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EFFICIENT AND INEFFICIENT COMMODITY "MONEY" EQUILIBRIA

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ABSTRACT

We study an overlapping generations model which contains a capital good that resembles actual gold. This capital good can be stored without physically depreciating and can, by using other resources, be converted back and forth between gold jewelry which yields utility directly and raw gold which does not. Under the assumption that the three utility-yielding objects--first and second period consumption and jewelry--are gross substitutes, stationary equilibria are shown to exist and are characterized; for some parameter values, there are inefficient equilibria, while for others there are efficient equilibria. Both types can be interpreted as commodity money equilibria.

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One widely-held notion about commodity money systems is that they are inefficient if using the commodity as money precludes using it as a consumption good or as a productive input. One coherent but not widely accepted interpretation of this inefficiency notion involves identifying it with the inefficient or overaccumulation-of-capital equilibria of overlapping generations models.[1/] This interpretation seems not to be widely accepted in part because the capital goods of most overlapping generations models do not resemble the objects used as commodity monies--for example, the precious metals. Our goal in this paper is to overcome this objection and, thereby, to make the above identification more palatable. We do this by displaying and analyzing an overlapping generations model in which the capital good displays some of the technological features of actual commodity monies.

The capital of our overlapping generations model, which we label "gold," has features that make it resemble a precious metal like gold. There is a fixed and indestructible stock of it which can potentially take three forms at each discrete date t: "new jewelry," "raw gold," and "old jewelry." The first gives utility to those who hold it from t to t + 1. However, at t + 1, it automatically becomes old jewelry, which does not give utility to other people who hold it subsequently. Old jewelry can, however, be turned into utility-yielding new jewelry by expending nongold resources. It must first, via a costly technology, be turned into raw gold and then the raw gold, via another costly technology, gets turned into new jewelry. As we will see, whether an equilibrium for our model is efficient or not depends on the form in which gold ends up being held. In particular, although neither old jewelry nor raw gold yields utility, only holdings of the latter (forever) imply inefficiency.

The paper is organized and its main results are as follows. The model is described in detail in Section 1. The conditions for a perfect foresight competitive equilibrium are developed and set out in Section 2. In Section 3, we prove the existence of and partly characterize stationary equilibria. It turns out that the assumption of gross substitution among the three utility-yielding objects--nongold consumption when young and when old and new jewelry--plays a crucial role in the existence proof. Indeed, as argued in Appendix 2, such an assumption is necessary for existence even of a steady state. In Section 4, we characterize stationary equilibria in terms of Pareto-optimality. It turns out that there is no price characterization of optimality, but that there is one in terms of the objects held in an equilibrium. A necessary and sufficient condition for inefficiency of a stationary equilibrium is that raw gold is held. In Section 5, we introduce fiat money and examine the relationship between its equilibrium value and the equilibrium relative price of gold and nongold goods. The implied relationship is consistent with the notion that a commodity (gold) is more valuable in terms of other commodities the more important is its role as a commodity money. Most of the proofs appear in Appendix 1.

1. The Model

The model is of a discrete time economy defined over integer dates $t \geq 1$. It is an overlapping generations model of two-period lived generations, which differs from other such models because of the role played by the capital good which we call gold.

Resources and Technologies

At each date t, the economy has a new social endowment of $W > 0$ units of a good, which we call time t bread. Time t bread cannot be converted via a production process into time $t + j$ bread, for any $j \neq 0$.

The only other resource of this economy is a positive stock of gold at $t = 1$. We assume that this is in the form of "old jewelry." Gold can potentially take three forms at each date; old jewelry, raw gold, and new jewelry. As described below, only new jewelry is an argument of utility functions.

There are simple, fixed proportion technologies available for producing each form of gold. For any $q \geq 0$, q units of new jewelry at t can be produced using q units of raw gold at t and $a_1 q$ units of time t bread, where $a_1 \geq 0$. There are two ways to produce raw gold at t. For any $q \geq 0$, q units of raw gold at t can be produced from q units of raw gold at $t - 1$ (by storing the raw gold at $t - 1$); it can also be produced using q units of old jewelry at t and $a_2 q$ units of time t bread, where $a_2 > 0$. There are also two ways to produce old jewelry at t. For any $q \geq 0$, q units of old jewelry at t can be produced from q units of old jewelry at $t - 1$ or from q units of new jewelry at $t - 1$. In other words, both raw gold and old jewelry can be stored without physically depreciating or appreciating and new jewelry at $t - 1$ automatically becomes old jewelry at t. Old jewelry at t can at a cost of a_2 per unit in terms of bread at t be turned into raw gold at t and the latter can at a cost of a_1 per unit in terms of bread at t be turned into new jewelry at t.

Our technological assumptions, represented schematically in Figure 1, imply the following definition for feasible, utility-yielding aggregates. In stating the definition, we use the following notation: $C_{t-1}(t)$ $[C_t(t)]$ is total consumption of time t bread by the members of generation $t - 1$ $[t]$, D_t is the stock of new jewelry at t, and G_t (K_t) is the stock of raw gold (old jewelry) held from t to $t + 1$.

<u>Definition.</u> Given $K_0 > 0$, $G_0 = D_0 = 0$, a nonnegative sequence $\{C_{t-1}(t), C_t(t), D_t\}$ is feasible if there exists a nonnegative sequence $\{G_t, K_t\}$ that for all $t \geq 1$ satisfies

(1) $$D_t + K_t + G_t = D_{t-1} + K_{t-1} + G_{t-1}$$

(2) $$C_{t-1}(t) + C_t(t) + a_1 D_t + a_2(D_t + G_t - G_{t-1}) \leq W$$

(3) $$D_t + G_t - G_{t-1} \geq 0$$

Equality (1) limits the stock of gold at t in all its forms to the stock at $t - 1$. Imposing equality here is without loss of generality since old jewelry (and raw gold) can be costlessly stored from one date to the next. Inequality (2) is the constraint on uses of time t bread. The first two terms are direct consumption by members of generations $t - 1$ and t, respectively. The third is bread used in producing new jewelry from raw gold, and the fourth is bread used in transforming part of the stock of old jewelry into raw gold. Inequality (3) expresses the irreversibility of the process for converting old jewelry into raw gold. Imposing it rules out allocations that call for converting raw gold at t into old jewelry at t and time t bread.

Preferences and Endowments

We distinguish between the "old" at t = 1, the members of generation 0, and everyone else. Each member of generation 0 maximizes his or her consumption of time 1 bread. Each is endowed with some time 1 bread and with some old jewelry. Among them, the members of generation 0 own K_0, the entire initial stock of old jewelry.

All other generations are identical as regards preference and endowment types and numbers. Member h in generation t, t $\geq$ 1, maximizes utility, denoted $u^h(c_t^h(t), c_t^h(t+1), d_t^h)$, where the subscript denotes generation and the first argument is time t bread, the second is time t + 1 bread, and the third is new jewelry at t that is held from t to t + 1 (at which time it becomes old jewelry). There are H members in each generation.

We assume that u^h is twice continuously differentiable, strictly quasi-concave, and is such that all goods are gross substitutes. As regards endowments, h in generation t is endowed with some time t bread, $w_t^h(t) > 0$, and with some time t + 1 bread $w_t^h(t+1) > 0$, but is not endowed with new jewelry or gold in any form.

As we will show, this specification implies that there is a stationary or constant-over-time equilibrium. The gross substitutes assumption turns out to be important for existence of such an equilibrium.

2. Choice, Demands, and Equilibrium

We assume competitive behavior and, because we will be looking at perfect foresight equilibria, will not distinguish between actual and anticipated prices.

For $t \geq 1$, h in generation t faces the following constraints:

$$c_t^h(t) + (p_t + a_1)d_t^h \leq w_t^h(t) - l_t^h - p_t g_t^h - v_t k_t^h \tag{4}$$

$$c_t^h(t+1) \leq w_t^h(t+1) + r_t l_t^h + p_{t+1} g_t^h + v_{t+1}(d_t^h + k_t^h) \tag{5}$$

where p_t is the price of raw gold at t in units of time t bread, l_t^h is loans granted by h, $g_t^h \geq 0$ is raw gold that h carries from t to t + 1, v_t is the price of old jewelry at t in units of time t bread, $k_t^h \geq 0$ is old jewelry that h carries from t to t + 1, and r_t is the gross real return on loans granted in time t bread and paid back in time t + 1 bread.

Constraints (4) and (5) are valid constraints on the arguments of u^h for nonnegative prices that satisfy $v_t \geq p_t - a_2$. This inequality is satisfied in any equilibrium because it expresses the requirement that profits from turning old jewelry into raw gold are not positive.

Since we assume that l_t^h is unconstrained, which is to say that h can either borrow or lend at r_t, (4) and (5) are equivalent to the following single constraint obtained by eliminating l_t^h:

$$(6) \qquad c_t^h(t) + (1/r_t)c_t^h(t+1) + (a_1+p_t-v_{t+1}/r_t)d_t^h \leq$$

$$w_t^h(t) + (1/r_t)w_t^h(t+1) + g_t^h\xi_t + k_t^h\delta_t$$

where

$$\xi_t = (p_{t+1}/r_t-p_t), \ \delta_t = (v_{t+1}/r_t-v_t).$$

In any equilibrium, h must face a budget set that is bounded in the objects that give utility. Therefore, in any equilibrium, the coefficient of g_t^h, ξ_t, and that of k_t^h, δ_t, cannot be positive. Moreover, $g_t^h > 0$ implies $\xi_t = 0$, $\xi_t < 0$ implies $g_t^h = 0$, $k_t^h > 0$ implies $\delta_t = 0$, and $\delta_t < 0$ implies $k_t^h = 0$. Thus, in any equilibrium

$$(7) \qquad \xi_t \leq 0, \ \delta_t \leq 0, \ g_t^h\xi_t = 0, \ k_t^h\delta_t = 0$$

It follows that for prices that satisfy the inequalities of (7), the budget set constraining the arguments of u^h can be written with the terms in g_t^h and k_t^h omitted or as

$$(8) \qquad s_{1t}c_t^h(t) + s_{2t}c_t^h(t+1) + s_{3t}d_t^h \leq s_{1t}w_t^h(t) + s_{2t}w_t^h(t+1)$$

where

$$(9) \qquad s_{1t} \equiv 1, \ s_{2t} \equiv 1/r_t, \ s_{3t} \equiv a_1 + p_t - v_{t+1}/r_t.$$

Under our assumptions about u^h, maximization of u^h subject to (8) implies (8) (and (4) and (5)) at equality and differentiable demand functions, $c^h(s_t)$ for $c_t^h(t+1)$ (second period consumption of bread) and $d^h(s_t)$ for d_t^h (new jewelry), where s_t =

(s_{1t}, s_{2t}, s_{3t}). The gross substitutes assumption says that $c^h(s_t)$ is nondecreasing in s_{1t} and s_{3t} and decreasing in s_{2t} and that $d^h(s_t)$ is nondecreasing s_{1t} and s_{2t} and decreasing in s_{3t}.

Now let $C(s_t)$ and $D(s_t)$ denote $\sum_h c^h(s_t)$ and $\sum_h d^h(s_t)$, respectively, where the summation is over all members of generation t. These aggregate functions possess the properties of the corresponding individual functions noted above.[2/] Moreover, as the notation indicates, these functions do not depend on time because different generations are identical. It is convenient to define $W_1 = \sum_h w_t^h(t)$ and $W_2 = \sum_h w_t^h(t+1)$, and to note that $G_t = \sum_h g_t^h$ and $K_t = \sum_h k_t^h$, where these summations are also over the members of generation t.

We are now ready to define a perfect foresight competitive equilibrium, or, simply, from now on, an equilibrium.

Definition. Given $K_0 > 0$, an equilibrium consists of a positive sequence for r_t and nonnegative sequences for p_t, v_t, G_t, and K_t that for all $t \geq 1$ satisfy (7), $v_t \geq p_t - a_2$, and

$$(10) \qquad C(s_t) - W_2 = p_{t+1}[G_{t+1}+D(s_{t+1})] + v_{t+1}K_{t+1} - a_2[D(s_t)+K_t-K_{t+1}]$$

$$(11) \qquad D(s_t) + K_t + G_t = K_0.$$

Equation (10) expresses equality between the excess demand for time t + 1 bread on the part of generation t (the left-hand side) and the net excess supply of that good by members of generation t + 1, net of the amount used to convert old jewelry into raw gold, the term in a_2 (the right-hand side). Equation

(11) requires that the demand for gold in all its forms be equal to the inherited amount. (Imposing equality in (11) is innocuous since at a zero price for old jewelry ($v_t = 0$), individuals are indifferent about the amount of old jewelry they hold.)

We will concern ourselves only with the existence and properties of stationary equilibria, equilibria for which the r_t p_t, v_t, G_t, and K_t sequences are constant sequences.

3. Existence of Stationary Equilibria

Let us define $C'(r,p) \equiv C(s_1,s_2,s_3)$ and $D'(r,p) \equiv D(s_1,s_2,s_3)$ where $s_1 = 1$, $s_2 = 1/r$, and $s_3 = a_1 + a_2/r + p(1-1/r)$. Note that this expression for s_3 follows from imposing $v_t = p_t - a_2$ for all t. (Stationary equilibria with $v_t > p_t - a_2$ do not exist, because the inequality implies that old jewelry is not converted into new jewelry at t.) Note that at $r = 1$, $s_3 = a_1 + a_2$, and, in particular, does not depend on p. Thus, $C'(1,p) = C(1,1,a_1+a_2) \equiv C^*$ and $D'(1,p) = D(1,1,a_1+a_2) \equiv D^*$. The magnitudes C^* and D^* are the quantities demanded of second period consumption of bread and of new jewelry, respectively, when the real rate of interest is zero ($r_t = 1$) and when the price of raw gold is not changing and is consistent with producing raw gold from old jewelry. By (7), these are necessarily the equilibrium quantities in a stationary equilibrium in which raw gold or old jewelry is stored. In proposition 1, we will be characterizing stationary equilibria in terms of the signs of $C^* - W_2$ and $D^* - K_0$.

Our definition of an equilibrium implies the following definition of a stationary equilibrium.

Definition. A stationary equilibrium consists of a positive scalar r and nonnegative scalars $p - a_2$, G, and K that satisfy

(12) $$C'(r,p) - W_2 = pG + (p-a_2)[K+D'(r,p)],$$

(13) $$D'(r,p) + K + G = K_0,$$

and $r \geq 1$ with $r = 1$ if $G + (p-a_2)K > 0$, these last being the stationary version of (7). (For $v_t = p_t - a_2$, a necessary condition for a stationary equilibrium, (12) and (13) are the stationary versions of (10) and (11).)

The following four lemmas (their proofs appear in Appendix 1) establish properties of the C' and D' functions. These are used in establishing existence and some properties of stationary equilibria.

Lemma 1. For any $p \geq a_2$, (a) $rW_1 \geq C'(r,p) - W_2$; (b) $C_1'(r,p) > 0$; and (c) $C'(r,p) \to \infty$ as $r \to \infty$.

In Figure 2, for an arbitrary $p \geq a_2$, we illustrate two alternative $C'(r,p)$ functions, one for $C^* - W_2 \geq 0$, the other for $C^* - W_2 < 0$. The positive slope--i.e., property (b)--is a consequence of our gross substitutes assumption.

Lemma 2. Let $p^* = \max[a_2,(C^*-W_2+a_2K_0)/K_0]$ and define $\psi(p)$ for $p \geq p^*$ by

(14) $$C'(\psi(p),p) - W_2 = (p-a_2)K_0$$

Then $\psi(p)$ is a continuous function such that $\psi(p) \geq (p-a_2)K_0/W_1$.[3/]

The locus $(p,\psi(p))$ consists of all pairs (p,r) consistent with (12) and (13) combined and with $G = 0$. In Figure 3, we illustrate the points $(p,r) = (p,\psi(p))$ for two cases, $C^* - W_2 \geq 0$ and $C^* - W_2 < 0$.

Lemma 3. (a) For any $r > 1$, $D'(r,p) \to \infty$ as $p \to -(a_1+a_2/r)r/(r-1)$, $D'(r,p) \to 0$ as $p \to \infty$, and $D_2' < 0$. (b) For any $p \geq a_2$, $D_1' < 0$.

Lemma 4. For any $r > 1$, let $\phi(r)$ be defined by

$$(15) \qquad D'(r,\phi(r)) = K_0.$$

Then (a) $\phi(r)$ is a continuous function that is decreasing whenever $\phi(r) \geq a_2$; (b) if $D^* > K_0$, then $\phi(r) \to \infty$ as $r \to 1$; (c) all pairs (r,p) with $p \geq \phi(r)$ satisfy $D'(r,p) \leq K_0$; (d) if $D^* \leq K_0$, then all pairs (r,p) with $r \geq 1$ and $p \geq a_2$ satisfy $D'(r,p) \leq K_0$ and with slack unless $D^* = K_0$ and $r = 1$.

The locus $(\phi(r),r)$ consists of all pairs (p,r) that equate the demand for jewelry to the total stock of gold, all pairs that satisfy (13) and $G = K = 0$. For the case $D^* > K_0$, we show two possible $\phi(r)$ functions in Figure 3. Note that parts (b) and (c) of Lemma 4 imply that if $D^* > K_0$, then any stationary equilibrium has $r > 1$ and, therefore, $G = K = 0$.

We are now ready to prove existence of and to partially characterize stationary equilibria.

Proposition 1.

(i) There exists at least one stationary equilibrium.

(ii) If $D^* > K_0$ or if $C^* < W_2$, then $r > 1$; otherwise $r = 1$.

(iii) If $D^* < K_0$ and $C^* > W_2$, then there exist equilibria with $G > 0$.

Proof. We proceed by considering cases delineated by the magnitudes of C^* and D^*.

Case 1. $D^* > K_0$.

(a) $C^* - W_2 \geq 0$. Here, by Lemmas 2 and 4, there exists an (r,p) that satisfies (14) and (15). This (r,p) and $G = K = 0$ is an equilibrium.

(b) $C^* - W_2 < 0$. If there is no (r,p) satisfying (14) and (15), one of the situations illustrated in Figure 3, then $p = a_2$, $r = \psi(a_2)$, $G = 0$, and $K = K_0 - D'(\psi(a_2),a_2)$ is an equilibrium. If there is an (r,p) satisfying (14) and (15), then it and $G = K = 0$ is an equilibrium.

Case 2. $D^* \leq K_0$.

(a) $C^* - W_2 \geq a_2(K_0-D^*)$: We show that $r = 1$, $K = 0$, $G = K_0 - D^*$ is an equilibrium. Since (13) is satisfied by construction, we have only to find a $p \geq a_2$ satisfying (12); namely, one that satisfies $C^* - W_2 = p(K_0-D^*) + (p-a_2)D^* = (p-a_2)K_0 + a_2(K_0-D^*)$. The hypothesis of this case implies that there is a solution satisfying $p \geq a_2$.

(b) $0 \leq C^* - W_2 < a_2(K_0-D^*)$: Here $r = 1$, $p = a_2$, and $G = (C^*-W_2)/a_2$ satisfy (12). The hypothesis implies that these and a positive K satisfy (13). Note that this and the given construction of a Case 2a equilibrium imply conclusion (iii).

(c) $C^* - W_2 < 0$. Here, $p = a_2$, $r = \psi(a_2)$, $G = K = 0$ is an equilibrium. In this case, $r = 1$ could not be an equilibrium because the right-hand side of (12) must be nonnegative. This and the remark made after the proof of Lemma 4 imply the first part of conclusion (ii).

Since Cases 1 and 2 and their subcases cover all possibilities, we have proved conclusion (i) of the proposition.

To complete the proof, we must establish the second part of conclusion (ii). That is, we must show that Cases 2a and 2b do not have an equilibrium with $r > 1$. If $r > 1$, then $G + (p-a_2)K = 0$ and by Lemma 4 part (d), $D'(r,p) < K_0$. These and (13) imply $p = a_2$. By (12) these imply $C'(r,a_2) - W_2 = 0$, which contradicts $C^* - W_2 \geq 0$ and $r > 1$. Δ

The reader may have noticed that there are additional stationary equilibria in Cases 2a and 2b, ones with smaller G and larger K than those described in the proof. Indeed, each economy satisfying the Case 2a or 2b hypotheses has a continuum of stationary equilibria. To see this, write (12) at $r = 1$ as

$$(16) \qquad C^* - W_2 = p(G+K) - a_2K + (p-a_2)D^*$$

Then, if we solve (13) at $r = 1$ for $G + K$ and substitute the result into (16), we get

$$(17) \qquad C^* - W_2 + a_2D^* = pK_0 - a_2K$$

In Figure 4, we show the (p,K) pairs that satisfy (17) and $p \geq a_2$.

The different equilibria have the following features. At any $p \geq a_2$ and $v = p - a_2$, the old are indifferent between supplying old jewelry and raw gold. They are, however, better off the higher is p. The higher is p, the smaller the quantity of old jewelry that gets converted into raw gold. The equilibrium with the highest p (and highest K) is one with $K = K_0 - D^*$ and $G = 0$. The welfare of the young does not vary across these equilibria.

The multiplicity of Case 2a and 2b equilibria displayed in Figure 4 depends on the assumption that a sufficient amount of the initial stock of gold is in the form of old jewelry. Having displayed the equilibria for an economy in which the entire initial stock is in the form of old jewelry, we can describe the equilibria for alternative economies, ones that differ only with regard to the composition of the starting stock of gold.

We let initial conditions, (K_0', G_0'), for such an alternative economy be given by $K_0' = K_0 - D^* - \alpha(K_0 - D^*)$ and $G_0' = D^* + \alpha(K_0 - D^*)$, where α is a parameter between 0 and 1. Notice that $K_0' + G_0' = K_0$. For $\alpha = 0$, the alternative economy has the same set of (p,K) equilibria as the original economy. For $\alpha > 0$, it does not. For $\alpha > 0$, the alternative economy does not have a (stationary) $G = 0$ equilibrium. More generally, as we increase α we eliminate equilibria from the right in Figure 4. Thus it is easy to construct a large class of economies all of whose stationary equilibria have $G > 0$.[4/]

4. Optimality Characterization of Stationary Equilibria

Before we state and prove a proposition, we restate the standard definitions.

Definitions. An allocation, $\{\bar{c}_{t-1}(t), \bar{c}_t(t), \bar{d}_t\}$, $t \geq 1$, is Pareto superior to the allocation, $\{\hat{c}_{t-1}(t), \hat{c}_t(t), \hat{d}_t\}$, $t \geq 1$, if for all $t \geq 1$ and all h, $u^h(\bar{c}_t^h(t), \bar{c}_t^h(t+1), \bar{d}_t^h) \geq u^h(\hat{c}_t^h(t), \hat{c}_t^h(t+1), \hat{d}_t^h)$ and $\bar{c}_0^h(1) \geq \hat{c}_0^h(1)$, with at least one strict inequality. (Here $c_{t-1}(t)$, $c_t(t)$, and d_t are H-element vectors with typical elements, respectively, $c_{t-1}^h(t)$, $c_t^h(t)$, and d_t^h.) An allocation is

Pareto optimal if there is no feasible allocation Pareto superior to it.

We now prove the following.

<u>Proposition 2.</u> $G = 0$ (zero storage of raw gold) is necessary and sufficient for Pareto optimality of a stationary equilibrium.

<u>Proof:</u> The proof of sufficiency is in Appendix 1. To prove necessity, we proceed by contradiction as follows. Suppose that the equilibrium has $G > 0$. Then for any date $t \geq 1$ and any h in generation t, it is feasible to raise the utility of h, while leaving intact the allocation of everyone else--everyone else in generation t and everyone else in all other generations. This can be done by reducing $c_t^h(t)$, h's consumption of time t bread, and using the freed time t bread and some of the time t raw gold to produce some time t jewelry for h. Since at the equilibrium allocation, the ratio of marginal utilities, u_3^h/u_1^h, is equal to $a_1 + a_2$ and since it is feasible to produce q units of extra jewelry by reducing $c_t^h(t)$ by a_1q units, it is possible to choose a $q \in (0,G]$ which raises h's utility. Although there is less raw gold available thereafter, that does not make infeasible anyone else's equilibrium allocation. Δ

Propositions 1 and 2 imply that if $D^* \geq K_0$ or if $C^* \leq W_2$, then any stationary equilibria is optimal. In these equilibria, no raw gold (or old jewelry) is stored. However, since the price of raw gold is determined and raw gold is present momentarily at each date, nothing would seem to prevent raw gold from serving as a numeraire in these equilibria.

Propositions 1 and 2 also imply that the conditions $D^* < K_0$ and $C^* > W_2$ are necessary and sufficient for the existence of a stationary nonoptimal equilibrium. They are not, however, sufficient to imply that any stationary equilibrium is nonoptimal. Such an implication does follow for the alternative class of economies described above, those with a sufficiently large portion of the initial gold stock in the form of raw gold, since all stationary equilibria for such economies have $G > 0$. Note that our proof of the necessity part of Proposition 2 applies to such economies because the proof does not rely at all on the waste involved in converting some of the initial stock of old jewelry into raw gold that gets stored.[5]

5. The Relative Price of a Commodity Money

It turns out that economies with $D^* < K_0$ and $C^* > W_2$ have a larger class of multiplicities than those displayed in Figure 4. They also have equilibria with a valued fixed stock of fiat money. It is in terms of this broader class of equilibria that we interpret the notion that a commodity is more valuable in terms of other commodities the more important is its role as a commodity money.

As we interpret this notion, it is about multiple equilibria and about an association, among equilibria, between relative prices, p in our case, and the extent or degree to which a commodity is playing the role of a commodity money. We interpret the degree to which gold plays a commodity money role as inversely related to the value of fiat money. In other words, we interpret the notion as one which calls for a negative association between p and the value of fiat money among multiple equilibria.

To examine whether there is such an association, we suppose that the members of generation 0 are endowed in the aggregate with one unit of fiat money in addition to their gold. We suppose also that this quantity does not change over time and we let M_t denote the value of this unit of fiat money at date t in units of time t bread. Note that the equilibria of Proposition 1 are also equilibria for the economy with the endowment of fiat money; they are equilibria with $M_t = 0$ for all $t \geq 1$.

Without going through the obvious amendments to (4)-(7), we will display all the r = 1 stationary equilibria for economies with $D^* \leq K_0$ and $C^* \geq W_2$. These are equilibria with $M_t = M \geq 0$ for all $t \geq 1$.

Instead of (16), we have

$$(18) \qquad C^* - W_2 = p(G+K) - a_2K + (p-a_2)D^* + M$$

Then, if we substitute for G + K from (13) at r = 1, we get

$$(19) \qquad C^* - W_2 + a_2D^* = pK_0 - a_2K + M$$

In Figure 5, the shaded area depicts all (p,M) pairs that are equilibria.

For each M, there is an interval of p's which are equilibria. This is the multiplicity remarked upon in the M = 0 case. There is an inverse association between M and p in the sense that for each M, the midpoint of the corresponding p interval is decreasing in M. Moreover, the following can be verified directly.

Proposition 3. If there exists a stationary equilibrium in which $K + G > 0$, then there is an equilibrium with $K = G = 0$, $M = C^* - W_2$ and $p = a_2$.

The equilibrium with $M = C^* - W_2$ and $p = a_2$ is one in which gold in any form is not serving as wealth (old jewelry is free) and in which the relative price of raw gold is at the minimum consistent with the production of new jewelry from old jewelry.

As was true for the continuum of equilibria depicted in Figure 4, the allocations corresponding to the equilibria depicted in Figure 5 differ by at most the amount that the $t = 1$ old consume, a difference that corresponds to the difference in the amount of consumption good used to turn old jewelry into raw gold. Moreover, the addition of fiat money adds no new consumption allocations; every allocation implied by a point in Figure 5 is present in the one-dimensional continuum of Figure 4. A more significant role for fiat currency would arise in settings in which storage and trading of old jewelry is not a perfect substitute for storage and trading of fiat currency.[6/]

6. Concluding Remarks

We have described efficient and inefficient equilibria that can be interpreted as commodity money equilibria in a model in which markets are perfect in the sense that borrowers are free to issue claims in any form. In particular, borrowers are free to issue claims in forms--analogous to banknotes, for example--that compete perfectly with any other nonutility yielding asset.[7/] In some, if not most, actual commodity money systems, unfettered private intermediation seems not to have been permitted. For example, in 19th century England, Peel's Acts (1844, 45) set a 100 percent marginal reserve requirement in gold against banknote issue. And, in the United States, the National Banking Act (1863) set a 100 percent reserve requirement in government bonds against banknote issue.[8/] We suspect that systems with binding restrictions on private intermediation are characterized by distortions, some of which have nothing to do with the overaccumulation distortion that characterizes our nonoptimal equilibria. Against the background of such restrictions, the choice of a standard--say, whether commodity or fiat--becomes a kind of second-best choice. We have not attempted to analyze such choices--in part because there is a strong presumption that no general results would emerge and in part because the role of the monetary standard would not be isolated if some essentially arbitrary distortion were taken as a given.

APPENDIX 1

Proof of Lemma 1

Property (a) is an immediate consequence of (8) summed over the members of generation t. Property (b) is a consequence of our gross substitutes assumption which gives us $C_2 < 0$, $C_3 \geq 0$. Hence, $C_1' \equiv \partial C'/\partial r = C_1(\partial s_1/\partial r) + C_2(\partial s_2/\partial r) + C_3(\partial s_3/\partial r) = -C_2/r^2 + C_3(p-a_2)/r^2 > 0$. To establish (c), for each $p \geq a_2$, let $I(p) = [a_1+a_2, a_1+p]$. Then, by the definition of s_3, for any $r \geq 1$, $s_3(r,p) \in I(p)$. For each $x \in I$ and any A, there exists $\bar{r}(x)$ such that $C(1,1/r,x) \geq A$ for any $r \geq \bar{r}(x)$. (This follows from the fact that $C(s_1,s_2,s_3) \to \infty$ as $s_2 \to 0$ for fixed (s_1,s_3).) Let $r^* = \max[\bar{r}(x)]$ for $x \in I(p)$. Then, if $r \geq r^*$ and $x \in I(p)$, $C(1,1/r,x) \geq A$. Δ

Proof of Lemma 2

For a fixed $p \geq p^*$, the right-hand side of (15) is a constant function. It is a minimum at $p = p^*$. If $C^* - W_2 \geq 0$, the minimum is $C^* - W_2$; otherwise it is 0. Thus for any $p \geq p^*$, $C'(1,p) - W_2 \leq (p-a_2)K_0$. This and property (c) of Lemma 1 imply existence of ψ. Property (b) of Lemma 1 implies that $\psi(p)$ is a function. The continuity of C' implies that ψ is continuous. Finally, the last fact says only that the $\psi(p)$ (or r) that solves (14) lies to the right in Figure 2 of the r that solves $rW_1 = (p-a_2)K_0$. This is an obvious consequence of Lemma 1(a). (Note, by the way, that $\psi(p) \geq 1$ and with equality only at $p = p^*$ if $C^* - W_2 \geq 0$.) Δ

Proof of Lemma 3

Since s_1 and s_2 do not depend on p, property (a) depends only on how $D(s_1,s_2,s_3)$ varies with s_3. The asserted limiting behavior is implied by $D \to \infty$ as $s_3 \to 0$ and $D \to 0$ as $s_3 \to \infty$ and by the following limiting behavior of s_3; $s_3 \to 0$ as $p \to -(a_1+a_2/r)$ $r/(r-1)$ and $s_3 \to \infty$ as $p \to \infty$. The asserted signs of the partial derivatives of D' follow from the chain rule and our gross substitutes assumption. Δ

Proof of Lemma 4

(a) Lemma 3(a) implies that $\phi(r)$ exists and is a function. Continuity of ϕ is implied by continuity of D'. That $\phi(r)$ is decreasing whenever $\phi(r) \geq a_2$ follows from Lemma 3(b).

(b) Suppose not and that $B \geq \phi(r)$ for all r. Then, $K_0 = D(r,\phi(r)) \geq D(r,B)$ for all r. In particular, the limit of $D(r,B)$ as $r \to 1$ does not exceed K_0. However, since this limit is D*, we have a contradiction.

(c) This is an immediate consequence of $D_2' < 0$ for $r > 1$ (Lemma 3).

(d) Since $D^* \leq K_0$, $D'(1,p) \leq K_0$ is satisfied by any p. For any $r > 1$, $D'(r,a_2) = D(1,1/r,a_1+a_2) < D^*$, where the inequality follows from $D_1' < 0$ (Lemma 3). Since $D_2' < 0$ (Lemma 3), it follows that $D'(r,p) \leq D^*$ for all $r \geq 1$ and $p \geq a_2$ and with slack unless $r = 1$. Δ

Proof of Proposition 2: Sufficiency

We derive a contradiction from the assumption that there is an allocation, denoted the "-" allocation, that is feasible and Pareto superior to the stationary $G = 0$ equilibrium allocation, denoted the "^" allocation. The proof proceeds by considering the first date at which "-" differs from "^" and by showing that the difference and the other properties of "-" contradict its feasibility. Infeasibility is established by showing that the sequence $\{\bar{C}_t(t+1)\}$ is not bounded. This is done using properties of a mapping g that, for each $C_{t-1}(t)$, determines the minimum value of $C_t(t+1)$ consistent with feasibility and with individual utility being as high as under the "^" allocation.

The proof relies on three lemmas. The first, Lemma 5, establishes that g is an increasing and strictly convex function which satisfies $g(\hat{C}_2) = \hat{C}_2$ and $g'(\hat{C}_2) \geq 1$, where $\hat{C}_2$ denotes the constant value of total second period consumption under the "^" allocation. One such function is displayed in Figure 6. Lemma 6 establishes that if t is the first departure of the "-" allocation from the "^" allocation, then $\bar{C}_t(t+1) > \hat{C}_2$ (it lies to the right of the fixed point of g). The details of the induction step are in Lemma 7.

The following notation and facts are used in the lemmas.

Let $f^h(c_1^h, d^h)$ be defined by $u^h(c_1^h, f(c_1^h, d^h), d^h) = \hat{u}^h$, where $\hat{u}^h$ is h's utility in the given $G = 0$ equilibrium, and let $F(c_1, d) = \sum_h f^h(c_1^h, d^h)$, the sum being over the H members of generation t, $t \geq 1$. Since u^h is twice continuously differentiable and strictly quasi-concave, the functions f^h and F are twice continuously differentiable and strictly convex.

Also, for $x \in (0,W)$, let $\Gamma(x) = \{(c_t(t),d_t) \in R_+^{2H}: C_t(t) + (a_1+a_2)D_t \leq W - x \text{ and } D_t \leq K_0\}$. As is well known, $\Gamma(x)$, which is a special case of a budget correspondence, is a continuous correspondence. Moreover, for fixed x, $\Gamma(x)$ is convex and compact.

Lemma 5. Let $g(x) = \min F(c_1,d)$ subject to $(c_1,d) \in \Gamma(x)$. Then (a) $g(x)$ is a differentiable, strictly convex, and increasing function of x; (b) $g(\hat{C}_2) = \hat{C}_2$ and is attained at, and only at, $(c_1,d) = (\hat{c}_1,\hat{d})$; (c) $g'(\hat{C}_2) \geq 1$. (Here $(\hat{c}_1,\hat{d})$denotes $(\hat{c}_t(t),\hat{d}_t)$, vectors of constants.)

Proof. (a) Given the strict convexity of $F(\cdot)$ and the convexity of the set $\Gamma(x)$, the values (c_1,d) that attain $g(x)$ are unique. Then differentiability follows from twice differentiability of $F(\cdot)$.

To establish convexity of g, let x_1 and x_2 be any two real numbers such that $\Gamma(x_1)$ and $\Gamma(x_2)$ are nonempty. Let (c_1^i,d^i) be the unique minimizer associated with x_i, i = 1, 2. Pick any $0 < \lambda < 1$ and define $x(\lambda) = \lambda x_1 + (1-\lambda)x_2$, $c_1(\lambda) = \lambda c_1^1 + (1-\lambda)c_1^2$, and $d(\lambda) = \lambda d^1 + (1-\lambda)d^2$. Then it is straightforward to verify that $(c_1^1,d^1) \in \Gamma(x_1)$ and $(c_1^2,d^2) \in \Gamma(x_2)$ imply $(c_1(\lambda),d(\lambda)) \in \Gamma(x(\lambda))$. This, the strict convexity of F, and the definition of g imply $g(x(\lambda)) \leq F(c_1(\lambda),d(\lambda)) < \lambda F(c_1^1,d^1) + (1-\lambda)F(c_1^2,d^2) = \lambda g(x_1) + (1-\lambda)g(x_2)$.

Finally, if $x_1 > x_2$, then $\Gamma(x_1)$ is a strict subset of $\Gamma(x_2)$, strict in the sense that any element on the boundary of the set $\Gamma(x_2)$ does not belong to $\Gamma(x_1)$. As $F(\cdot)$ is decreasing in each argument, its minimum is achieved on the boundary. Therefore, g is increasing.

(b) From the definition of $\Gamma(\hat{C}_2)$ and feasibility of the "^" allocation (see (2)), it follows that $(\hat{c}_1,\hat{d}) \in \Gamma(\hat{C}_2)$. Given the strict convexity of $F(c_1,d)$ and the convexity of the feasible set $\Gamma(\hat{C}_2)$, the first order conditions are necessary and sufficient for a minimum and are attained at a unique point. These conditions are: $c_1^h(f_1^h+\lambda) = 0$ and $d^h[f_2^h+(a_1+a_2)\lambda+\mu] = 0$ for each h; $\lambda\{\sum c_1^h+(a_1+a_2)\sum d^h-W-\hat{C}_2\} = 0$; and $\mu\{\sum d^h - K_0\} = 0$, where λ and μ are the nonnegative Lagrange multipliers associated with the constraints that define the set $\Gamma(\hat{C}_2)$. We now show that these 2H + 2 equations are satisfied by $c_1 = \hat{c}_1$, $d = \hat{d}$, $\lambda = \hat{r}$, and $\mu = \hat{p}(\hat{r}-1) + \hat{r}a_1 + a_2 - \hat{r}(a_1+a_2)$.

Let $\hat{f}_1^h \equiv f_1^h(\hat{c}_1^h,\hat{d}^h)$ and $\hat{f}_2^h \equiv f_2^h(\hat{c}_1^h,\hat{d}^h)$. By construction $f^h(\hat{c}_1^h,\hat{d}^h) = \hat{c}_2^h$. As the "^" allocation is an equilibrium, the following marginal conditions hold: $\hat{f}_1^h = \hat{u}_1^h/\hat{u}_2^h = \hat{r} \geq 1$ and $\hat{f}_2^h = \hat{u}_3^h/\hat{u}_2^h = \hat{p}(\hat{r}-1) + \hat{r}a_1 + a_2$, where $\hat{u}_i^h = u_i^h(\hat{c}_1^h,\hat{c}_2^h,\hat{d}^h)$. It follows that our conjectured solution satisfies the first 2H equations. Since the "^" allocation satisfies (2) with equality, the next to the last equation is also satisfied. As for the last equation, notice that if $\hat{r} = 1$ or $\hat{p} = a_2$, then $\mu = 0$. If $\hat{r} > 1$ and $\hat{p} > a_2$, then the proof of Proposition 1 shows that $(\hat{r},\hat{p}) = (\hat{r},\phi(\hat{r}))$ and, consequently, $\sum \hat{d}^h = K_0$.

(c) By the envelope theorem, $g'(\hat{C}_2) = \lambda = \hat{r} \geq 1$. Δ

<u>Lemma 6</u>. If $t = \bar{t}$ is the <u>first</u> date at which $(\bar{c}_{t-1}(t),\bar{c}_t(t),\bar{d}_t) \neq (\hat{c}_{t-1}(t),\hat{c}_t(t),\hat{d}_t)$, then for $t = \bar{t}$, $\bar{C}_t(t+1) \geq x_t + a_2\bar{G}_t$ for some $x_t > \hat{C}_2$.

<u>Proof</u>. By the definition of $\bar{t}$ and feasibility (see (2)), $\bar{G}_t = 0$ for all $t < \bar{t}$. In addition, by the definition of $\bar{t}$, $\bar{C}_{t-1}^h(t) \geq$

$\hat{C}^h_{t-1}(t)$ for $t = \bar{t}$. (If not, then some old person at $t = \bar{t}$ has less consumption under the "-" allocation then under the "^" allocation, which violates the assumed Pareto superiority of the former over the latter.) These two facts and (2) imply that for $t = \bar{t}$, $\bar{C}_t(t) + (a_1+a_2)\bar{D}_t \leq W - \hat{C}_2 - a_2\bar{G}_t$ and, therefore, that $(\bar{c}_t(t),\bar{d}_t) \in \Gamma(\hat{C}_2+a_2\bar{G}_t)$. We complete the proof of this lemma by dealing separately with two cases.

$\bar{G}_t = 0$ for $t = \bar{t}$: In this case, $(\bar{c}_t(t),\bar{d}_t) \in \Gamma(\hat{C}_2)$. We know that $g(\hat{C}_2) = \hat{C}_2$ and that this minimum of F is attained uniquely at $(\hat{c}_t(t),\hat{d}_t)$. It follows that $F(\bar{c}_t(t),\bar{d}_t) > \hat{C}_2$. Since $\bar{C}_t(t+1) \geq F(\bar{c}_t(t),\bar{d}_t)$, we have our result for this case.

$\bar{G}_t > 0$ for $t = \bar{t}$: In this case, $\bar{C}_t(t+1) \geq g(\hat{C}_2+a_2\bar{G}_t) > \hat{C}_2 + a_2\bar{G}_t$ for $t = \bar{t}$, where the strict inequality is a consequence of Lemma 5. Δ.

<u>Lemma 7</u>. If (i) $\bar{C}_t(t+1) \geq x_t + a_2\bar{G}_t$ and (ii) $x_t > \hat{C}_2$, then $\bar{C}_{t+1}(t+2) \geq g(x_t) + a_2\bar{G}_{t+1}$.

<u>Proof</u>. We have $\bar{C}_{t+1}(t+2) \geq g[\bar{C}_t(t+1)+a_2(\bar{G}_{t+1}-\bar{G}_t)] \geq g(x_t+a_2\bar{G}_{t+1})$, where the first inequality follows from (2) and the assumed Pareto superiority of "-" and where the second follows from hypothesis (i) of the lemma and $g' > 0$. By hypothesis (ii) and Lemma 5 (see Figure 6), $g(x_t+a_2\bar{G}_{t+1}) \geq g(x_t) + a_2\bar{G}_{t+1}$. Δ

<u>Proof of Sufficiency</u>. Lemmas 6 and 7 provide the ingredients of an induction argument that implies that $\{\bar{C}_t(t+1)\}$ is unbounded and, hence, not feasible. For $t = \bar{t}$ and all positive integers k, they imply that $\bar{C}_{t+k}(t+k+1) \geq x_{t+k}$ where for all such k, $x_{t+k} = g(x_{t+k-1})$ with $x_t > \hat{C}_2$. By Lemma 5, the sequence $\{x_{t+k}\}$ is unbounded. Δ

APPENDIX 2

The main purpose of this appendix is to indicate the sense in which our gross substitutes assumption is necessary for existence of a stationary equilibrium (or even for existence of a stationary point). The case of concern, the one for which the assumption is used, is $D^* > K_0$ and $C^* \geq W_2$ (Case 1a of Proposition 1).

The crucial implication of gross substitutes is Lemma 1(b): $C_1'(r,p) > 0$. Absent this property, instead of the situation illustrated in Figure 2, we could have a $C'(r,p)$ function with the properties illustrated in Figure 7 (recall that $C'(1,p) = C^*$ for all p). In such a case, $\psi(p)$--which is defined in Lemma 2 (equation (14))--is not a function and, in particular, is a correspondence which is not lower hemicontinuous (see Figure 8). If, in addition, $\phi(r)$--which is a function under very general assumptions--is as shown in Figure 8, then no stationary point exists.

A sufficient condition for nonlower hemicontinuity of $\psi(p)$ is $C_1'(1,p) < 0$ for all $p \geq p^*$ in a neighborhood of p^*. To emphasize that nothing bizarre is required in order for this to happen, we suggest conditions under which it happens even if there is one person per generation.

Let $h(s_1,s_2,s_3,y)$ be the standard individual demand function for second period bread as a function of prices and income, $y = s_1W_1 + s_2W_2$. Then, denoting the corresponding substitution terms of the Slutsky matrix by h_i^*, it follows that

$$(20) \qquad C_1'(1,p) = -[h_2^* - (C^* - W_2)h_4 - (p - a_2)h_3]$$

where all partial derivatives are evaluated at $r = 1$ and, therefore, at $(s_1,s_2,s_3,y) = (1,1,a_1+a_2,W_1+W_2)$, which does not depend on p.

Assuming normal goods, so that $h_4 > 0$, then (since $C^* \geq W_2$ and $p \geq a_2$) a necessary condition for $C_1'(1,p) < 0$ is $h_3 < 0$. If this necessary condition holds, then $C_1'(1,p) < C_1'(1,p^*)$ for all $p > p^*$. Thus, it suffices to examine $C_1'(1,p^*)$.

By decomposing h_3 into income and substitution terms, we find that the right-hand side of (20) evaluated at $p = p^*$ is negative if and only if

(21) $$[(C^*-W_2)/K_0][(K_0-D^*)h_4+h_3^*] < h_2^*.$$

Since $D^* > K_0$, the income effect contributes to satisfying (21). And, since h_3^* can be anything (negative if second period bread and jewelry are net complements), it is clear that well-behaved preferences can satisfy (21).

Although, as this discussion indicates, gross substitution is necessary for existence of a stationary equilibrium (and point), it is not necessary for existence of an equilibrium. Our model satisfies the assumptions under which Möller (1983) establishes existence of equilibrium for a general class of overlapping generations models with production. Unfortunately, as is usual with general existence results, Möller's result does not describe the equilibria. So, for example, it is unclear whether $D^* > K_0$ is sufficient for existence of an optimal equilibrium in the absence of the gross substitutes assumption.

FOOTNOTES

[1]For examples of overlapping-generations models with the possibility of capital over-accumulation, see Diamond (1965), Cass-Yaari (1966), Wallace (1980), and Sargent-Wallace (1983). Wallace and Sargent-Wallace also suggest identifying the capital goods of these models with commodity monies.

[2]Naturally, it is sufficient for all our results that C and D satisfy the gross substitutes assumption.

[3]If it were not true that $C_1' > 0$, then we could not establish that $\psi(p)$ is a function. It would be a correspondence. Since lower hemicontinuity could not be established, a stationary equilibrium would, in general, fail to exist. See Appendix 2 for further details.

[4]One way to rule out equilibria in which old jewelry is stored is to impose a "transaction cost" on trade in old jewelry--a cost which could be thought of as depicting the need to have old jewelry appraised each time it is traded. If such a cost is modeled as a cost in terms of bread resources that is proportional to the amount of old jewelry traded and if it exceeds any corresponding cost for the trading of raw gold, then there could not be a stationary equilibrium in which old jewelry is stored.

[5]For our original economy, an alternative proof of the necessity part of Proposition 2 involves noting that some additional time 1 bread can be provided without sacrificing any other utility-yielding object by producing less raw gold at $t = 1$. Both our proof and this alternative rely on $a_2 > 0$.

6/ The "transaction cost" model described in footnote 4 is an example.

7/ Our model can easily accommodate real resource costs of converting safe private loans into assets like banknotes provided that the technology is consistent with a competitive equilibrium. If it is, then adding it to our model would not substantially change the model or, therefore, Propositions 1 and 2.

8/ As these examples suggest, there is more to the design of a financial system than the choice of a standard. The debates about Peel's Act and those about a successor system to the National Banking system in the United States all presumed a given standard.

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Figure 1: Static and intertemporal technologies

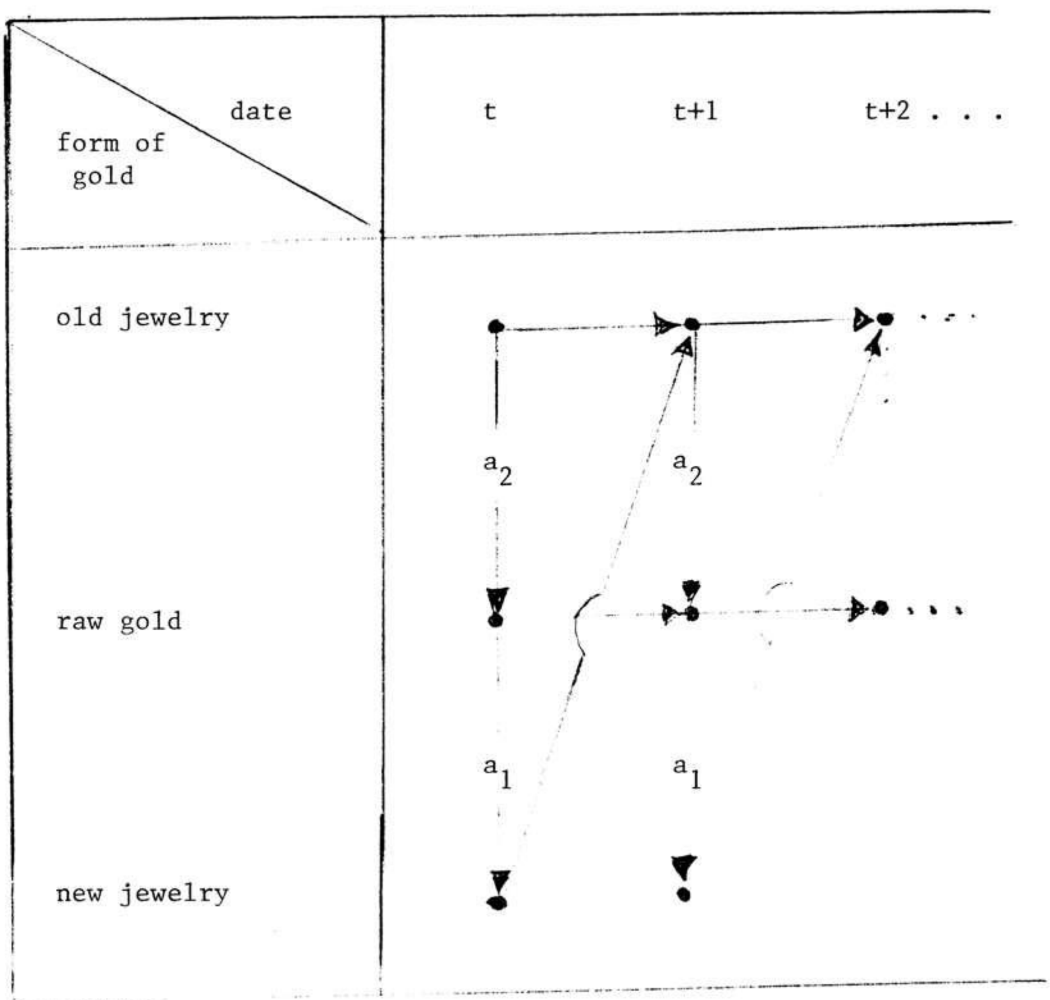

Figure 2: Illustrative $C'(r,p)-W_2$ functions (p fixed)

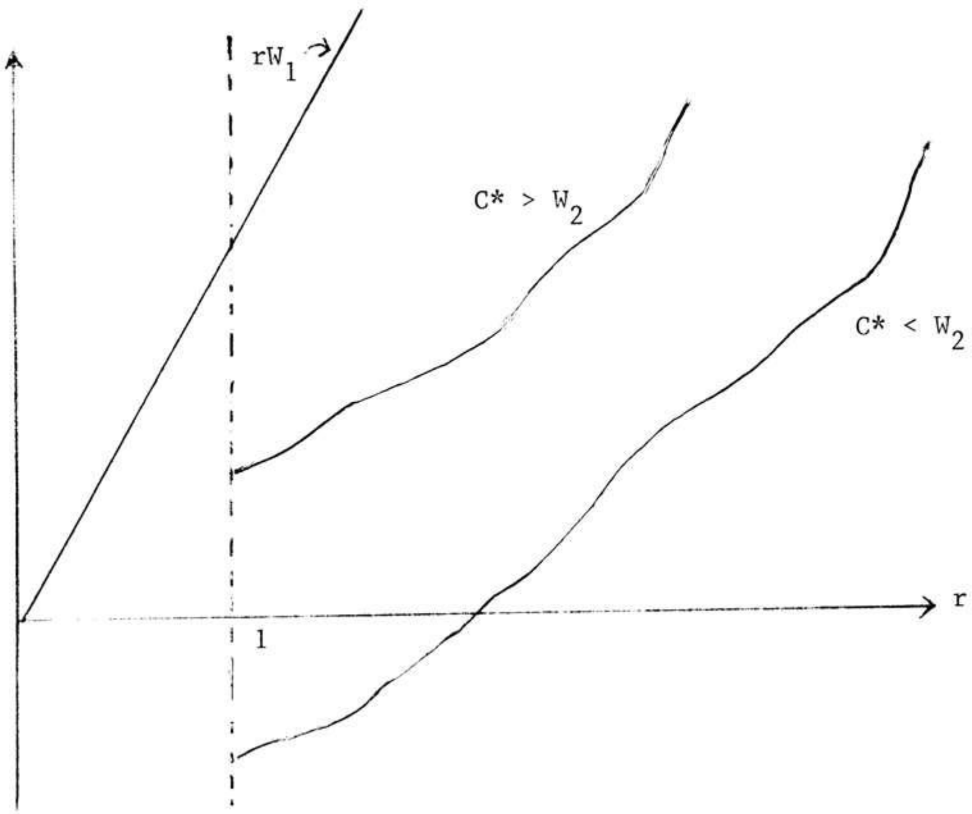

Figure 3: Illustrative Case 1 Equilibria

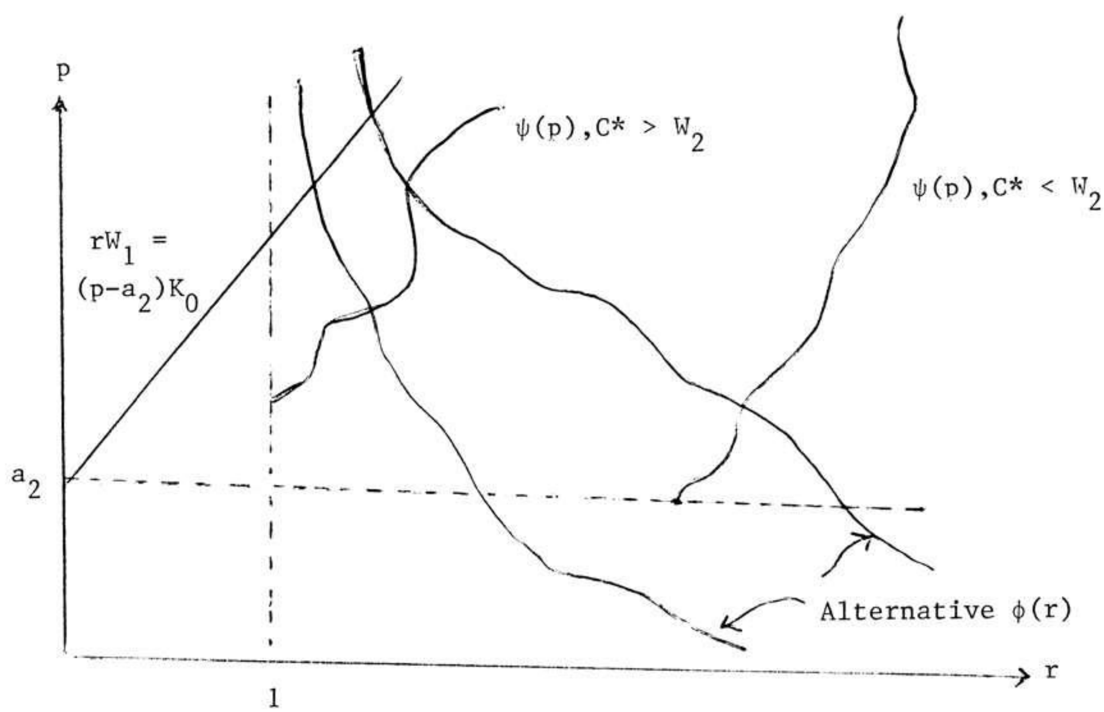

Figure 4: Equilibria for economies with $D^* < K_0$ and $C^* > W_2$

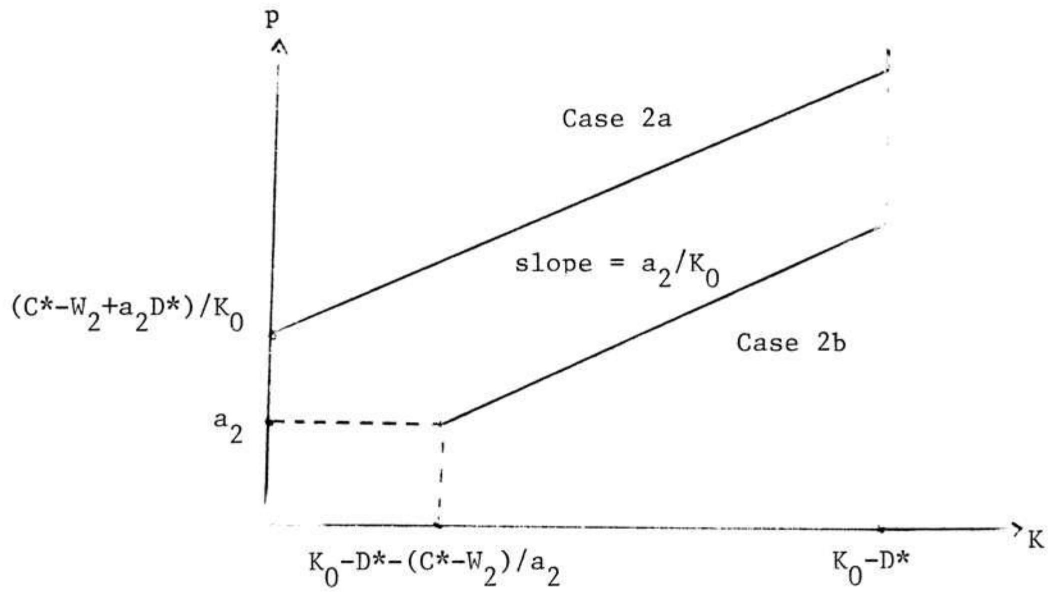

Figure 5: Monetary equilibria for economies with $D^* < K_0$ and $C^* > W_2$

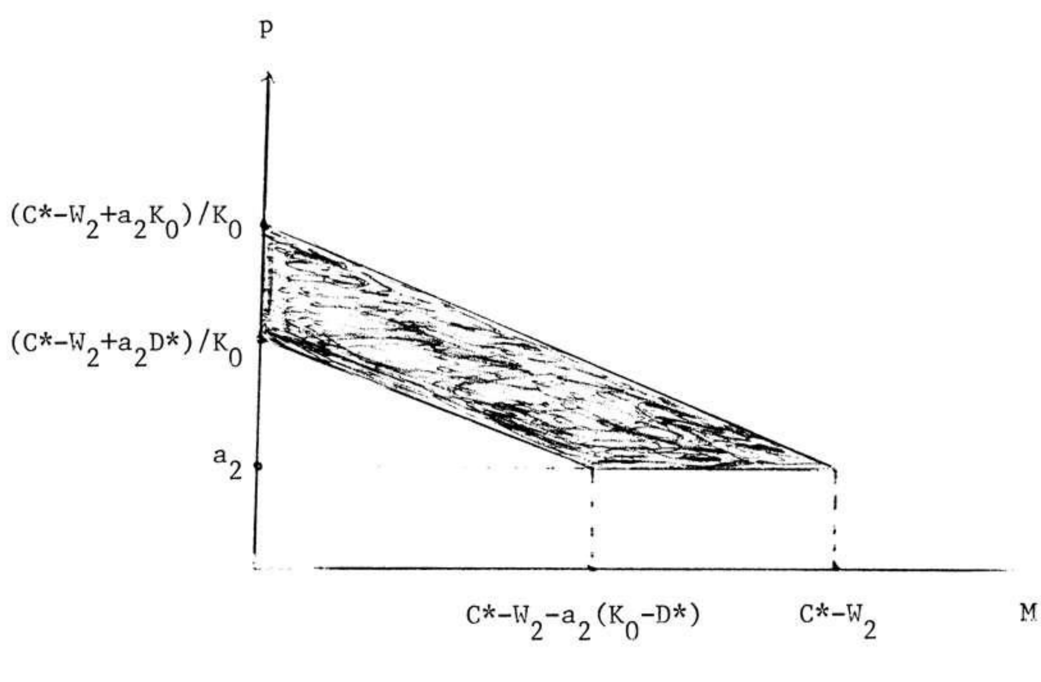

Figure 6 : An illustrative g function

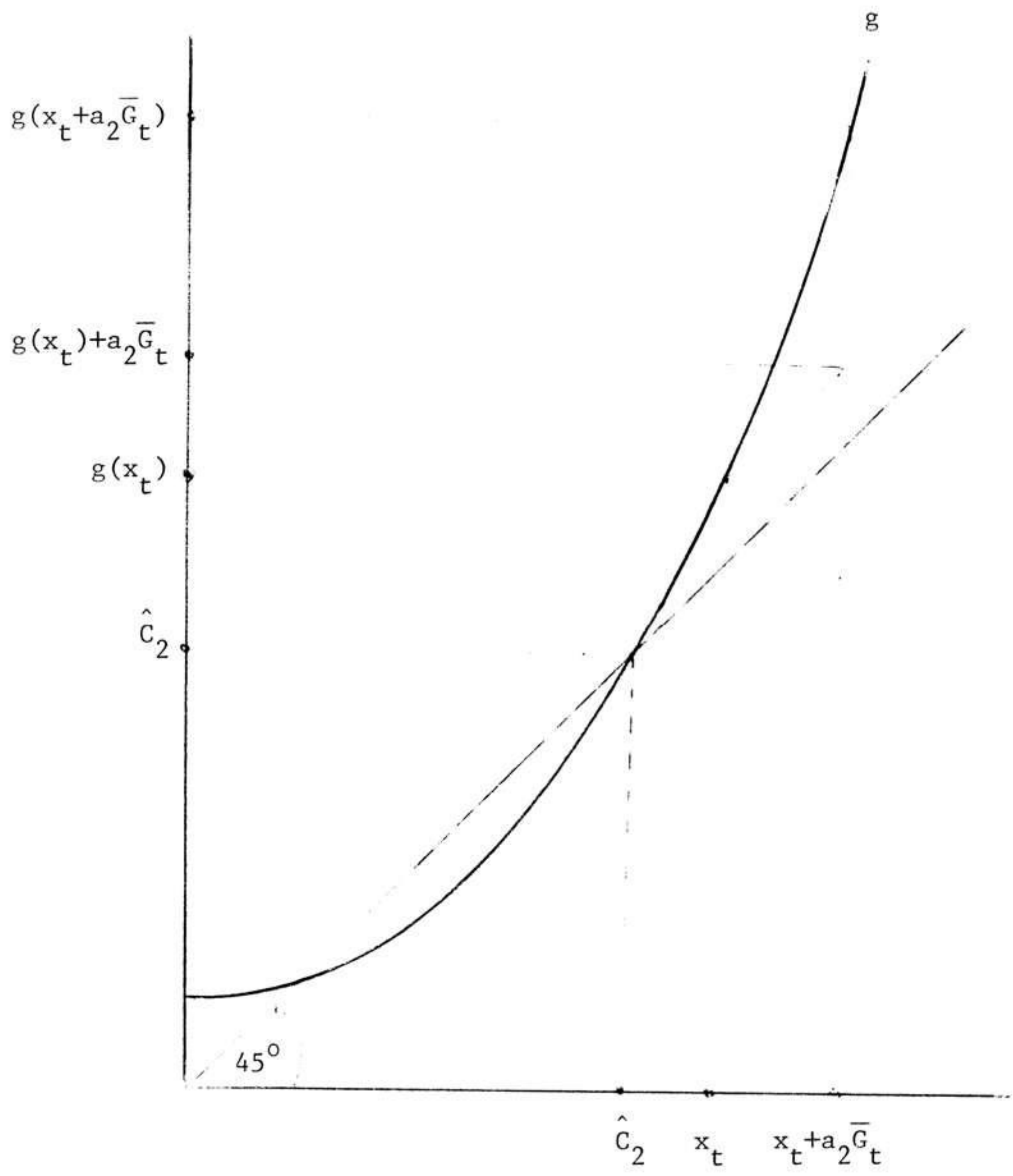

Figure 7

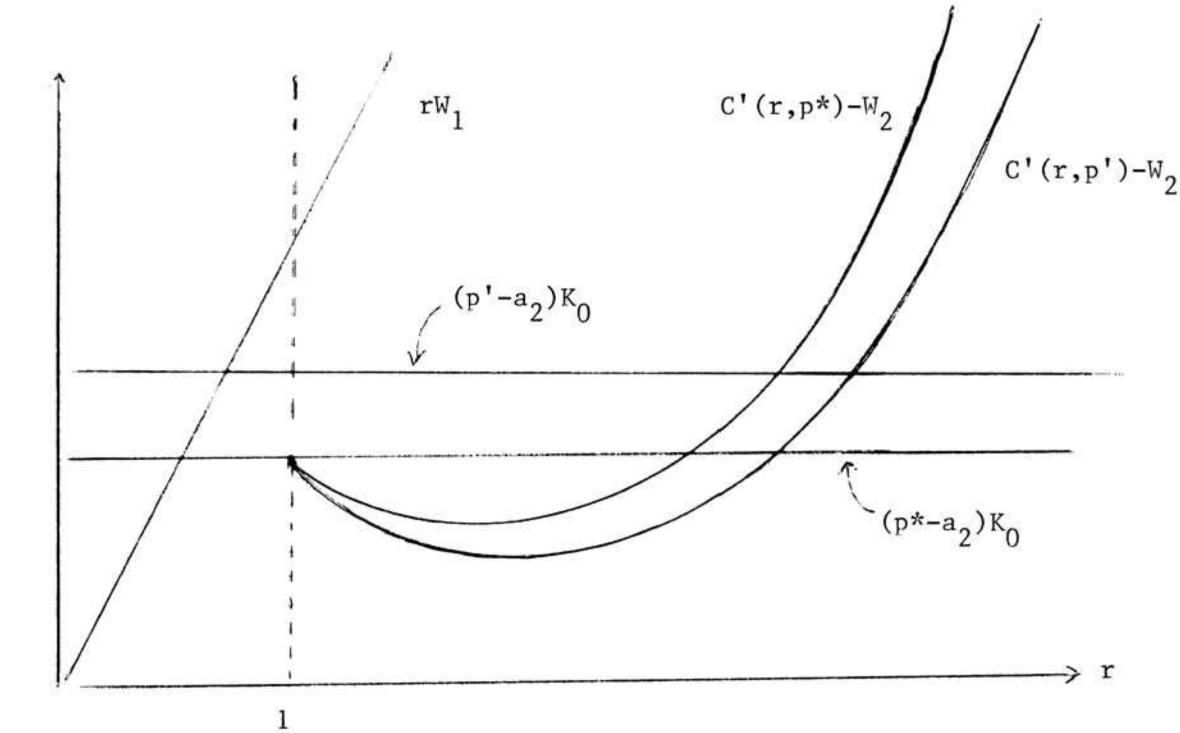

Figure 8