

Internal

Federal Reserve Bank of Minneapolis
Research Department Working Paper

NEW COOPERATIVE AND NONCOOPERATIVE EQUILIBRIUM
CONCEPTS FOR SOME SETTINGS WITH PRIVATE INFORMATION

John H. Boyd, Edward C. Prescott, and Bruce D. Smith

Working Paper 257
FACS File 3275

July 1984

NOT FOR DISTRIBUTION
WITHOUT AUTHORS' APPROVAL

The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. The material contained is of a preliminary nature, is circulated to stimulate discussion, and is not to be quoted without permission of the authors.

The R.S. Insurance Environment

We begin by discussing a version of the insurance environment familiarized by Rothschild and Stiglitz (1976), Wilson (1977), and Spence (1978). In this environment, there are a continuum of agents who may be divided into a finite number of types indexed by i ; $i = 1, \dots, n$. Each agent is faced with the possibility of either of two states of nature occurring; a "loss" state and a "no-loss" state. Let $s = 1$ be the "no-loss" state, and $s = 2$ be the "loss" state. Realizations of these states are independent across agents, and a type i agent faces probability p_i of the "no-loss" state occurring. The p_i obey $0 < p_1 < p_2 < \dots < p_n < 1$. If any agent ends up in state s , he receives endowment of the single good in that state e_s . Since $s = 1$ is the "no-loss" state, $e_1 > e_2$, with endowments being identical across types.

Let c_{is} denote the consumption of a type i agent in state s . All agents have a common utility function defined on R_+ denoted $U(c)$, with $U'(c) > 0$, $U''(c) < 0 \forall c \in R_+$. Finally, let $\mu^* \equiv (\mu_1^*, \dots, \mu_n^*)$ be a vector consisting of the measures of each type of agent, so that $\sum \mu_i^* = 1$, and $\mu_i > 0 \forall i$.

Having described the environment, it is necessary to describe the information structure for this economy. Each agent knows his own type prior to trade, but this is private information ex ante. Hence, this is a standard adverse selection insurance environment.

A Two-Stage Game

In addition to the agents described above, let there be a set of firms $F = \{1, \dots, M\}$, indexed by m , who can costlessly enter the activity of selling insurance policies. These policies offer type i agents state contingent consumption pairs (c_{i1}, c_{i2}) ; $i = 1, \dots, n$. Then we imagine insurance firms involved in the following game, which evolves in two stages. Let $\theta_k = (\theta_{k1}, \dots, \theta_{kn})$ be a vector specifying the measure of type i agents who purchase a policy from firm k . Thus, $\theta_{ki} \in [0, \mu_i^*]$. Firms then announce, in stage 1 of the game, an allocation rule which specifies the consumption pairs received by type i agents contingent on (a) θ_k , and (b) the allocations received by agents at other insurance firms. Let F denote the set of firms (which may be infinite), let $\theta \equiv (\theta_1, \dots, \theta_{k-1}, \theta_k, \theta_{k+1}, \dots)$, and let $\theta_{-k} \equiv (\theta_1, \dots, \theta_{k-1}, \theta_{k+1}, \dots)$. An allocation rule is denoted as follows. Let c_{kis} be the state contingent consumption level offered to type i agents by firm k . Let $c_{ki} \equiv (c_{ki1}, c_{ki2})$, $c_k = (c_{k1}, \dots, c_{kn})$, $c = (c_1, \dots, c_{k-1}, c_k, c_{k+1}, \dots)$, and $c_{-k} = (c_1, \dots, c_{k-1}, c_{k+1}, \dots)$, where each of these is to be understood as a vector of functions, and let Δ^n be that subset of R_+^n obeying $\sum_{r=1}^n y_r \leq 1$: $y \equiv (y_1, \dots, y_n)$. Then an allocation rule for firm k is a mapping

$$c_{kis}: \Delta^n \times \prod_{\substack{m \in F \\ m \neq k}} \Delta^n \times \prod_{m \in F} C \rightarrow R_+$$

specifying the consumption level of a type i agent in state s at firm k , where C denotes the space of allocation rules. We will write an allocation rule as $c_{kis}(\theta_k; \theta_{-k}, c_{-k})$.

Of course, certain restrictions must be imposed on the allocation rules announced by firms. An allocation rule must be resource feasible, i.e., satisfy

$$(1) \quad \sum_{i=1}^n \theta_{ki} [p_i (e_{1-ki1}(\theta_k; \theta_{-k}, c_{-k})) + (1-p_i)(e_{2-ki2}(\theta_k; \theta_{-k}, c_{-k}))] > 0$$

$$\forall \theta \in \prod_{k \in F} \Delta^n, \forall c_{-k} \in \prod_{\substack{m \in F \\ m \neq k}} C.$$

Also, since agent types are not observable ex ante, it must be the case that type i and only type i agents wish to obtain the consumption pair (c_{ki1}, c_{ki2}) . An allocation rule is incentive feasible if

$$(2) \quad p_i U[c_{ki1}(\theta_k; \theta_{-k}, c_{-k})] + (1-p_i) U[c_{ki2}(\theta_k; \theta_{-k}, c_{-k})] >$$

$$p_i U[c_{mj1}(\theta_m; \theta_{-m}, c_{-m})] + (1-p_i) U[c_{mj2}(\theta_k; \theta_{-k}, c_{-k})];$$

$$\forall i, j = 1, \dots, n, \forall m \in F.$$

Henceforth, we restrict attention to resource and incentive feasible (henceforth feasible) allocation rules.

At stage 1 of our game, then, each firm announces a θ -contingent allocation rule. In the second stage of our game, given the rules announced by each firm and taking the vector θ as given, each agent decides which firm to "purchase a policy" from. Since all firms announce incentive feasible allocation rules, each agent simply purchases a policy from that firm offering him the most preferred pair $[c_{ki1}(\theta_k; \theta_{-k}, c_{-k}), c_{ki2}(\theta_k; \theta_{-k}, c_{-k})]$.

It is now possible to say something about the values θ_k . Clearly

$$(3a) \quad \theta_{ki} = \mu_i^* \text{ if } p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}) > \\ p_i U(c_{mi1}) + (1-p_i)U(c_{mi2}) \quad \forall m \in F, m \neq k.$$

$$(3b) \quad \theta_{ki} = 0 \text{ if } p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}) < p_i U(c_{mi1}) + \\ (1-p_i)U(c_{mi2}) \text{ for some } m \in F.$$

However, there is considerable arbitrariness in specifying the values θ_{ki} if type i agents are indifferent regarding which firm they purchase a policy from. First, then, in keeping with standard practice, we assume that if all firms announce the same allocation rules, all agents are divided among firms according to their population proportions. Then, without loss of generality, we could proceed as if there were one firm. As a notational convention, then

$$(3c) \quad \theta_i = \frac{\mu_i^*}{M} \equiv \mu_i \text{ if } c_k(\theta) = c_m(\theta) \quad \forall k, m \in F, \quad \forall \theta \in \Delta^n.$$

If firms announce different rules, however, there is no equally obvious convention to adopt. One possibility would be to let a particular firm pick the value θ_{ki} anywhere in the interval $[0, \mu_i]$. A second possibility is that if all firms but one (say firm k) announce the same allocation rule, then $\theta_{ki} > 0$ iff type i agents are made strictly better off by purchasing their policy from firm k . We proceed as follows then. Our focus below will be on symmetric equilibria. Hence, θ_{ki} will fail to be proportional to μ_i only for a firm which is deviating from a candidate set of

equilibrium strategies. Therefore, we will only require a specification of the θ_{ki} in the case of type i indifference when all firms but k announce the same rule. We employ alternately the following two assumptions:

$$(3d) \quad \theta_{ki} \in [0, \mu_i^*] \text{ if } p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}) = \\ p_i U(c_{mi1}) + (1-p_i)U(c_{mi2}) \text{ and if } c_{mis}(\theta) = c_{tis}(\theta) \\ \forall m, t \in F, m, t \neq k, \forall \theta \in \Delta^n.$$

$$(3e) \quad \theta_{ki} = 0 \text{ if } p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}) \leq p_i U(c_{mi1}) + \\ (1-p_i)U(c_{mi2}) \text{ for some } m \in F.$$

It remains to say something about firm profits, and about what strategies firms can pursue. Let π_k denote the profits of firm k , with π_k given as follows:

$$(4) \quad \pi_k = \pi(\theta_k, c_k; \theta_{-k}, c_{-k}) = \\ \sum_{i=1}^n \theta_{ki} \{ p_i [e_1 - c_{ki1}(\theta_k; \theta_{-k}, c_{-k})] \\ + (1-p_i) [e_2 - c_{ki2}(\theta_k; \theta_{-k}, c_{-k})] \},$$

with the values θ_k determined either by (3a)-(3d), or by (3a)-(3c) and (3e). Finally, firm strategies are choices of feasible allocation rules.

A Nash Equilibrium

Prior to defining a Nash equilibrium for the game just described, we impose one additional restriction on the set of allocation rules which firms can announce. We begin with the following

Definition. An allocation $c_{ki}(\theta_k)$; $k \in F$, is θ_k -Pareto optimal if $\forall k \in F$ there does not exist another feasible, incentive compatible allocation $\hat{c}_{kis}(\theta_k)$ such that

$$p_i U[\hat{c}_{kil}(\theta_k)] + (1-p_i)U[\hat{c}_{ki2}(\theta_k)] > \\ p_i U[c_{kil}(\theta_k)] + (1-p_i)U[c_{ki2}(\theta_k)],$$

$\forall i = 1, \dots, n$, with strict inequality for some i , and if

$$\sum_{i=1}^n \theta_i \{p_i [\hat{c}_{kil}(\theta_k) - c_{kil}(\theta_k)] + \\ (1-p_i)[\hat{c}_{ki2}(\theta_k) - c_{ki2}(\theta_k)]\} \leq 0,$$

with θ_{-k} and $c_{mis}(\theta_{-k})$, $m \in F$, $m \neq k$ taken as parametric (where an obvious abbreviation of notation has been employed).

Finally, let C^* denote the set of admissible allocation rules for firms. Then $c_{iks}(\theta) \in C^*$ if c_{iks} is θ_k Pareto optimal $\forall \theta \in \times_{k \in F} \Delta^n$, and satisfies (1) and (2). The reason for requiring admissible allocation rules to be θ_k Pareto optimal $\forall k$ is that this prevents a firm from threatening to give poor allocations if their policies are purchased by (too many of) certain types of agents. This restriction might be justified as follows. Suppose θ_k represented the actual measure of types purchasing policies from firm k . Then it must not be possible for some firm, taking the allocations received by customers of other firms and θ_{-k} as given, to attract away all of the customers of firm k and thereby earn a profit.

Having defined the set of admissible allocation rules, we may now define a Nash equilibrium for our two-stage game.

Definition. A Nash equilibrium is a set of announced allocation rules $c_{kis}^* \in C^*$; $k \in F$, and a vector $\theta^* \in X \Delta^n$ such that

$$(5) \quad \sum_{k \in F} \theta_k^* = \mu^*.$$

$$(6) \quad p_i U[c_{ki1}(\theta_k^*; \theta_{-k}^*, c_{-k}^*)] + (1-p_i) U[c_{ki2}(\theta_k^*; \theta_{-k}^*, c_{-k}^*)] >$$

$$p_i U[c_{mi1}(\theta_m^*; \theta_{-m}^*, c_{-m}^*)] + (1-p_i) U[c_{mi2}(\theta_m^*; \theta_{-m}^*, c_{-m}^*)]$$

$\forall i, \forall k, m \in F.$

$$(7) \quad \pi_k(\theta_k^*, c_k^*; \theta_{-k}^*, c_{-k}^*) \geq \pi_k(\theta_k, c_k; \theta_{-k}, c_{-k}^*)$$

$\forall c_k \in C^*$, for all (θ_k, θ_{-k}) consistent with (either) (3a)-(3d) (or (3a)-(3c) and (3e)), and such that $(\theta_k, \theta_{-k}) \in X \Delta^n$, $\sum_{m \in F} \theta_m = \mu^*$.

Hence, a Nash equilibrium is a set of announcements by firms which are admissible and which leave no incentive for any firm to change its announcement, and an allotment of agents among firms given these announcements such that no agents have an incentive to purchase a policy from any other firm, given the purchases of other agents.

A Set of Allocation Rules

A much studied allocation rule is one which solves the problem (for fixed $\theta \gg 0$)

$$(8) \quad \max p_n U(c_{n1}) + (1-p_n) U(c_{n2})$$

subject to

$$(9) \quad p_i U(c_{i1}) + (1-p_i) U(c_{i2}) >$$

$$p_i U(c_{j1}) + (1-p_i) U(c_{j2}) \quad \forall i, j = 1, \dots, n,$$

$$(10a) \quad p_1 U(c_{11}) + (1-p_1)U(c_{12}) \geq \bar{U}_1$$

$$(10b) \quad p_i U(c_{i1}) + (1-p_i)U(c_{i2}) \geq \bar{U}_i(\theta); i = 2, \dots, n-1.$$

$$(11) \quad \sum_{i=1}^n \theta_i [p_i (e_1 - c_{i1}) + (1-p_i)(e_2 - c_{i2})] \geq 0,$$

where the values $\bar{U}_i(\theta); i = 1, \dots, n-1$ are defined recursively

by

$$(12) \quad \bar{U}_1 = \max p_1 U(c_{11}) + (1-p_1)U(c_{12})$$

subject to $p_1(c_{11} - e_1) + (1-p_1)(c_{12} - e_2) = 0.$

$$(13) \quad \bar{U}_i(\theta) = \max p_i U(c_{i1}) + (1-p_i)U(c_{i2})$$

subject to

$$(14) \quad p_j U(c_{j1}) + (1-p_j)U(c_{j2}) \geq \bar{U}_j(\theta); 1 < j < i-1$$

$$(15) \quad \sum_{j=1}^i \theta_j [p_j (e_1 - c_{j1}) + (1-p_j)(e_2 - c_{j2})] \geq 0$$

$$(16) \quad p_j U(c_{j1}) + (1-p_j)U(c_{j2}) \geq$$

$$p_h U(c_{h1}) + (1-p_h)U(c_{h2}); \forall j, h; j, h < i.$$

Solutions to this problem have been associated with "Wilson-equilibria" by Spence (1978) and Miyazaki (1977). We will claim that a subset of solutions to the problem (8)-(11) is a Nash equilibrium allocation for our two-stage game. Prior to stating this result, however, we need to make two remarks about this problem.

First, in our setting it is possible that $\theta_{ki} = 0$ for some i , for some firm k . Then it is necessary to say how firm k

treats the incentive compatibility conditions (9) (and (16)) involving this i . Clearly, if $\theta_{ki} = 0$, and if this is a preferred state of affairs for firm k , then firm k must structure its allocation rule so that type i agents will not purchase its policies designed for type j agents, $j \neq i$. Hence, we append the following constraint to the problem (8)-(11):

$$(17) \quad p_i U[c_{kj1}(\theta_k; \theta_{-k}, c_{-k})] + (1-p_i) U[c_{kj2}(\theta_k; \theta_{-k}, c_{-k})] \leq$$

$$p_i U[c_{mi1}(\theta_m; \theta_{-m}, c_{-m})] + (1-p_i) U[c_{mi2}(\theta_m; \theta_{-m}, c_{-m})]$$

$$\forall j, \forall m \in F \text{ such that } \theta_{mi} > 0, \forall i \text{ such that } \theta_{ki} = 0,$$

where firm k takes the values (c_{mi1}, c_{mi2}) as parametric. It is necessary to assume that these values are taken as parametric to prevent firms from attempting to manipulate other firms' allocations.

The second remark we need to make is as follows. As Miyazaki (1977) points out, the solution to (8)-(11) (and (17)) need not be unique. Henceforth, let $c_{kis}^*(\theta_k; \theta_{-k}, c_{-k})$ denote the allocation rule which solves (8)-(11) and (17) $\forall \theta \in \prod_{m \in F} \Delta^n$, and where c_{kis}^* is that solution which gives the highest expected utility to type $n - 1$ agents if there is no unique solution. Similarly, if there are two solutions which result in identical values of expected utility for type $n - 1$ agents, c_{kis}^* is that which gives highest expected utility to type $n - 2$ agents, etc. Notice, then, that c_{kis}^* need not produce exactly the Miyazaki-Spence-Wilson equilibrium allocation, since Miyazaki argues that the logic of the Wilson equilibrium concept results in the follow-

ing: if the solution of (8)-(11) is not unique, an equilibrium allocation will be the solution least preferred by type $n - 1$ agents, etc.

Existence of Equilibrium

The primary result of this section is

Proposition 1. If all firms announce the allocation rule $c_{kis}^*(\theta_k; \theta_{-k}, c_{-k})$, this along with $\theta_k = \mu$ constitutes a Nash equilibrium.

This is, of course, in contrast to Rothschild and Stiglitz (1976) or Wilson (1977), where no Nash equilibrium need exist in pure strategies. The proof of Proposition 1 for the case of $n > 2$ is somewhat involved, and makes use of the assumption (3e). Hence, we begin by proving our result for the case of $n = 2$, for which we need only use (3a)-(3d).

Proof of Proposition 1: 2 Types

The proof proceeds as follows. Suppose all firms announce the allocation rule c_{kis}^* . If this is not an equilibrium for some economy, then there exists for this economy a firm $d \in F$, an alternate admissible allocation rule \hat{c}_{dis} and a set of values $\hat{\theta}_d$ and $\hat{\theta}_{-d}$ (with $\sum_{\substack{k \in F \\ k \neq d}} \hat{\theta}_k = \mu^* - \hat{\theta}_d$) such that

$$p_i U[c_{kil}(\hat{\theta}_k; \hat{\theta}_{-k}, c_{-k})] + (1-p_i) U[c_{kil}(\hat{\theta}_k; \hat{\theta}_{-k}, c_{-k})] >$$

$$p_i U[c_{mil}(\hat{\theta}_m; \hat{\theta}_{-m}, c_{-m})] + (1-p_i) U[c_{mi2}(\hat{\theta}_m; \hat{\theta}_{-m}, c_{-m})];$$

$$\forall k \in F \text{ such that } \hat{\theta}_k > 0, \forall m \in F,$$

and

$$(18) \quad \pi_d(\hat{\theta}_d, \hat{c}_d; \hat{\theta}_{-d}, \hat{c}_{-d}) > \pi_d(\mu, c_d^*; \mu, c_{-d}^*)$$

when all other firms announce c_{kis}^* . We show that assuming (18) leads to a contradiction. In order to see this, we will need a lemma, and in order to state the lemma we will require some additional notation.

Suppose, following Prescott and Townsend (1984), that insurance firms could offer consumption lotteries contingent on the realization of the state of nature for each agent. Let X denote the set of possible realizations of the lottery, with typical element $x \in X$. Let $c_{is}(x)$ denote the consumption of a type i agent in state s if the realization of the lottery is x , and let us think of some set of agents choosing probabilities q_{ixs} of x occurring for type i agents, subject to $\sum_{x \in X} q_{ixs} = 1 \quad \forall i = 1, \dots, n$. A feasible lottery satisfies (for μ)

$$(19) \quad \sum_{i=1}^n \mu_i \sum_{x \in X} \{q_{ix1} p_i [c_{i1}(x) - e_1] + q_{ix2} (1-p_i) [c_{i2}(x) - e_2]\} < 0,$$

and an incentive compatible lottery satisfies

$$(20) \quad \sum_{x \in X} \{q_{ix1} p_i U[c_{i1}(x)] + q_{ix2} (1-p_i) U[c_{i2}(x)]\} > \\ \sum_{x \in X} \{q_{jx1} p_i U[c_{j1}(x)] + (1-p_i) q_{jx2} U[c_{j2}(x)]\}; \\ i, j = 1, \dots, n; i \neq j.$$

Suppose we now consider the problem

$$\max_{x \in X} \sum \{q_{nx1} p_n U[c_{n1}(x)] + q_{nx2} (1-p_n) U[c_{n2}(x)]\}$$

subject to (19), (20),

$$(21) \quad \sum_{x \in X} \{q_{ix1} p_i U[c_{i1}(x)] + (1-p_i) q_{ix2} U[c_{i2}(x)]\} \\ > \bar{U}_i; \quad i = 1, \dots, n-1,$$

with \bar{U}_i defined in a manner analogous to equations (13)-(16) and subject to

$$(22) \quad \sum_{x \in X} q_{ixs} = 1; \quad i = 1, \dots, n; \quad s = 1, 2.$$

Then we have the following result.

Lemma. The solution to this problem has $q_{ixs} = 1$ for some x ; $i = 1, \dots, n$, $s = 1, 2$.

Proof. The solution to the above problem is a Pareto optimum (Prescott and Townsend (1984), p. _____), and any Pareto optimal consumption lottery has $q_{ixs} = 1$ for some x , $\forall i, s$. (Prescott and Townsend, p. _____).

We may now prove Proposition 1. First, suppose $n = 2$ and suppose there is some $d \in F$, some allocation rule \hat{c}_d , and some set of values $(\hat{\theta}_d, \hat{\theta}_{-d})$ such that (18) holds when all firms other than d announce c_{kis}^* . Since the profit function of firm d is given by (4), $\hat{\theta}_{id} > 0$ must hold for some i . Consider, then, what must be the case for $\hat{\theta}_{id} > 0$ to hold. In order to attract type 2 agents, clearly firm d must offer an allocation rule such that

$$(23) \quad p_2 U[\hat{c}_{d21}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] + (1-p_2) U[\hat{c}_{d22}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] > \\ p_2 U[c_{k21}^*(\mu; \mu, c_{-k}^*)] + (1-p_2) U[c_{k22}^*(\mu; \mu, c_{-k}^*)].$$

Then, if type 2 agents are attracted in some nonnegative number by firm d, type 1 agents can be attracted iff

$$(24) \quad p_1 U[\hat{c}_{d11}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] + (1-p_1) U[\hat{c}_{d12}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] > \\ p_1 U[c_{k11}^*(\mu; \mu, c_{-k})] + (1-p_1) U[c_{k12}^*(\mu; \mu, c_{-k})], \quad \forall k \in F, \\ k \neq d$$

or

$$(25) \quad p_1 U[\hat{c}_{d11}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] + (1-p_1) U[\hat{c}_{d12}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] > \\ p_1 U[c_{k11}^*(\mu - \hat{\theta}_d; \hat{\theta}_{-k}, c_{-k})] + (1-p_1) U[c_{k12}^*(\mu - \hat{\theta}_d; \hat{\theta}_{-k}, c_{-k})] \\ \forall k \in F, k \neq d.$$

Thus, type 2 agents are attracted by firm d only if they are made better off, and type 1 agents are attracted if they are better off at firm d than they would have been if firm d announced c_d^* , or if they are better off at firm d after the defection of some type 2 agents from other firms to firm d. Finally, since any admissible allocation rule must be θ -Pareto optimal $\forall \theta \in \prod_{k \in F} \Delta^n$, clearly if (23) and either (24) or (25) hold with equality, then $\pi_d(\hat{\theta}_d, \hat{c}_d; \hat{\theta}_{-d}, c_{-d}^*) = 0$ would hold. Hence for (18) to hold, either (23) or (25) must hold with strict inequality.

There are now four possible cases to consider.

Case 1. $(\hat{\theta}_{1d}, \hat{\theta}_{2d}) = (\hat{\theta}_{1d}, 0)$, $\hat{\theta}_{1d} > 0$. Then firm d attracts only type 1 agents, and not type 2 agents. Therefore, (24) holds with strict inequality. But clearly

$$(26) \quad p_1 U[\hat{c}_{d11}(\hat{\theta}_d)] + (1-p_1) U[\hat{c}_{d12}(\hat{\theta}_d)] < \bar{U}_1,$$

where \bar{U}_1 is defined by (12) (with an obvious abbreviation of notation in (26)). (26) contradicts either (24) or (25) (with strict inequality in either), so that this case is impossible.

Case 2. $(\hat{\theta}_{1d}, \hat{\theta}_{2d}) = (\hat{\theta}_{1d}, \mu_2)$, $\hat{\theta}_{1d} < \mu_1$. Then given the form of c_k^* , $k \neq d$,

$$(27) \quad p_1 U[c_{k11}^*(\mu - \hat{\theta}_d)] + (1-p_1)U[c_{k12}^*(\mu - \hat{\theta}_d)] = \bar{U}_1; k \in F.$$

Therefore

$$(28) \quad p_2 U[\hat{c}_{d21}(\hat{\theta}_d)] + (1-p_2)U[\hat{c}_{d22}(\hat{\theta}_d)] \leq \bar{U}_2,$$

where \bar{U}_2 maximizes $p_2 U(c_{21}) + (1-p_2)U(c_{22})$ subject to

$$(29) \quad p_1 U(c_{21}) + (1-p_1)U(c_{22}) = \bar{U}_1$$

$$(30) \quad p_2(c_{21} - e_1) + (1-p_2)(c_{22} - e_2) \leq 0.$$

Clearly, then

$$(31) \quad p_2 U[c_{k21}^*(\theta; \theta_{-d}, c_{-d})] + (1-p_2)U[c_{k22}^*(\theta; \theta_{-d}, c_{-d})] \geq \bar{U}_2$$

$$\forall \theta \in \prod_{k \in F} \Delta^n, \forall c_{-d} \in \prod_{k \in F} C,$$

as the maximization problem defining \bar{U}_2 is more tightly constrained than (8)-(11). But (28) and (31) contradict (23).

Hence, this case is impossible.

Case 3. $(\hat{\theta}_{1d}, \hat{\theta}_{2d}) = (\mu_1, \mu_2)$, so that firm d attracts all agents.

Now clearly

$$p_2 U[\hat{c}_{d21}(\hat{\theta}_d)] + (1-p_2)U[\hat{c}_{d22}(\hat{\theta}_d)] \leq$$

$$p_2 U[c_{k21}^*(\mu)] + (1-p_2)U[c_{k22}^*(\mu)]$$

for this value of $\hat{\theta}_d$, by the definition of c_{kis}^* . Therefore, (23) must hold with equality. Hence there are at least two allocations giving the level of type 2 utility which emerges as the solution to (8)-(11) when $\theta = \mu$. Also, since (23) holds with equality, (24) must hold with strict inequality. But then c_{kis}^* is not the allocation rule among the two or more solutions to (8)-(11) which gives type 1 agents the highest level of expected utility. This contradicts the definition of c_{kis}^* , so this case is also impossible.

Case 4. $\mu_2^* > \hat{\theta}_{d2} > 0$, $\hat{\theta}_{d1} \in [0, \mu_1^*]$. Now, since $(\mu_2^* - \hat{\theta}_{d2}) \hat{\theta}_{d2} > 0$,

$$(32) \quad p_2 U[\hat{c}_{d21}(\hat{\theta}_d)] + (1-p_2)U[\hat{c}_{d22}(\hat{\theta}_d)] = p_2 U[c_{k21}^*(\hat{\theta}_k)] + \\ (1-p_2)U[c_{k21}^*(\theta_k)] > p_2 U[c_{k21}^*(\mu)] + (1-p_2)U[c_{k22}^*(\mu)];$$

where the second inequality follows from (23). Also, if the inequality in (32) is not strict, then (24) holds (and $\hat{\theta}_{1d} > 0$). Then consider a consumption lottery which allocates type i agents in state s $\hat{c}_{dis}(\hat{\theta}_d)$ with probability $\hat{\theta}_{di}/\mu_i^*$, and $c_{kis}^*(\mu - \hat{\theta}_d)$ with probability $1 - (\hat{\theta}_{di}/\mu_i^*)$. (This lottery is feasible, since each allocation is individually. Obviously, it also assigns type 1 agents an expected utility at least as great as \bar{U}_1 .) Then, from (32), type 2 agents are at least as well off under this lottery as they are receiving $c_{2s}^*(\mu)$ with certainty in state s , and if they are not strictly better off then (from (24)) type 1 agents are. However, this implies that there exist values $q_{ixs} \in (0,1)$ that solve the problem discussed in the lemma. The existence of such a solution contradicts the lemma, so this case is impossible. Thus, we have proved Proposition 1 for the case of $n = 2$.

Proof of Proposition 1: $n \geq 2$ Types

Again the proof proceeds by assuming that (18) holds, and deriving a contradiction. We begin by noting that the lemma of the previous section holds for arbitrary n , and by stating the analogs of (23)-(25): $\hat{\theta}_{dn} > 0$ iff

$$(33) \quad p_n U[\hat{c}_{dn1}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] + (1-p_n) U[\hat{c}_{dn2}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] > \\ p_n U[c_{kn1}^*(\mu; \mu, c_{-k})] + (1-p_n) U[c_{kn2}^*(\mu; \mu, c_{-k})]; \quad \forall k \in F, \\ k \neq d$$

and $\hat{\theta}_{di} > 0$; $i < n$, iff

$$(34) \quad p_i U[\hat{c}_{di1}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] + (1-p_i) U[\hat{c}_{di2}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] > \\ p_i U[c_{ki1}^*(\mu; \mu, c_{-k})] + (1-p_i) U[c_{ki2}^*(\mu; \mu, c_{-k})]; \quad \forall k \in F, \\ k \neq d,$$

or

$$(35) \quad p_i U[\hat{c}_{di1}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] + (1-p_i) U[\hat{c}_{di2}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] > \\ p_i U[c_{ki1}^*(\hat{\theta}_k; \hat{\theta}_{-k}, c_{-k})] + (1-p_i) U[c_{ki2}^*(\hat{\theta}_k; \hat{\theta}_{-k}, c_{-k})]; \quad \forall k \in F, \\ k \neq d.$$

Thus to attract type n agents, firm d must make them (weakly) better off than they were initially. Also, to attract type i agents, $i < n$, it must either make them weakly better off than they were initially, or better off than they would be after the defection of type j agents, $j > i$. Also, as before, for (18) to hold, it is necessary that either (33) hold with strict inequality, or that (34) hold with strict inequality for some i .

Now suppose there exists a firm d , an allocation rule \hat{c}_{dis} , and a vector $(\hat{\theta}_d, \hat{\theta}_{-d})$ (such that $\sum_{k \in F} \hat{\theta}_k = \mu^*$) such that (18) holds. There are then several cases to consider.

Case 1. Suppose $\hat{\theta}_{di} > 0 \forall i$ satisfying $j \leq i \leq \ell$, and $\hat{\theta}_{di} = 0$ otherwise.

(a) Suppose $\ell = n$. Then clearly a contradiction results if $j = 1$. Hence $j > 1$. Now since $\hat{\theta}_{di} > 0$, and if (3.e) holds, $\hat{\theta}_{di} = \mu_i^*$. Therefore

$$p_{j-s} U[c_{k,j-s,1}^*(\hat{\theta}_k; \hat{\theta}_{-k}, c_{-k})] + (1-p_{j-s}) U[c_{k,j-s,2}^*(\hat{\theta}; \hat{\theta}_{-k}, c_{-k})] \geq \bar{U}_{j-s}(\mu) \quad \forall s = 1, \dots, j-1$$

since $\hat{\theta}_{dj-s} = 0$ by assumption, and since types with indices less than j receive the allocation specified by c_{kis}^* . Moreover, since (35) holds,

$$p_i U[c_{dil}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] \geq \bar{U}_i(\mu) \quad \forall i \text{ such that } j \leq i \leq n.$$

Hence the allocation giving agents with indices $j \leq i \leq n$ the consumption pairs $[\hat{c}_{dil}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*), \hat{c}_{di2}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)]$, and agents with indices $i < j$ the pairs $[c_{i1}^*(\mu - \hat{\theta}_d), c_{i2}^*(\mu - \hat{\theta}_d)]$ satisfies the constraints (10), and clearly is feasible. Also, by (33) and (34), either type n agents strictly prefer $(\hat{c}_{dn1}, \hat{c}_{dn2})$ to (c_{n1}^*, c_{n2}^*) , or else they are indifferent and type $n-1$ agents prefer $(\hat{c}_{d,n-1,1}, \hat{c}_{d,n-1,2})$ to $(c_{n-1,1}^*, c_{n-1,2}^*)$, etc. In particular, one type i ; $j \leq i \leq n$, is made strictly better off, with no type $i+s$; $s = 1, \dots, n-i$, made worse off. But this contradicts the definition of c_{is}^* , and is impossible.

(b) Then $\ell < n$. Since no agents of type $i > \ell$ defect to firm d , type ℓ agents must weakly prefer $[\hat{c}_{d\ell 1}(\hat{\theta}_d), \hat{c}_{d\ell 2}(\hat{\theta}_d)]$ to $[c_{\ell 1}^*(\mu), c_{\ell 2}^*(\mu)]$, and some agent type i ; $j \leq i \leq \ell$ is made strictly better off.

In particular,

$$(36) \quad p_\ell U[\hat{c}_{d\ell 1}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] + (1-p_\ell)U[\hat{c}_{d\ell 2}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d}^*)] > \\ p_\ell U[c_{k\ell 1}^*(\mu)] + (1-p_\ell)U[c_{k\ell 2}^*(\mu)]; \quad k \in F, k \neq d.$$

Also, by assumption (3e), $\hat{\theta}_{di} = \mu_i \forall i; j < i < \ell$. We now make use of the following lemma.

Lemma 2. Suppose some firm k has $\theta_{kq} = 0$. Then $c_{kis}^*(\theta_k; \theta_{-k}, c_{-k})$ sets $U_{q-1} \leq \bar{U}_{q-1}$ (if $\theta_{kq-1} > 0$).

Proof: it is easy to show that the incentive constraint (9) for $i = q - 1$ binds only on the choices c_{qs}^* in the problem (8)-(11). If $\theta_{kq} = 0$, then c_{qs} is taken as parametric by firm k . Therefore, setting $U_{q-1} > \bar{U}_{q-1}$ does not relax any constraints in the problem (8)-(11), and uses resources. Hence $U_{q-1} \leq \bar{U}_{q-1}$ (if $\theta_{kq-1} > 0$). As is well known (Spence (1978)), this implies

$$(37) \quad \sum_{i=1}^{q-1} \theta_{ki} [p_i [c_{i1}(\theta_k) - e_1] + (1-p_i)[c_{i2}(\theta_k) - e_2]] \leq 0.$$

Thus, since $\hat{\theta}_{d\ell} > 0$ (and hence, by (3e), since $\hat{\theta}_{k\ell} = 0 \forall k \in F, k \neq d$), (37) holds $\forall k \neq d$ for $q - 1 = \ell - 1$. Therefore, the following two statements are true:

$$(38) \quad \sum_{i=1}^{\ell} \hat{\theta}_{ki} [p_i c_{i1}(\hat{\theta}_k) + (1-p_i)c_{i2}(\hat{\theta}_k)] \leq \\ \sum_{i=1}^{\ell} \mu_i [p_i c_{i1}^*(\mu) + (1-p_i)c_{i2}^*(\mu)],$$

(from (37) with $q = \ell$ and from $\hat{\theta}_{k\ell} = 0$), and (36) holds. Hence

$$(39) \quad p_n U[c_{kn1}^*(\hat{\theta}_k; \hat{\theta}_{-k}, c_{-k})] + (1-p_n)U[c_{kn2}^*(\hat{\theta}_k; \hat{\theta}_{-k}, c_{-k})] > \\ p_n U[c_{n1}^*(\mu)] + (1-p_n)U[c_{n2}^*(\mu)],$$

and if (39) holds with equality, some type i , $\ell < i < n$, is made strictly better off (with no type greater than i made worse off at firm k , $k \neq d$). This is true since (by (36)) an incentive constraint is relaxed in the problem (8)-(11), and (by (38)) no greater amount of resources are consumed by agents with $i \leq \ell$ after the change in announcement by firm d . Also, as before, no type i agent receives expected utility less than $\bar{U}_i(\mu)$. But then (39) (with strict inequality, or (39) at equality along with the fact that some other type is made better off) contradicts that c_{is}^* solves (8)-(11). Hence, this case is also impossible.

Case 2. Therefore, there must be indices i , j , and ℓ satisfying $j < i < \ell$, such that $\hat{\theta}_{dj} \hat{\theta}_{d\ell} > 0$ and $\hat{\theta}_{di} = 0$. Repeating the previous argument, if ℓ is the largest index such that $\hat{\theta}_{d\ell} > 0$, we derive the same contradiction as before. Hence, this case is also impossible, and Proposition 1 is proved.

A Cooperative Equilibrium Concept

We now consider the imposition of a fairly standard cooperative equilibrium concept on the same Rothschild-Stiglitz insurance environment described above. In particular, a coalition k is a set of indices $k \subset \{1, \dots, n\}$ and an associated vector of measures $(\theta_{k1}, \dots, \theta_{kn})$. Let K denote the set of possible coalitions. Again, our focus is on allocation rules as above, for reasons we discuss momentarily. Recall that an allocation rule $c_{is}(\mu)$ specifies an allocation to be received by type i agents in state s , if μ describes the population for the economy. Then we say that an allocation rule $c_{is}(\mu)$ is blocked if there exists a coalition $k \in K$, and an allocation rule $c_{kis}(\theta_k; \theta_{-k}, c_{-k})$ specify-

ing the allocation to be received by a type i member of coalition k (where θ_k is the vector of measures of agent types belonging to coalition k , θ_{-k} is the vector of members belonging to other coalitions, and c_{-k} is the vector of allocation rules announced by other coalitions) with the following properties:

$$(40) \quad \sum \theta_{ki} \{ p_i [c_{i1}(\theta_k; \theta_{-k}, c_{-k}) - e_1] + (1-p_i) [c_{i2}(\theta_k; \theta_{-k}, c_{-k}) - e_2] \} < 0$$

$$(41) \quad p_i U[c_{ki1}(\theta_k; \theta_{-k}, c_{-k})] + (1-p_i) U[c_{ki2}(\theta_k; \theta_{-k}, c_{-k})] > p_i U[c_{mi1}(\theta_m; \theta_{-m}, c_{-m})] + (1-p_i) U[c_{mi2}(\theta_m; \theta_{-m}, c_{-m})] \\ \forall i \text{ such that } \theta_{ki} > 0, \forall m \in K, m \neq k.$$

$$(42) \quad p_i U[c_{ki1}(\theta_k; \theta_{-k}, c_{-k})] + (1-p_i) U[c_{ki2}(\theta_k; \theta_{-k}, c_{-k})] < p_i U[c_{mi1}(\theta_m; \theta_{-m}, c_{-m})] + (1-p_i) U[c_{mi2}(\theta_m; \theta_{-m}, c_{-m})] \\ \forall i \text{ such that } \theta_{ki} < \mu_i, \text{ for some } m \in K, m \neq k.$$

$$(43) \quad p_i U[c_{\ell i1}(\theta_\ell; \theta_{-\ell}, c_{-\ell})] + (1-p_i) U[c_{\ell i2}(\theta_\ell; \theta_{-\ell}, c_{-\ell})] > p_i U[c_{mj1}(\theta_m; \theta_{-m}, c_{-m})] + (1-p_i) U[c_{mj2}(\theta_m; \theta_{-m}, c_{-m})] \\ \forall \ell, m \in K, \forall i, j = 1, \dots, n.$$

$$(44) \quad p_n U[c_{n1}(\theta_k; \theta_{-k}, c_{-k})] + (1-p_n) U[c_{n2}(\theta_k; \theta_{-k}, c_{-k})] > p_n U[c_{n1}(\mu)] + (1-p_n) U[c_{n2}(\mu)],$$

if $\theta_n > 0$,

$$(45) \quad p_{n-1} U[c_{n-1,1}(\theta_k; \theta_{-k}, c_{-k})] +$$

$$(1-p_{n-1})U[c_{n-1,2}(\theta_k; \theta_{-k}, c_{-k})] > \\ p_{n-1}U[c_{n-1,1}(\mu)] + (1-p_{n-1})U[c_{n-1,2}(\mu)]$$

if $\theta_n = 0$, $\theta_{n-1} > 0$, or if $\theta_n > 0$ and (44) holds with equality, etc.

A word about conditions (40)-(45) is in order. Equation (40) requires the allocation rule for coalition k to be resource feasible. Equations (41) and (42) require that coalition membership be individually rationale, and also that exclusion of individuals from a coalition be voluntary, i.e., in (42) individuals are not members of k iff they do not wish to be. This is natural for our private information setting, since otherwise it is not clear on what basis individuals are to be excluded from a coalition. Equation (43) requires that all announced allocation rules be incentive feasible both within and across coalitions. In particular, type j agents in the complementary coalition to k cannot wish to either join coalition k and truthfully reveal their type, or join it and claim to be of some other type. Again, this seems a natural requirement in this private information setting. Finally, (44), (45), etc., require that some agent types in coalition k strictly prefer the allocation received in that coalition to the allocation they would receive under the allocation rule $c_{is}(\mu)$.

As a final definition, a core allocation rule is a feasible, incentive compatible allocation rule which is not blocked. It remains, then, to explain our focus on allocation rules in this setting. It will be recalled that we require that

blocking coalitions announce incentive compatible allocations. Hence, the set of allocations available to any coalition depends on the actions of the complementary coalition. This is a feature familiar from the literature on the cores of economies with externalities/public goods (e.g., Foley (), Richter (), Starrett ()). Hence, all coalitions must announce allocation rules, so that they specify what they will do contingent on the actions of other coalitions. Again, we focus here only on what we call admissible allocation rules (defined essentially as above), so that all allocation rules must satisfy (40)-(43) $\forall (\theta_k, \theta_{-k}) \in X \times \Delta^n$, and must also satisfy the following condition. Given $k \in K$ (θ_k, θ_{-k}) , and given the allocations of other coalitions, there must not exist an alternate feasible allocation rule \hat{c}_{is} such that

$$(46) \quad p_i U[\hat{c}_{kil}(\theta_k; \theta_{-k}, c_{-k})] + (1-p_i) U[\hat{c}_{ki2}(\theta_k; \theta_{-k}, c_{-k})] > p_i U[c_{kil}(\theta_k; \theta_{-k}, c_{-k})] + (1-p_i) U[c_{ki2}(\theta_k; \theta_{-k}, c_{-k})]$$

$\forall i$ such that $\theta_{ki} > 0$, with strict inequality for some such i . This condition is imposed to prevent coalitions from threatening to take actions in certain contingencies which their memberships would unanimously reject if the contingency actually arose.

Consider now the allocation rule $c_{is}^*(\theta)$, which solves (8)-(11) if $\theta_i \neq 0$ for any i , and the associated allocation rule $c_{kis}^*(\theta_k; \theta_{-k}, c_{-k})$ which solves (8)-(11) subject to the assumptions discussed above if $\theta_{ki} = 0$ for some i . Our first result regarding cooperative equilibria is

Proposition 2. The allocation rule $c_{is}^*(\theta)$ is a core allocation rule.

For the proof we make use of assumption (3e), i.e., if a blocking coalition exists for some allocation rule, with d denoting the blocking coalition, $\theta_{di} = \mu_i^*$ or $\theta_{di} = 0 \forall i = 1, \dots, n$.

Proof: Again we assume there exists a coalition d which announces an allocation rule $\hat{c}_{dis}(\theta_d; \theta_{-d}, c_{-d})$ and has an associated measure of vectors $\hat{\theta}_d$ which blocks $c_{is}^*(\theta)$. We then show that this assumption leads to a contradiction. As before there are several cases to consider.

Case 1. $\hat{\theta}_{di} > 0 \forall i$ such that $j \leq i \leq \ell$. First, suppose $\ell = n, j = 1$. Then all types are attracted. By (43), type n agents weakly prefer the allocation received under $\hat{c}_{dis}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d})$ to that received under $c_{is}^*(\mu)$. If they are indifferent between the two, type $n - 1$ agents weakly prefer the $\hat{c}_{dis}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d})$ allocation to that received under $c_{is}^*(\mu)$, etc., with some type being strictly better off, and no type above that being worse off. But this contradicts the definition of c_{is}^* . Hence this case is impossible.

Suppose, then, that $\ell = n, j > 1$. Since $\hat{\theta}_{di} = \mu_i \forall i > j$, all types with $q < j$ receive the allocation specified by $c_{is}^*(\mu - \hat{\theta}_d)$. Therefore

$$(47) \quad p_{j-1}U[c_{j-1,1}(\mu - \hat{\theta}_d)] + (1-p_{j-1})U[c_{j-1,2}(\mu - \hat{\theta}_d)] = \bar{U}_{j-1}(\mu - \hat{\theta}_d),$$

with \bar{U}_{j-1} defined by (14)-(16). Then type n agents cannot be made better off under $\hat{c}_{dis}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d})$ than they would be if they received the allocation which solved

$$\max p_n U(c_{n1}) + (1-p_n)U(c_{n2})$$

subject to (40), (43), (10), and (47). But this problem is more heavily constrained than (8)-(11) with $\theta = \mu$. Hence, type n agents cannot be strictly better off under $\hat{c}_{dis}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d})$ than under $c_{is}^*(\mu)$. If they are indifferent between the two allocations type $n - 1$ agents cannot be strictly better off, etc. Hence, no type $i, j < i < n$ can be strictly better off. Therefore, $\ell < n$.

Then suppose $\hat{\theta}_{di} > 0; 1 < j < i < \ell < n$. As we noted above, the incentive constraints associated with types $q < j$ do not bind on types $r > \ell$ in the problem (8)-(11) with $\theta_\ell = 0$, and with type ℓ agents receiving a given allocation specified by $\hat{c}_{d\ell s}(\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d})$. Therefore, (since $\hat{\theta}_{-d}$ is the complementary coalition)

$$(48) \quad \sum_{\alpha=1}^{\ell} \hat{\theta}_{-d\alpha} \{ p_{\alpha} [c_{\alpha 1}^*(\hat{\theta}_{-d}; \hat{\theta}_d, c_d) - e_1] + (1-p_{\alpha}) [c_{\alpha 2}^*(\hat{\theta}_{-d}; \hat{\theta}_d, c_d) - e_2] \} \leq 0$$

(see, e.g., Spence (1978)). Also, by (45),

$$(49) \quad p_{\ell} U[\hat{c}_{d\ell 1}(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)] + (1-p_{\ell}) U[\hat{c}_{d\ell 2}(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)] > p_{\ell} U[c_{\ell 1}^*(\mu)] + (1-p_{\ell}) U[c_{\ell 2}^*(\mu)].$$

But (48) and (49) imply that

$$(50) \quad p_n U[c_{n1}^*(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)] + (1-p_n) U[c_{n2}^*(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)] > p_n U[c_{n1}^*(\mu)] + (1-p_n) U[c_{n2}^*(\mu)].$$

If (50) holds with equality the analog of (50) holds for $n - 1$, etc., and there is strict inequality for some $r > \ell$. Now it was initially feasible to give all agents of type $i; j < i < \ell$, the

allocation $\hat{c}_{dis}(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)$, and all other agents the allocation $c_{is}^*(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)$. Moreover, this allocation would clearly satisfy (10). Therefore, (50) (or its analog for other types if (50) holds with equality) implies that $c_{is}^*(\mu)$ does not solve (8)-(11) with $\theta = \mu$. This contradicts the definition of $c_{is}^*(\mu)$, so that this case is impossible.

Case 2. There exist indices satisfying $j < i < \ell$ such that $\hat{\theta}_{dj} \hat{\theta}_{d\ell} > 0$, $\hat{\theta}_{di} = 0$. Also, without loss of generality, let ℓ be the largest index with $\hat{\theta}_{d\ell} > 0$. First, suppose $\ell = n$. Then (44) holds for type n agents. In addition, since each coalition has a feasible, incentive compatible allocation,

$$(51) \quad p_n U[c_{n1}^*(\mu)] + (1-p_n)U[c_{n2}^*(\mu)] \geq p_n U(\hat{c}_{n1}) + (1-p_n)U(\hat{c}_{n2}),$$

where $(\hat{c}_{n1}, \hat{c}_{n2})$ solves

$$\max p_n U(c_{n1}) + (1-p_n)U(c_{n2})$$

subject to (9), (10), (11), (with $\theta = \mu$), and

$$p_{i^*} U(c_{i^*1}) + (1-p_{i^*})U(c_{i^*2}) = \bar{U}_{i^*}(\mu - \hat{\theta}_d),$$

where i^* is the largest index such that $\hat{\theta}_{di} = 0$ and $\hat{\theta}_{di+1} > 0$.

(51) is true since the problem defining $(\hat{c}_{n1}, \hat{c}_{n2})$ involves more constraints than that defining (c_{n1}^*, c_{n2}^*) . But clearly

$$(52) \quad p_n U(\hat{c}_{n1}) + (1-p_n)U(\hat{c}_{n2}) \geq p_n U[\hat{c}_{dn1}(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)] + \\ (1-p_n)U[\hat{c}_{dn2}(\hat{\theta}_{-d}; \hat{\theta}_d, c_d)].$$

Now (51) and (52) contradict (44) with strict inequality. If $\hat{\theta}_{dn-i} = 0 \forall i$ this is a contradiction. If $\hat{\theta}_{dn-1} > 0$ and if (44) holds with equality, we may repeat the argument for $n - 1$, etc. Hence, this case is impossible.

Therefore, $\ell < n$. But now using (45) for ℓ , and using (by previous arguments) that

$$(53) \quad \sum_{\alpha=1}^{\ell} \hat{\theta}_{-d\alpha} \{p_{\alpha} [c_{\alpha 1}^* (\hat{\theta}_{-d}; \hat{\theta}_d, c_d) - e_1] + (1-p_{\alpha}) [c_{\alpha 2}^* (\hat{\theta}_{-d}; \hat{\theta}_d, c_d) - e_2]\} \leq 0,$$

clearly type n agents are not worse off under $c_{ns}^* (\hat{\theta}_{-d}; \hat{\theta}_d, c_d)$ than under $c_{ns}^* (\mu)$. Similarly, if they are not better off, type $n - 1$ is not worse off, etc. Hence, there exists a type $h > \ell$ which is strictly better off than under $c_{is}^* (\mu)$, and no type i , $h < i \leq n$, is worse off than under $c_{is}^* (\mu)$. But since both $c_{is}^* (\hat{\theta}_{-d}; \hat{\theta}_d, c_d)$ and $\hat{c}_{is} (\hat{\theta}_d; \hat{\theta}_{-d}, c_{-d})$ are feasible and incentive compatible, this contradicts the definition of $c_{is}^* (\mu)$. Therefore, this case is impossible.

Case 3. There is just one index, ℓ , with $\hat{\theta}_{d\ell} > 0$. But then, by (45) (for type ℓ) and (53), we can reproduce the argument above. Thus, this case is impossible as well, proving Proposition 2.

Notice that we have relied heavily on (3e), which says a blocking coalition must attract all or none of each type. For the case of $n = 2$ a proof of Proposition 2 can be constructed for any $\theta_{di} \in [0, \mu_i^*]; i = 1, 2$. This proof is analogous to the proof of Proposition 1 for $n = 2$, and is omitted here.

Our second result on cooperative equilibrium allocation rules is

Proposition 3. Let $c_{is}^*(\theta)$ be the allocation rule solving (8)-(11). Then $c_{is}^*(\theta)$ is the unique core allocation rule.

As before, the proof for $n = 2$ is straightforward, whereas the proof for $n > 2$ requires that the complementary coalition either has all or none of each type of agent. Hence, we first present the proof for $n = 2$.

Proof of Proposition 3 (n=2). By Proposition 2, $c_{is}^*(\theta)$ is a core allocation rule. Suppose there also exists some other core allocation rule $\hat{c}_{is}(\theta)$. We now derive a contradiction by constructing a blocking coalition for $\hat{c}_{is}(\theta)$.

Let d denote the blocking coalition, and θ_d^* its associated vector of measures. Suppose $\theta_{d2}^* = \mu_2^*$, and suppose d announces the allocation rule $c_{dis}^*(\theta_{-d}; \theta_d, c_d)$. Since $\theta_{d2}^* = \mu_2^*$, if $\theta_{d1}^* < \mu_1$ clearly

$$(54) \quad p_1 U[\hat{c}_{11}(\mu^* - \theta_d^*)] + (1-p_1) U[\hat{c}_{12}(\mu^* - \theta_d^*)] < \bar{U}_1,$$

with \bar{U}_1 defined by (12). Since type 1 agents obtain expected utility no less than \bar{U}_1 with coalition d (by constraint (10a)), we could arbitrarily assign all type 1 agents to d and make them no worse off than they were in the complementary coalition. There are now two cases to consider.

Case 1. $\hat{c}_{is}(\mu^*)$ does not solve (8)-(11) for $\theta = \mu^*$. Then clearly

$$p_2 U[c_{21}^*(\mu^*)] + (1-p_2) U[c_{22}^*(\mu^*)] > \\ p_2 U[\hat{c}_{21}(\mu^*)] + (1-p_2) U[\hat{c}_{22}(\mu^*)].$$

Also, by constraint (10a),

$$(55) \quad p_1 U[c_{11}^*(\mu^*)] + (1-p_1)U[c_{12}^*(\mu^*)] > \bar{U}_1.$$

Moreover, by construction $c_{is}^*(\theta)$ is feasible. Hence $c_{is}^*(\theta)$ satisfies all of the conditions required of a blocking allocation rule, so this case is impossible.

Case 2. There is more than one solution to (8)-(11). $\hat{c}_{is}(\mu^*)$ is one such solution, but is not the solution which gives type 1 agents the greatest utility. Then clearly the grand coalition blocks $\hat{c}_{is}(\theta)$. Thus $c_{is}^*(\theta)$ is the unique core allocation rule, as claimed.

For $n > 2$, we again make use of (3e), so that when a blocking coalition forms the complementary coalition either has $\theta_{-d} = \mu_i^*$ or $\theta_{-d} = 0 \forall i = 1, \dots, n$.

Proof of Proposition 3 ($n > 2$). As before, we know $c_{is}^*(\theta)$ is a core allocation rule. Again, we suppose there is some other core allocation rule $\hat{c}_{is}(\theta)$, and derive a contradiction by constructing a blocking coalition. We continue to let d denote the blocking coalition, θ_d^* its associated vector of measures, and we continue to suppose $\theta_{dn}^* = \mu_n^*$, and that d announces allocation rule $c_{dis}^*(\theta_{-d}; \theta_d, c_d)$. There are two cases to consider.

Case 1. $\hat{c}_{is}(\mu^*)$ does not solve (8)-(11). Then clearly

$$p_n U[c_{n1}^*(\mu^*)] + (1-p_n)U[c_{n2}^*(\mu^*)] >$$

$$p_n U[\hat{c}_{n1}(\mu^*)] + (1-p_n)U[\hat{c}_{n2}(\mu^*)].$$

Also, $c_{is}^*(\theta)$ is feasible by construction. Hence, if we show that we can always select $\theta_{di}^* = \mu_i^*$; $1 \leq i < n$, we will have constructed a blocking coalition.

By (3e), if $\theta_{di}^* \neq \mu_i^*$, $\theta_{di}^* = 0$. Then suppose we cannot select $\theta_{di}^* = \mu_i^* \forall i$. Let q denote the largest index such that $\theta_{dq}^* = 0$. There are now two possibilities.

a) $\theta_{dq-h}^* = \mu_{q-h}^*$ for some h ; $1 \leq q-h < q$. Then as we know, $c_{d,q-h,s}^*(\theta_d; \theta_{-d}, c_{-d})$ obeys

$$(56) \quad \sum_{j=1}^{q-1} \theta_{dj}^* \{p_j [c_{j1}^*(\theta_d^*; \theta_{-d}^*, c_{-d}^*) - e_1] + (1-p_j) [c_{j2}^*(\theta_d^*; \theta_{-d}^*, c_{-d}^*) - e_2]\} \leq 0,$$

and

$$(57) \quad p_{q-h} U[c_{q-h,1}^*(\theta_d^*; \theta_{-d}^*, c_{-d}^*)] + (1-p_{q-h}) U[c_{q-h,2}^*(\theta_d^*; \theta_{-d}^*, c_{-d}^*)] \leq \bar{U}_{q-h}(\theta_d^*),$$

with $\bar{U}_{q-h}(\theta_d^*)$ defined by (14)-(16). Thus it is not hard to see that

$$(58) \quad p_q U[\hat{c}_{q1}(\theta_d^*; \theta_{-d}^*, c_{-d}^*)] + (1-p_q) U[\hat{c}_{q2}(\theta_d^*; \theta_{-d}^*, c_{-d}^*)] \leq \bar{U}_q(\mu^*).$$

This is true since $p_q U[\hat{c}_{q1}(\theta_d^*; \theta_{-d}^*, c_{-d}^*)] + (1-p_q) U[\hat{c}_{q2}(\theta_d^*; \theta_{-d}^*, c_{-d}^*)]$ is bounded above by the solution to the problem that defines $\bar{U}_q(\mu^*)$, but with a set of constraints of the form (57) added (since $\theta_{di}^* = 0$ for at least one i). Now if $\theta_{di}^* = \mu_i^* \forall i$, type q agents obtain at least $\bar{U}_q(\mu^*)$ by (10). Hence, we may assign these agents to coalition d without violating (42). Thus, d would satisfy all the requirements for a blocking coalition.

b) $\theta_{di}^* = 0 \forall 1 \leq i \leq q$. Then clearly (58) holds. Hence the argument above applies, and we may assign all agents to coalition d without violating (42). Thus, again a blocking coalition d has been constructed, so that this case is impossible.

Case 2. $\hat{c}_{is}(\mu^*)$ does solve (8)-(11) for $\theta = \mu^*$, but there are multiple such solutions and \hat{c}_{is} is not the one which assigns highest utility to type $n - 1$ agents, etc. Then obviously the grand coalition blocks $\hat{c}_{is}(\theta)$. Hence, we can always construct a blocking coalition for any allocation rule other than $c_{is}^*(\theta)$, establishing the proposition.

The logic of the proposition is straightforward. Under any allocation rule other than $c_{is}^*(\theta)$ which is Pareto optimal $\forall \theta \in \Delta^n$ (and hence a candidate for a core allocation rule), type n agents subsidize agents of other types. Hence, type n agents can always form a blocking coalition by defecting, and offering agents of other types an allocation weakly preferred by them to any allocation they could attain on their own. Since this is at least as good as what these agents can attain in the absence of type n agents, they are in essence "forced" to join the blocking coalition. This intuition also suggests why $c_{is}^*(\theta)$ is an unblocked allocation rule.

Finally, there is an obvious corollary to Propositions 1-3.

Corollary. The set of core allocation rules is contained in the set of Nash equilibrium allocation rules.

Thus core allocation rules can be "decentralized" here.