_1^t.$$

Note that in the representation for  $h(t)$ , the polynomials in  $L$  on  $w_1(t)$  and  $w_2(t)$  are proportional to one another, while in the representation for  $p(t)$ , the polynomials in  $w_1(t)$  and  $w_2(t)$  are not proportional. This implies that  $h(t)$  Granger causes  $p(t)$ , while  $p(t)$  fails to Granger cause  $h(t)$ .<sup>6/</sup>

This structure of Granger causality is a special feature of the singular case in which  $(1-\pi_2 L)$  has been cancelled out. In the general case,  $p(t)$  and  $h(t)$  Granger cause each other, which can be proved by studying the structure of the solution (5) or (22). We now seek special cases in which  $p(t)$  Granger causes  $h(t)$ , with no Granger causality extending from  $h(t)$  to  $p(t)$ .

We first consider a case in which there is no spurious indicator impinging on the solution, so that in (22)  $d_{03} = 0$ . Suppose that  $a_{11}(L)$  and  $a_{22}(L)$  satisfy the restriction



$$a_{22}(L) = \frac{d_{02}}{\gamma d_{01}} \left(1 - \frac{1}{1+n}L\right) a_{11}(L).$$

The reader can verify that under the above restriction, the polynomials on  $w_1(t)$  and  $w_2(t)$  in  $d(L)$  are proportional to one another, while the polynomials in  $g(L)$  are not proportional to one another. Under this special condition, it follows that  $p(t)$  Granger causes  $h(t)$ , but that  $h(t)$  fails to Granger cause  $p(t)$ .

To motivate the next special case, notice that real balances demanded at a constant expected gross return on money of unity are given by  $(1-\lambda)/\gamma$ . Let us reparameterize the system by setting  $\gamma = (1-\lambda)\theta$ , where  $\theta = (p/h)(1)$ , the inverse of real balances at a gross inflation rate of unity. To achieve the following special case, we shall think of holding  $\theta$  fixed as we vary  $\lambda$ , so that as  $\lambda \rightarrow 1$ ,  $\gamma \rightarrow 0$ . In the limiting case with  $\lambda = 1$ ,  $\gamma = 0$ , (22) implies

$$\begin{pmatrix} d(L) \\ g(L) \end{pmatrix} = \begin{pmatrix} \frac{d_{01}-L}{1-L} & , & \frac{d_{02}}{1-L} & , & \frac{d_{03}}{1-L} \\ \frac{\xi(d_{01}-L)}{(1-L)(1 - \frac{1}{1+n}L)} & , & \frac{\xi d_{02}+(1-L)}{(1-L)(1 - \frac{1}{1+n}L)} & , & \frac{\xi d_{03}}{(1-L)(1 - \frac{1}{1+n}L)} \end{pmatrix}$$

where we are using the fact that when  $\gamma = 0$ ,  $\pi_1 = (1+n)^{-1}$ ,  $\pi_2 = 1$ . In the special case that  $w_{1t} = 0$ , so that the portfolio balance schedule is exact,  $d(L)$  is such that  $p_t$  is a martingale, so that  $p(t)$  is not Granger-caused by  $h(t)$ . However,  $p(t)$  Granger causes  $h(t)$  so long as the spurious indicator is present. (The martingale characterization of  $p(t)$  under these conditions can also be deduced directly from equation (a).)

These examples constitute singular cases in which  $h(t)$  fails to Granger cause  $p(t)$ . They indicate the presence of a range of examples close to these in which 0 ranges causality extends from  $h(t)$  to  $p(t)$ , but is difficult to detect in short samples. The model thus appears to be potentially capable of accommodating Granger causality patterns such as those detected in earlier work (Sargent and Wallace [    ]).

### 7. Identification in Equilibria with Spurious Indicators

Suppose that a solution of the class (22) is guiding the system. In particular, we have that  $(p(t), h(t))$  is evolving according to

$$\begin{aligned}
 (23) \quad & (1-\pi_1 L)(1-\pi_2 L) \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} \\
 & = -\lambda^{-1} L \begin{pmatrix} (1 - \frac{1}{1+n} L)(a_{11}(L) - \lambda d_{01} L^{-1}), -(1 - \frac{1}{1+n} L)\lambda d_{02} L^{-1} + \gamma a_{22}(L), \\ \xi(a_{11}(L) - \lambda d_{01} L^{-1}) \quad , \quad -\xi \lambda d_{02} L^{-1} + a_{22}(L)(1 - \lambda L^{-1}) , \\ -\lambda d_{03} L^{-1} (1 - \frac{1}{1+n} L) \\ -\xi \lambda d_{03} L^{-1} \end{pmatrix} \\
 & \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} + \pi_1^t \begin{pmatrix} \frac{\gamma J_1}{1 - \pi_1 \lambda} \\ J_1 \end{pmatrix} + \pi_2^t \begin{pmatrix} \frac{\gamma J_2}{1 - \pi_2 \lambda} \\ J_2 \end{pmatrix}
 \end{aligned}$$

We assume that

$$Ew(t)w(t)^T = \pi_2^t \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix},$$

so that the spurious white noise is orthogonal to the "fundamentals." We represent  $w(t)$  as

$$w(t) = \pi_2^{t/2} \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{pmatrix}$$

where  $Ez_t z_t^T = I$  and where  $\pi_2^{t/2} G$  is a matrix that orthonormalizes the  $w(t)$  vector. The parameters  $\pi_2$  and  $\varepsilon_{ij}$  represent the covariance structure of the  $w(t)$  process.

To begin, assume as above that  $a_{11}(L) = a_{22}(L) = 1$ . The moving average part of the solution now has as free parameters ( $\lambda, \gamma, \xi, n, \varepsilon_{11}, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{33}, d_{01}, d_{02}, d_{03}$ ) while the deterministic part has ( $J_1, J_2$ ), for a total of 13 parameters to be identified.

When  $a_{11}(L) = a_{22}(L) = 1$ , the identifiable parameters of (23) continue to be displayed in (21). Let the moving average part of (23) be represented as  $\phi(L)G\tilde{z}_t$  and let the moving average part of (21) be  $c(L)\tilde{\eta}_t$ , where  $\tilde{z}_t = \pi_2^{t/2} z_t$ . Then by using the same steps used to obtain a Wold representation by factoring a spectral density matrix,  $C(L)$  can be determined by solving  $C(L)C(L^{-1})^T = \phi(L)G G^T \phi(L^{-1})^T$ . These equations supply a total of nine equations. Four more equations are given by matching the

deterministic parts of (21) and (23), while two more equations are given by (15). Thus we have a total of 15 restrictions on 13 free parameters. According to such order considerations, the model seems overidentified even with  $a_{11}(L) = a_{22}(L) = 1$  in the presence of a spurious noise.

In general, richer specifications for  $a_{11}(L)$  and  $a_{22}(L)$  will lead to stronger overidentification. As inspection of (+) shows, the higher is the order of  $a_{jj}(L)$ , the more cross-equation restrictions are there on the moving average representation of the  $(p(t), h(t))$  process.

### 8. Estimation

We describe how to estimate the system (22), in which a spurious indicator  $w_3(t)$  is included among the noises driving the system. We let  $w(t)^T = (w_1(t), w_2(t), w_3(t))$ . Let us represent the system as

$$(24) \quad (1-\pi_1 L)(1-\pi_2 L)y(t) = (D_0 + D_1 L + D_2 L)w(t) + H_1 \pi_1^t + H_2 \pi_2^t \\ = D(L)w(t) + H_1 \pi_1^t + H_2 \pi_2^t$$

where  $D(L) = \begin{pmatrix} d(L) \\ g(L) \end{pmatrix}$ , where  $y(t) = \begin{pmatrix} p(t) \\ h(t) \end{pmatrix}$ , and where  $H_1 = \begin{pmatrix} \gamma J_1 \\ \frac{1-\pi_1^\lambda}{1-\pi_1} \\ J_1 \end{pmatrix}$ ,

$$H_2 = \begin{pmatrix} \gamma J_2 \\ \frac{1-\pi_2^\lambda}{1-\pi_2} \\ J_2 \end{pmatrix}.$$

As in the preceding section, we suppose that

$$Ew(t)w(t)^T = \mu^t V,$$

where  $V$  is a positive semi-definite matrix, and where we shall set  $\mu = \pi_2$ . We define the transformed variables

$$\tilde{y}(t) = \mu^{-t/2} y(t)$$

$$\tilde{w}(t) = \mu^{-t/2} w(t).$$

In terms of these transformed variables, (24) becomes

$$(25) \quad (1 - \pi_1 \mu^{-1/2L})(1 - \pi_2 \mu^{-1/2L}) \tilde{y}(t) = D(\mu^{-1/2L}) \tilde{w}(t) \\ + H_1 \pi_1 t \mu^{-t/2} + H_2 \pi_2 t \mu^{-t/2}.$$

In (25),  $D(\mu^{-1/2L}) = D_0 + D_1 \mu^{-1/2L} + D_2 \mu^{-2/2L^2}$ .

The moving average component  $D(\mu^{-1/2L}) \tilde{w}(t)$  is covariance stationary, and has covariance generating function

$$D(\mu^{-1/2z}) V D(\mu^{-1/2z^{-1}})^T,$$

where recall that  $D$  is a  $2 \times 3$  matrix and  $V$  is a  $(3 \times 3)$  matrix.

We obtain a fundamental Wold representation for  $D(\mu^{-1/2L}) \tilde{w}(t)$  by factoring the above covariance generating equation; that is, by solving

$$(26) \quad D(\mu^{-1/2z}) V D(\mu^{-1/2z^{-1}})^T = F(z) \Omega F(z^{-1})^T$$

where  $\Omega$  is a  $2 \times 2$  positive semi-definite matrix, where  $F(z) = I + F_1 z + F_2 z^2$ , where the  $F_j$  are  $(2 \times 2)$  matrices, and where the zeroes of  $\det F(z)$  lie outside the unit circle. With these side conditions, there is a unique  $F(z)$  that solves (26). The matrix  $\Omega$  is the covariance matrix of  $\tilde{a}_t$ , the vector of one-step ahead errors

in predicting  $\tilde{y}_t$  from its own infinite past:  $\tilde{a}_t = \tilde{y}_t - E\tilde{y}_t | \tilde{y}_{t-1}, \tilde{y}_{t-2}, \dots$

Given the solution to (26), we have that

$$(27) \quad D(\mu^{-1/2}L)\tilde{w}(t) \equiv F(L)\tilde{a}_t.$$

Equation (27) expresses the moving average  $D(\mu^{-1/2})\tilde{w}(t)$  in  $\tilde{w}(t)$ , which is possibly unobservable to the econometrician possessing only observations on  $y(t)$ , in terms of a moving average of the (2x1) vector of noises  $\tilde{a}_t$  that are fundamental for  $\tilde{y}_t$ .

Substituting (27) into (25) gives

$$(28) \quad (1-\pi_1\mu^{-1/2}L)(1-\pi_2\mu^{-1/2}L)\tilde{y}(t) = \tilde{a}_t + F_1\tilde{a}_{t-1} + F_2\tilde{a}_{t-2} \\ + H_1\pi_1 t_{\mu^{-t/2}} + H_2\pi_2 t_{\mu^{-t/2}}.$$

Solving for  $\tilde{a}_t$  gives

$$(29) \quad \tilde{a}_t = -F_1\tilde{a}_{t-1} - F_2\tilde{a}_{t-2} - H_1\pi_1 t_{\mu^{-t/2}} - H_2\pi_2 t_{\mu^{-t/2}} + \tilde{y}(t) \\ - (\pi_1 + \pi_2)\mu^{-1/2}\tilde{y}(t-1) + \pi_1\pi_2\mu^{-1}\tilde{y}(t-2).$$

This equation expresses innovations  $\tilde{a}_t$  in  $\tilde{y}_t$  in terms of values of the free parameters of the model. Given a sample on  $y(t)$  running over  $t = 1, \dots, T$  and assuming that the  $w(t)$ 's are distributed according to the multivariate normal distribution, maximum likelihood estimates can be obtained by maximizing

$$\log L = -T - 1/2 \log \det \Omega - 1/2 \tilde{a}_t^T \Omega^{-1} \tilde{a}_t$$

over the free parameters subject to (26), and the restriction that  $\mu = \pi_2$ . This minimization is to be accomplished by using a hill

climbing procedure. For a given set of values in the space of free parameters, (26) is to be solved for  $F(L)$ , then (29) is to be used to calculate the  $\tilde{a}_t$ 's implied by those parameters, and (30) is to be computed.

We now consider in the special case in which there is no spurious indicator, so that the model is (20). In this case, (24) is to be reinterpreted with  $w(t)^T$  now being the  $(2 \times 1)$  vector  $(w_1(t), w_2(t))$ . The estimation procedure described above then remains appropriate. Notice that in this case, the step of factoring the spectral density matrix of the moving average part via equation (26) plays a role even though  $w(t)$  is now  $(2 \times 1)$ . In particular, it is possible that  $\det D(\mu^{-1/2}z)$  has zeroes inside the unit circle. The step of using (26) insures that  $\det F(z)$  has all of its zeroes outside the unit circle. This condition is required for (29) to be a valid way of recovering the  $\tilde{a}_t$ 's that appear in the likelihood function (30).

In applying the above method, it would be fortunate if  $F(L)$  were independent of the parameters of  $V = E\tilde{w}(t)\tilde{w}(t)^T$ , depending only on the parameters of  $D(L)$ . (There seems to be a chance for this in the no-spurious indicator case, but little chance when  $w(t)$  is of dimension  $3 \times 1$ .) In this special case, maximum likelihood estimates can be obtained by minimizing

$$\det \sum_{t=1}^T \tilde{a}_t \tilde{a}_t^T$$

over the free parameters that determine  $D(L)$ ,  $\pi_1, \pi_2$  subject to  $\mu = \pi_2$ . Then the covariance matrix  $\Omega$  can be estimated by

$$\hat{\Omega} = T^{-1} \sum_{t=1}^T \tilde{a}_t \tilde{a}_t^T,$$

where the  $\tilde{a}_t$  are evaluated at the maximum likelihood values of the parameters determining  $D(L)$ ,  $\pi_1$  and  $\pi_2$ . This procedure has the advantage of a reduced number of parameters over which the search for an optimum is conducted, the parameters in  $V$  not being searched over.

#### Procedures for Starting Estimates

To obtain initial estimates of some of the parameters with which to pursue maximization of the likelihood function, we could proceed as follows. First, set  $n = 0$ . Second, estimate  $\xi$  from the sample average of

$$(h(t)-h(t-1))/p(t) = \hat{\xi}$$

Third, estimate the variance of  $\varepsilon(t)$  by computing the variance of

$$\hat{\varepsilon}(t) = (h(t)-h(t-1) - \hat{\xi}p(t)).$$

Fourth, estimate  $\gamma^{-1}(1-\lambda)$  as  $(\overline{h/p})_1$ , the mean of  $h(t)/p(t)$  during a preinflationary period over which the gross inflation rate averaged about unity. Fifth, estimate  $\lambda$  from

$$\begin{aligned} \left(\frac{\overline{h}}{\overline{p}}\right)_1 &= \gamma^{-1}(1-\lambda) \\ \left(\frac{\overline{h}}{\overline{p}}\right)_2 &= \gamma^{-1} - \frac{\lambda}{\gamma} \left(\frac{\overline{p+1}}{\overline{p}}\right)_2 \end{aligned}$$

where  $(\overline{h/p})_2$  is the mean  $(h/p)$  near the end of the hyperinflation and  $(\overline{p+1/p})_2$  is the gross inflation rate over the same period. Sixth, estimate the variance of  $u$  by using the preceding estimates of  $\lambda$  and  $\gamma$  together with the "variance bounds" inequality

$$\sigma_u^2 < \text{var} [p(t)-\lambda p(t+1)-\gamma h(t)].$$



9. A System with Bonds

We now describe how to compute the solution of an expanded and reinterpreted system, a version of which was studied by Sargent and Wallace [ ]. The method is related to the one used by Novales to study nonlinear rational expectations models.

The system is now

- (a) 
$$p_t = \lambda E_t p_{t+1} + \gamma h_t + u_t$$
- (b) 
$$h_t = \frac{1}{1+n} h_{t-1} + \xi p_t + \epsilon_t$$
- (c) 
$$b_t = \frac{1 + R_{t-1}}{1+n} b_{t-1} + d_t - (\xi + \epsilon_t / p_t)$$
- (d) 
$$R_{t-1} = R(p_{t-1} b_{t-1} / h_{t-1}) + \rho_t, R^1 > 0.$$

Equations (a) and (b) are identical with those studied above, only now  $(\xi p_t + \epsilon_t)$  has an interpretation as the amount of the nominal deficit per capita that is to be financed by printing high-powered money. In (c),  $R_{t-1}$  is the real rate of return on one-period interest bearing government bonds, while  $d_t$  is the real per capita deficit, net of interest payments. We think of  $d_t$  as a stochastic process that is exogenous with respect to  $p_t, h_t$ . In (d),  $\rho_t$  is an exogenous, possibly stochastic process, while  $R(\cdot)$  is a nondecreasing function, designed to reflect a dependence of the interest rate on government debt on the ratio of bonds to base money that is outstanding.

This system has a recursive structure. Equations (a) and (b) can be solved for stochastic processes for  $(p_t, h_t)$ . Then (c) and (d) can be solved for stochastic processes for  $b_t, R_t$ .

This system could be used to create simulations of more complicated and stochastic versions of the unpleasant arithmetic examples studied by Sargent and Wallace. Various iterative devices could be used to incorporate the idea that there is an upper bound on  $b_t$  that can create a tradeoff between tight money today and loose money tomorrow. For example, for a given  $d_t$  process, we could study an  $\epsilon_t$  process with a moving average representation of the form  $\epsilon_t = a_{22}(L)w_{2t} \equiv c(L)w_{2t}$ ,  $w_{2t} = \epsilon_t - E_{t-1}\epsilon_t$ , with  $c_j = -1$  for  $j = 0, 1, \dots, M$ ;  $c_j = +1$  for  $j = M + 1, \dots, 2M$ . A large negative innovation in  $\epsilon_t$  generates a string of  $M$  large money supply decreases, which are followed by  $M$  money supply increases.

Simulations could also be generated roughly to match data by working backwards from observed time series for  $p_t, h_t, d_t, b_t, R_{t-1}$  to processes for  $\epsilon_t, u_t, p_t$ , the function  $R(\cdot)$ , and the free parameters  $\lambda, \xi, \lambda$ .

Footnotes

1/For settings of  $\xi > \xi_{\max}$ , solutions of the expectational difference equation system (1) still exist, and are of exactly the form given in the text. However, when  $\xi > \xi_{\max}$ , the roots  $\pi_1$  and  $\pi_2$  are complex conjugates, and the deterministic part of the system

$$\pi_1^t \begin{pmatrix} F_1 \\ J_1 \end{pmatrix} + \pi_2^t \begin{pmatrix} F_2 \\ J_2 \end{pmatrix}$$

can be shown to oscillate among positive and negative values of both  $p(t)$  and  $h(t)$ . The system only makes economic sense when  $p(t)$  and  $h(t)$  are restricted to remain positive, which is the reason that we require  $\xi < \xi_{\max}$ . (The reader is invited to construct the complete version of figure 1, filling in the remaining quadrants, and to use it to analyze the movements of the deterministic part of the system when  $\xi > \xi_{\max}$ .)

2/As for the moving average part of the solution, the existence of a continuum of solutions of mean exponential order less than  $\lambda^{-1}$  is predicted by a modified version of Whiteman's theorem [p. 91]. Modifications of the conditions of the statement of Whiteman's theorem are needed to allow for (a) the fact that in our system, Whiteman's  $F_1$  is singular, and (b) the fact that we are seeking a solution of mean exponential order less than  $\lambda^{-1}$ , while Whiteman sought a solution of mean exponential order less than unity.

3/The solution is unique only up to the choice of  $J_1$ , which can be regarded as reflecting a choice of units in which to express money and price.

4/ Our assumption is that the covariance matrix of the  $w(t)$  process grows geometrically at the same rate as the maximal root, which is generally  $\pi_2$ , but is  $\pi_1$  in the special case now under analysis. This assumption delivers a kind of eventual homoskedasticity for gross inflation and money creation rates.

5/ By "adjusting" our solutions by multiplying the solutions by a factor  $\theta^t$  where  $\theta > \pi_2$ , we could transform our system to one to which Whiteman's results would apply.

6/ For a proof, see exercise \_\_\_\_\_ of Sargent [ , Chapter XI].

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