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# THE ADVERSE SELECTION PROBLEM REVISITED

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## Introduction

The problems that arise in a wide variety of markets characterized by the presence of private information have received a great deal of attention in recent literature. This paper focuses specifically on a widely considered class of adverse selection and signalling environments. While these environments have been much studied, no study to date has attempted to incorporate (or even formally to recognize) what Rothschild and Stiglitz (1976, p. 646) term "the peculiar provision of many insurance contracts, that the effective premium is not determined until the end of a period. . . ." In fact, there exists a wide variety of mutual forms of organization (mutual insurance companies and mutual banking institutions are examples) in which participants not only pay premiums and receive insurance payments, but in which they periodically receive dividends contingent on the aggregate experiences of the organization. Similar examples arise in organizations that are not explicitly mutual in nature, e.g., profit sharing plans in labor markets, etc. Hence models of insurance and labor markets should attempt to incorporate (and explain) such forms of organization.

This paper represents an attempt to do so. In particular, we consider two kinds of settings. The first is one in which the types of organizations that exist are not given exogenously, but in which agents are free to cooperate and form arbitrary coalitions. Here we consider a solution concept which is a straightforward translation of standard core concepts. The second is one in which there exists a set of firms (whose existence is taken as a datum of the economy) that compete in a noncooperative manner. Here we employ a standard Nash equilibrium concept. What we show is that, in each case, there is a simple rule that an organization can employ such that

(a) an equilibrium (in pure strategies) exists, and

(b) this equilibrium results in a Pareto optimal allocation of resources.

Moreover, in each case, we are able to show the existence of a unique equilibrium. Finally, the core allocation coincides with the Nash equilibrium of our noncooperative game.

The simple rule that organizations use in each case is one that takes exactly the form that rules specifying allocations take in mutual forms of organization. In particular, these rules specify the allocation received by each agent as a function of the population experience of the organization, and as a function of events specific to himself. Hence we show that once organizations are allowed to take on a "mutual aspect," existence problems that arise in adverse selection settings, and to a certain extent problems of multiplicity of equilibria in signalling environments, can be resolved in a straightforward manner with an obvious economic interpretation. Also, this demonstrates that a commonly held view of the mutual form of organization, that it is an anomaly, is false in settings with private information.

Having roughly described the nature of our approach and the results obtained, it is probably useful to describe in some detail the economic environments to which our approach applies and to provide some perspective on why these are of special interest. Moreover, the modeling strategy employed here is related to several different strands of literature. Hence, it is appropriate to relate our approach to its antecedents. The remainder of this section is addressed to this task.

The discussion in the paper is concerned primarily with the adverse selection insurance environment introduced by Rothschild and Stiglitz (1976) and Wilson (1977). While the discussion is phrased in terms of this specific context, it should be clear that it also applies to the labor market contexts of Miyazaki (1977) and Smith (1984). We will also briefly show that a version of the Spence (1973) signalling environment [with our particular version being due to Prescott and Townsend (1984)] can be analyzed via the same methods, and, therefore, that the same results apply.

While the Rothschild-Stiglitz and the Spence settings are formally quite similar [as has been pointed out previously; see, e.g., Riley (1979)], models of the two settings have encountered somewhat different problems. In particular, when standard Nash or competitive equilibrium concepts are applied to the Rothschild-Stiglitz environment, equilibria often fail to exist. When similar concepts are applied to the Spence environment, on the other hand, a large multiplicity of equilibria typically arises. A virtue of our approach (in either a cooperative or a noncooperative context) is that it always produces an equilibrium in the Rothschild-Stiglitz case, and at the same time results in a unique equilibrium in the Spence case.

More generally, this paper is one of a number of papers that attempts to produce mechanisms which result in (i) existence of an equilibrium, (ii) optimality of equilibrium, and (iii) uniqueness of equilibrium. In spirit our approach is similar to that of Prescott and Townsend (1984), who examine the properties of competitive mechanisms. What they show is that the competitive mechanism successfully produces objectives (i) and (ii) above when applied to economies where agents engage in trade prior to obtaining their private information. However, when agents are aware of their type (i.e., possessed of private information) prior to engaging in trade, an equilibrium may fail to exist (in the adverse selection setting). Moreover, they consider the following possible extension of the competitive mechanism. Since the presence of private information introduces externalities into the economy, one could attempt to use the devise of augmenting the commodity space which is employed in standard competitive settings with externalities [e.g., Starrett

(1972)]. However, this approach is unsuccessful in the private information setting. Finally, it should be noted that when a competitive approach is applied to the Spence economy, multiple equilibria still arise.

In light of the failure of both competitive equilibria and the Nash equilibrium (in pure strategies) considered by Rothschild and Stiglitz to exist, a large literature has considered "reactive" equilibrium concepts. Wilson (1977) was able to produce a general existence result for the adverse selection insurance setting using this approach, and a modification of the Wilson equilibrium concept by Miyazaki (1977) and Spence (1978) results in equilibria also being Pareto optimal. In addition, Dasgupta and Maskin (1982) have proved a general existence result regarding Nash equilibria in mixed strategies for these economies [see also Rosenthal and Weiss (1984)] when the set of "insurance firms" is fixed exogenously (and is finite). However, these equilibria are not typically optimal (as is clear from the discussion in Rosenthal and Weiss). In the signalling context, on the other hand, there have been attempts to reduce the set of equilibria by invoking the concept of stable sets of equilibria [Kreps (1984)]. However, while this approach does reduce the set of (presumably economically) relevant equilibria, it does not result in a unique equilibrium (again, see Kreps).

While these approaches partially resolve the problems that motivate us, there are some well-known problems with them which motivate us to employ a different approach. With respect to reactive equilibrium concepts, the shortcomings (and arbitrariness) of the approach are well articulated by Riley (1979) and Rothschild and Stiglitz (1976, p. 647) who refer to it as "a peculiar halfway house; firms respond to competitive entry by dropping policies, but not by adding new policies." Moreover, since Riley (1979) discusses the arbitrariness of the notion of reaction employed, this need not be done here. With respect to the mixed strategy approach, Rosenthal and Weiss (1984) point out that the equilibrium they construct (which they conjecture is unique) leaves rent opportunities for new entrants. Hence, one might wish to construct an equilibrium concept for which this was not the case.

In light of these remarks we adopt two different, yet related, approaches to modeling these environments. As indicated above, one is to avoid the notion that firms are an exogenous part of the economic environment, and instead to employ a cooperative approach with a standard notion of blocking and the core. However, as is widely pointed out elsewhere, the presence of privately informed agents gives rise to "informational externalities." Hence, we face a problem quite similar to one faced in the literature on cores of economies with externalities or public goods: what is feasible for any coalition may depend on the make-up and actions of its complement. In the earlier literature on the cores of such economies, it was recognized that how members of one coalition reacted to the formation of a potential blocking coalition could crucially affect the set of core allocations. We face a similar problem and resolve it by in effect unifying the earlier approaches. In particular, we consider what we call "core allocation rules." These must specify not only the allocations agents receive, but what allocations they will receive if any potential blocking coalition forms. Hence, coalitions can react to the formation of a blocking coalition, but must specify in advance the form their reaction will take. These allocation rules then make resulting arrangements look a great deal like mutual forms of organizations. In particular, they specify allocations as a function of group membership and performance (and also of the membership and actions of other groups). We show that, for the environments we consider, a unique core allocation rule exists.

It will be noted that in the cooperative context the notion of reaction arises naturally. Of course, the notion of firm reactions has been employed to produce existence of equilibrium when one proceeds from the assumption of an exogenously given set of firms. What we would like to do is give these firms the opportunity to specify in advance how they will react to any attempt to draw off any set of their customers. Thus again, we consider a set of incumbent firms who specify in advance a rule by which the allocations their customers receive is determined. This rule takes the same form as the rule discussed above: each agent's allocation depends on the make-up and experience of his firm's clientele, as well as the actions of any potential entrants who attempt to attract a portion of the incumbents' customers. When the space of strategies for the incumbents is the space of such rules (appropriately restricted), this allows all firms to adopt a single (fixed) strategy, which nonetheless specifies how they will react to the introduction of new "contracts" by potential entrants. Hence a noncooperative, nonreactive (Nash) equilibrium concept can be considered which, as in the case of our cooperative concept, unifies the Nash approach of Rothschild-Stiglitz (1976) with later reactive approaches. Hence a consideration of a rich enough set of potential arrangements, which has the plausible interpretation of a mutual form of organization, allows us to produce a result regarding the general existence of a Nash equilibrium. Moreover, the allocation associated with this equilibrium is the same as that specified by the unique core allocation rule.

## The Rothschild-Stiglitz Insurance Environment

Our focus in this paper is on the insurance environment discussed by Rothschild and Stiglitz (1976), Wilson (1977), and Spence (1978). Below we will point out that a version of the signalling environment discussed by Spence (1973, 1974)[1/] can be dealt with by the same methods we now discuss.

In the insurance environment considered here, there are a continuum of agents who can be divided into a finite number of "types." We let there be n such types, with type indexed by $i = 1, \ldots, n$. Each agent, regardless of type, is faced with the possibility of either of two states occurring. Let s index the set of states which any individual might face; $s = 1, 2$. Then there is a single consumption good, and an agent in state s finds himself with an endowment $e_s$ of the good. We let $e_1 > e_2$, so that $s = 1$ is what is commonly referred to in this context as the "no-loss" state, and $s = 2$ is the "loss" state. Realizations of s are independent across agents, with a type i agent facing probability $p_i$ of the "no-loss" state occurring. The $p_i$ obey $0 < p_1 < p_2 < \ldots < p_n$. Hence, agents with larger indices are "lower risk" than agents with lower indices.

Let $c_{is}$ denote the consumption of a type i agent who finds himself in state s. All agents have a common utility function defined on $R_+$ denoted $U(c)$, with $U'(c) > 0$, $U''(c) < 0 \; \forall \; c \in R_+$. Finally, let $\mu = (\mu_1, \ldots, \mu_n) \in R^n_{++}$ be a vector consisting of the measure of each type of agent, with $\Sigma\mu_i = 1$.

It remains to describe the nature of information in this economy. It is assumed that each agent knows his own type prior to realization of the state, and also that this is private information <u>ex ante</u>. Moreover, all "trades" are assumed to be observable. Hence this is a standard adverse selection setting.

We now wish to consider what kinds of insurance arrangements will be made by agents in this economy. We begin by considering a cooperative equilibrium concept in which groups of agents coalesce for the purpose of creating insurance opportunities. Notice that they do so in an environment in which there is no aggregate uncertainty.

A Cooperative Equilibrium Concept

While a number of cooperative equilibrium concepts have been put forward for application to private information settings of this type,[2/] we consider the imposition of a fairly standard cooperative equilibrium concept on this environment. As will be clear shortly, our concept most closely resembles traditional core concepts applied to economies with public goods/externalities.[3/]

To begin, a coalition is a set of indices $K \subset \{1,\ldots,n\}$ and an associated vector of measures $(\theta_{k1}, \ldots, \theta_{kn}) \varepsilon R^n_+$. For technical reasons that will be elaborated on below, it is convenient to adopt the view that initially all agents are members of the grand coalition. The grand coalition is required to specify an allocation, and also to specify the allocations its members will receive in the event of defection by any subset of its members. Hence, the grand coalition announces an allocation rule, which specifies the allocation received by each of its members in the event of the defection of any coalition. Without loss of generality, we may consider the case where only one coalition may defect at any point in time. Let k denote the complementary coalition to the one announcing an allocation rule, which we henceforth term the incumbent. Then k has associated with it a vector of measures $(\theta_{k1}, \ldots, \theta_{kn})$. The incumbent coalition, then, must select a rule specifying the allocations its members receive as a function of (i) its membership, (ii) the membership of k, and (iii) the allocation received by members of k. Let $c_{kis}$ denote the allocation received by type i agents in state s who belong to k. (If $\theta_{ki} = 0$, by convention set $c_{kis} = 0$; s = 1, 2), let $c_{ki} = (c_{ki1}, c_{ki2})$, and let $c_k = (c_{k1}, \ldots, c_{kn}) \varepsilon R^{2n}_+$. Also, let $\Delta^n$ denote the set of vectors $y \varepsilon R^n_+$ such that $\sum_{i=1}^{n} y_i \leq 1$. Then $\theta_k \equiv (\theta_{k1}, \ldots, \theta_{kn}) \varepsilon \Delta^n$. Therefore, the allocation rule of the incumbent is a mapping $c_{is}$: $\Delta^n \times \Delta^n \times R^{2n}_+ \rightarrow R_+$ specifying the consumption of a type i agent in state s, which may depend on the actions of the complementary coalition. We will often denote the allocation rule by $c_{is}(\theta;\theta_k,c_k)$, where the absence of a subscript related to coalitions indicates the incumbent coalition.

The reason for requiring the incumbent (grand) coalition to announce an allocation rule rather than an allocation can best be exposited after developing our results. We begin our analysis by stating the requirements we impose on allocation rules. First, consider an arbitrary coalition r with vector of members $(\theta_{r1}, \ldots, \theta_{rn}) \equiv \theta_r$. An allocation is resource feasible for r if

$$(1) \qquad \sum_{i=1}^{n} \theta_{ri}\,[p_i(e_1-c_{ri1})+(1-p_i)(e_2-c_{ri2})] \geqslant 0.$$

An allocation rule is resource feasible if the allocations specified by $c_{is}(\theta;\theta_k,c_k)$ satisfy (1) for all $(\theta,\theta_k) \in \Delta^{2n}$, for all $c_k \in R_+^{2n}$. Second, since agents' types are not directly observable, then the allocations $c_{ris}$ are incentive feasible if two conditions are satisfied. In particular, if $\theta_{ri} > 0$ and $\theta_{rj} > 0$, type i and j agents within r must prefer the allocation meant for them to that they could obtain by claiming to be of a different type. Hence the allocation vector $c_r$ is incentive feasible only if

$$(2.a) \qquad p_iU(c_{ri1}) + (1-p_i)U(c_{ri2}) \geqslant p_iU(c_{rj1}) + (1-p_i)U(c_{rj2})$$

$\forall$ i, j such that $\theta_{ri}\theta_{rj} > 0$. Moreover, we also impose the following incentive compatibility condition. If there are two coalitions, r and t, type i agents in coalition r cannot wish to join coalition t and claim to be of type j. Then the allocation vector $c_r$ is incentive feasible if (2.a) holds and if

$$(2.b) \qquad p_iU(c_{ri1}) + (1-p_i)U(c_{ri2}) \geqslant p_iU(c_{tj1}) + (1-p_i)U(c_{tj2})$$

$\forall$ r, t, r $\neq$ t, $\forall$ i, j such that $\theta_{ri}\theta_{tj} > 0$, where our convention is that $c_{tjs} = 0$ if $\theta_{tj} = 0$. Also, as before, an allocation rule is incentive feasible if

$c_{is}(\theta;\theta_k,c_k)$ satisfies (2.a) and (2.b) $\forall$ $(\theta,\theta_k) \in \Delta^{2n}$, $\forall$ $c_k \in R^+$ with $c_k$ incentive feasible. Finally, an allocation vector is feasible iff it is resource and incentive feasible, and similarly for an allocation rule.

It remains to say something about how coalition membership is determined in our setting. This requires comment, since with private information, membership in a coalition cannot be based on privately observed characteristics. Thus if a coalition wishes either to include or exclude certain individuals, it must create appropriate incentives to produce this result. Therefore, for a coalition k,

$$\text{(3.a)} \qquad \theta_{ki} = \mu_i \text{ if } p_iU(c_{ki1}) + (1-p_i)U(c_{ki2}) > p_iU(c_{mi1}) + (1-p_i)U(c_{mi2})$$

for all other coalitions m. Also

$$\text{(3.b)} \qquad \theta_{ki} = 0 \text{ if } p_iU(c_{ki1}) + (1-p_i)U(c_{ki2}) < p_iU(c_{mi1}) + (1-p_i)U(C_{mi2})$$

for some coalition $m \neq k$. Conditions (3.a) and (3.b) are fairly obvious, stating that all (no) type i agents are members of coalition k if no (some) coalition offers them a strictly preferred allocation. However, there is considerable arbitrariness in specifying the values $\theta_{ki}$ if type i agents are indifferent regarding coalition membership. Recall that we begin with an incumbent coalition which is the grand coalition, so initially $\theta_i = \mu_i$ (the incumbent coalition has no subscript). Suppose that some coalition k forms and defects from the grand coalition. One possibility, then would be to allow

$$\text{(3.c)} \qquad \theta_{ki} \in [0,\mu_i] \text{ if } p_iU(c_{ki1}) + (1-p_i)U(c_{ki2}) = p_iU(c_{i1}) + (1-p_i)U(c_{i2}).$$

(3.c) effectively allows the defecting coalition to select any value of $\theta_{ki}$ it desires, so long as type i agents are indifferent regarding coalition member ship. A second possible assumption is

$$\text{(3.d)} \qquad \theta_{ki} \in \{0,\mu_i\} \text{ if } p_iU(c_{ki1}) + (1-p_i)U(c_{ki2}) = p_iU(c_{i1}) + (1-p_i)U(c_{i2}).$$

(3.d) simply says that a defecting coalition must either attract all members of a given type or none of them. We will alternately employ (3.c) and (3.d) below.[4/]

Having described the conditions required for type i agents to become members of any coalition, we now impose one additional condition on the announced allocation rule of the incumbent coalition. In order to do this, we offer the following

Definition. Consider the incumbent coalition with associated vector of measures $\theta$, and possibly another coalition k with vector of measures $\theta_k$. Then an allocation $c_{is}(\theta;\theta_k)$ is Pareto optimal for $(\theta,\theta_k)$ iff there does not exist another feasible allocation $\tilde{c}_{is}(\theta;\theta_k)$ such that

$$p_i U[\tilde{c}_{i1}(\theta;\theta_k)] + (1-p_i)U[\tilde{c}_{i2}(\theta;\theta_k)] \geq p_i U[c_{i1}(\theta;\theta_k)] + (1-p_i)U[c_{i2}(\theta;\theta_k)]$$

with strict inequality for some i, where the allocations $c_{kis}$ have been taken as parametric.

The condition we impose on announced allocation rules is that any allocation rule announced by the incumbent coalition must specify an allocation which is Pareto optimal for $(\theta,\theta_k)$ $\forall$ $(\theta,\theta_k) \in \Delta^{2n}$ such that $\theta_k = \mu - \theta$. Formally, let C* denote the set of admissible allocation rules. Then $c_{is}(\theta;\theta_k,c_k) \in C^*$ iff $c_{is}(-)$ is feasible and specifies an allocation which is Pareto optimal $\forall$ $(\theta,\theta_k)$. The reason for requiring the announced allocation rule to be Pareto optimal for $(\theta,\theta_k)$ is as follows. Suppose $(\theta,\theta_k)$ represented the actual vectors of measures of agents belonging to the incumbent coalition and coalition k respectively. Then the requirement that announced allocation rules be Pareto optimal for the remaining members of the incumbent coalition prevents that coalition from specifying allocations in certain contingencies which would be unanimously rejected by its membership if that contingency arose. In other words, this requirement rules out "threats" by the incumbent coalition which it would not actually wish to carry out if the relevant event occurred.

We are now prepared to present our notion of blocking and our definition of the core of this economy. First, we say an allocation rule $c_{is} \in C^*$ is blocked if there exists a coalition k (with associated vector of measures $\theta_k$) and an allocation $c_{kis}$ with the following properties:

(4) $$\sum_{i=1}^{n} \theta_{ki} [p_i (c_{ki1}-e_1)+(1-p_i)(c_{ki2}-e_2)] \leq 0$$

(5) If $\theta_{ki} > 0$,

$$p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}) \geq p_i U[c_{i1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_i)U[c_{i2}(\mu-\theta_k;\theta_k,c_k)],$$

(6) while if $\theta_{ki} = 0$,

$$p_i U[c_{i1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_i)U[c_{i2}(\mu-\theta_k;\theta_k,c_k)] \geq p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}).$$

(7) $$p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}) \geq p_i U(c_{kj1}) + (1-p_i)U(c_{kj2})$$

$\forall$ i, j such that $\theta_{ki}\theta_{kj} > 0$.

(8.a) $$p_i U(c_{ki1}) + (1-p_i)U(c_{ki2}) \geq p_i U[c_{j1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_i)U[c_{j2}(\mu-\theta_k;\theta_k,c_k)]$$

$\forall$ i, j such that $\theta_{ki}(\mu-\theta_{kj}) > 0$.

(8.b) $$p_iU[c_{i1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_i)U[c_{i2}(\mu-\theta_k;\theta_k,c_k)] \geq p_iU(c_{kj1}) + (1-p_i)U(c_{kj2})$$

$\forall$ i, j such that $(\mu-\theta_{ki})\theta_{kj} > 0$.

(9) $$p_nU(c_{kn1}) + (1-p_n)U(c_{kn2}) \geq p_nU[c_{n1}(\mu)] + (1-p_n)U[c_{n2}(\mu)]$$

if $\theta_{kn} > 0$, while

(10) $$p_{n-1}U(c_{n-1,1}) + (1-p_{n-1})U(c_{n-1,2}) \geq p_{n-1}U[c_{n-1,1}(\mu)] + (1-p_{n-1})U[c_{n-1,2}(\mu)]$$

if $\theta_{kn} = 0$, $\theta_{kn-1} > 0$, or if $\theta_{kn} > 0$ and (9) holds with equality, and similarly for types n - s; s = 2, ..., n - 1 if (9) and (10) hold with equality (or if $\theta_{kn}\theta_{kn-1} = 0$), etc.

Some explanation of conditions (4)-(10) is in order. Equation (4) simply requires $c_k$ to be resource feasible. (5) and (6) require coalition membership to be voluntary, as discussed above, while (7) requires $c_k$ to be incentive feasible within coalition k. (8) states that type i agents in coalition k must not wish to join the incumbent coalition and claim to be of a type other than i, and conversely.

Equations (9) (and (10)) are our analogs of standard conditions for blocking. In particular an allocation can be blocked in either of two ways here. One is that type n agents are made strictly better off by joining the blocking coalition, subject only to the condition that any other members of coalition k join voluntarily. The idea here is that type n agents can "force" a better allocation for themselves by defecting, where the possibility is considered that after their defection agents of other types may wish to defect as well, even though they would have preferred the initial arrangement. The second way in which blocking can occur is either that type n agents do not defect (or are indifferent regarding coalition membership), but type n - 1 agents can "force" an improved allocation for themselves by defection, where again types with indices less than n - 1 may be "induced" to defect. Hence blocking requires that type n agents be made (weakly) better off if they defect, and if they are not strictly better off (or if they do not defect), that type n - 1 agents be made (weakly) better off if they defect, etc. Notice that all agents in the blocking coalition need not be better off than they were initially.

The strong role played by the ordering of types in our definition of blocking is based on the idea that some types have greater command over resources (per capita, and in an average sense) than others, and thus have the ability to induce certain outcomes. While this will be made clearer below, we note now that the idea that certain players occupy special positions in cooperative games has been widely used in recent developments in models of games with private information.[5/] However, an admitted shortcoming of this approach is that it requires an obvious ordering of types, although clearly a number of economies of interest have this feature.

As a final matter of definition, an allocation rule is a core allocation rule iff it is not blocked by any coalition. It is also appropriate, then, that we say something regarding why we phrase the discussion in terms of core allocation rules. Notice that, through condition (2.b), the actions of a potential blocking coalition affect the set of feasible actions for the incumbent coalition. Put somewhat informally, this is a setting in which standard "informational externalities" arise. The problem of defining the core for an economy with externalities has been put forth elsewhere.[6/] In particular, since the actions of a potential blocking coalition affect the feasible set of an incumbent coalition and conversely, it is necessary to state explicitly how the incumbent coalition responds to blocking threats. Our focus on allocation rules simply requires that the incumbent coalition specify in advance how it will respond to <u>any</u> attempts at blocking. This is its allocation rule. Thus our use of this concept allows a unification of previous treatments of the core with externalities in that the incumbent coalition may react to attempts at blocking through its rule, but must state in advance and hold constant the rule by which it reacts.

## A Set of Allocation Rules

A much studied allocation rule is one which solves the problem (for fixed $\theta >> 0$)

$$(11) \qquad \max\ p_n U(c_{n1}) + (1-p_n)U(c_{n2})$$

subject to

$$(12) \qquad p_i U(c_{i1}) + (1-p_i)U(c_{i2}) \geq$$

$$p_i U(c_{j1}) + (1-p_i)U(c_{j2}) \quad \forall\ i,\ j = 1,\ \ldots,\ n,$$

$$(13.a) \qquad p_1 U(c_{11}) + (1-p_1)U(c_{12}) \geq \overline{U}_1$$

$$(13.b) \qquad p_i U(c_{i1}) + (1-p_i)U(c_{i2}) \geq \overline{U}_i(\theta);\ i = 2,\ \ldots,\ n-1.$$

$$(14) \qquad \sum_{i=1}^{n} \theta_i\,[p_i(e_1-c_{i1})+(1-p_i)(e_2-c_{i2})] \geq 0,$$

where the values $\overline{U}_i(\theta)$; $i = 1, \ldots, n-1$ are defined recursively by

$$(15) \qquad \overline{U}_1 = \max\ p_1 U(c_{11}) + (1-p_1)U(c_{12})$$

subject to $p_1(c_{11}-e_1) + (1-p_1)(c_{12}-e_2) = 0.$

$$(16) \qquad \overline{U}_i(\theta) = \max\ p_i U(c_{i1}) + (1-p_i)U(c_{i2})$$

subject to

(17) $p_j U(c_{j1}) + (1-p_j)U(c_{j2}) \geq \overline{U}_j(\theta); \ 1 \leq j \leq i - 1$

(18) $\sum_{j=1}^{i} \theta_j [p_j(e_1 - c_{j1}) + (1-p_j)(e_2 - c_{j2})] \geq 0$

(19) $p_j U(c_{j1}) + (1-p_j)U(c_{j2}) \geq p_j U(c_{h1}) + (1-p_j)U(c_{h2}); \ \forall\, j,\, h;\ j,\, h \leq i.$

Solutions to this problem have been associated with "Wilson equilibria" by Spence (1978) and Miyazaki (1977). We will claim that a subset of solutions to the problem (11)-(19) is a core allocation rule. Prior to stating this result, however, we need to make two remarks about this problem.

First, in our setting it is possible that $\theta_i = 0$, i.e., that the incumbent coalition (which is the coalition announcing an allocation rule) has no agents of certain types. Then it is necessary to say how the incentive compatibility conditions (12) (and (19)) involving this i are treated. Clearly if $\theta_i = 0$, and if the incumbent coalition does not wish to retain type i agents, then its allocation rule must be such that type i agents will not wish to return to it and claim to be of type j; $j \neq i$. Similarly, if $\theta_i \theta_{kj} > 0$, type i agents cannot wish to join the blocking coalition and claim to be of type j. Hence, we append the following constraints to the problem (11)-(14):

(20.a) $p_i U[c_{j1}(\theta; \mu-\theta, c_k)] + (1-p_i)U[c_{j2}(\theta; \mu-\theta, c_k)]$

$$\leq p_i U(c_{ki1}) + (1-p_i)U(c_{ki2})$$

(20.b) $p_i U[c_{i1}(\theta; \mu-\theta, c_k)] + (1-p_i)U[c_{i2}(\theta; \mu-\theta, c_k)] \geq p_i U(c_{kj1})$

$$+ (1-p_i)U(c_{kj2}),$$

where (20.a) holds for all i, j such that $\theta_i = 0$ and $\theta_j > 0$, where (20.b) holds for all i, j such that $\theta_i \theta_{kj} > 0$, and where the values $c_k$ are taken as parametric in both (20.a) and (20.b).

The second remark we need to make is as follows. As Miyazaki (1977) points out, the solution to (11)-(19) (and (20)) need not be unique. Henceforth let $c^*_{is}(\theta;\theta_k,c_k)$ be the allocation rule solving (11)-(19) (and (20)) which gives the highest expected utility to type n - 1 agents if there is more than one solution. Similarly, if there is more than one solution which gives identical expected utility for type n - 1 agents, $c^*_{is}$ is that solution giving the highest expected utility to type n - 2 agents, etc. Notice, then, that $c^*_{is}$ need not produce exactly the Miyazaki-Spence-Wilson equilibrium allocation, since Miyazaki argues that the logic of the Wilson equilibrium concept results in the following: if the solution to (11)-(19) is not unique, an equilibrium allocation will be the solution least preferred by type n - 1 agents.

## A Result on Lotteries

Prior to stating our results, it will be useful to produce a preliminary result due to Prescott and Townsend (1984). However, since the statement of this result requires some investment in additional notation and already appears elsewhere, this section can be omitted by the reader without loss of continuity.

The result that we require concerns the use (or absence of use) of consumption lotteries in any Pareto optimal allocation. Suppose, then, following Prescott and Townsend (1984), that it is possible to offer consumption lotteries contingent on the realization of the state of nature for each agent. In particular, let X denote the (finite) set of possible realizations of the lottery, with typical element $x \in X$. Further, let $c_{is}(x)$ denote the consumption of a type i agent in state s if the realization of the lottery is x, and let us think of choosing the probabilities $q_{is}(x)$ of x occurring if an agent is of type i and state s occurs. These choices, of course, must satisfy

$\sum_{x \varepsilon X} q_{is}(x) = 1 \;\forall\; i = 1, \ldots, n;\; s = 1, 2.$ Then a resource feasible lottery satisfies (for an economy with a vector of population measures $\mu$)

$$(21) \qquad \sum_{i=1}^{n} \mu_i \sum_{x \varepsilon X} \{p_i q_{i1}(x)[c_{i1}(x)-e_1]+q_{i2}(x)(1-p_i)[c_{i2}(x)-e_2]\} \leq 0,$$

and an incentive feasible lottery satisfies

$$(22) \qquad \sum_{x \varepsilon X} \{q_{i1}(x)p_i U[c_{i1}(x)]+q_{i2}(x)(1-p_i)U[c_{i2}(x)]\} \geq \sum_{x \varepsilon X} \{q_{j1}(x)p_i U[c_{j1}(x)] + q_{j2}(x)(1-p_i)U[c_{j2}(x)]\}$$

$\forall\; i, j = 1, \ldots, n;\; i \neq j.$

Suppose we now consider the following problem:

$$\max \sum_{x \varepsilon X} \{q_{n1}(x)p_n U[c_{n1}(x)]+q_{n2}(x)(1-p_n)U[c_{n2}(x)]\}$$

by choice of $q_{is}(x)$, subject to (21), (22),

$$(23) \qquad \sum_{x \varepsilon X} \{q_{i1}(x)p_i U[c_{i1}(x)]+q_{i2}(x)(1-p_i)U[c_{i2}(x)]\} \geq \overline{U}_i;$$

$i = 1, \ldots, n - 1$, with $\overline{U_i}$ defined in a manner analogous to equations (15)-(19),[7] and subject to

$$(24) \qquad \sum_{x \varepsilon X} q_{is}(x) = 1;$$

$i = 1, \ldots, n;\; s = 1, 2.$ Then we have the following result.

Lemma 1. The solution to the above problem has $q_{is}(x) = 1$ for some $x$; $i = 1, \ldots, n;\; s = 1, 2.$

Proof. Prescott and Townsend (1984) consider the problem

$$\max \sum_{i=1}^{n} \lambda_i \sum_{x \in X} \{q_{i1}(x)p_i U[c_{i1}(x)] + q_{i2}(x)(1-p_i)U[c_{i2}(x)]\}$$

by choice of values $q_{is}(x)$, subject to (21), (22), and (24). They show that this problem has $q_{is}(x) = 1$ for some x, $\forall$ i, s, for arbitrary values $\lambda_i$ (p. ). By reinterpreting the $\lambda_i$ as the optimal Lagrange multipliers associated with the constraints (22) on our problem, their result applies.

## Core Allocation Rules

In this section, we show that $c^*_{is}(\theta;\theta_k,c_k)$, the solution to the problem defined by (11)-(19) and (20), is the unique core allocation rule for this economy. It is useful to begin by stating a preliminary result.

Lemma 2. Consider an incumbent coalition solving the problem (11)-(19) (and (20)) in the presence of another coalition k with associated vector of measures $\theta_k$ and allocation vector $c_k$. Suppose $\theta_q(=\mu-\theta_{kq}) = 0$. Then

$$(25) \qquad p_{q-1}U[c^*_{q-1,1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_{q-1})U[c^*_{q-1,2}(\mu-\theta_k,\theta_k,c_k)]$$

$$= \overline{U}_{q-1}(\mu-\theta_k;\theta_k,c_k)$$

if $\theta_{q-1}$ $(=\mu-\theta_{k,q-1}) > 0$, with an obvious notation.

Proof. Clearly (abbreviating notation)

$$p_{q-1}U(c^*_{q-1,1}) + (1-p_{q-1})U(c^*_{q-1,2}) \geq \overline{U}_{q-1}(\mu-\theta_k;\theta_k,c_k)$$

from (13). Further, it is easy to show that the incentive constraints (12) associated with type i hold with equality only for j = i + 1; i = 1, ..., n - 1. Therefore, since the incumbent coalition takes $c_k$ as given in solving the problem (11)-(19) and (20), setting $c_{q-1,s}$ such that

$$p_{q-1}U(c^*_{q-1,1}) + (1-p_{q-1})U(c^*_{q-1,2}) > \overline{U}_{q-1}(\mu-\theta_k;\theta_k,c_k)$$

does not relax any constraints in the problem, and consumes resources. Hence (25) holds.

There is also a corollary to lemma 2. This is that, as is well known[8/]

$$(26) \qquad \sum_{i=1}^{q-1} \theta_i \{p_i [c^*_{i1}(\theta;\theta_k,c_k)-e_1]+(1-p_i)[c^*_{i2}(\theta;\theta_k,c_k)-e_2]\} \leq 0.$$

We are now prepared to state the first of our principal results.

Proposition 1. The allocation rule $c^*_{is}(\theta;\theta_k,c_k)$ solving (11)-(19) and (20) is a core allocation rule.

The proof of this for $n > 2$ makes use of assumption (3.d) for the values $\theta_{ki}$, i.e., a blocking coalition must attract all or none of the agents of each type. Therefore, we begin by proving the proposition for $n = 2$, in which case we can employ the weaker assumption (3.c) on the values $\theta_{ki}$.

Proof of Proposition 1 (n = 2). For purposes of the proof, we assume the existence of a coalition k which blocks $c^*_{is}$, and then show that the assumed existence of such a coalition leads to a contradiction. The blocking coalition announces the allocation $\hat{c}_k$ and has associated vector of measures $\hat{\theta}_k$. There are then four possible configurations of $\hat{\theta}_k$ to consider.

Case 1. $(\hat{\theta}_{k1},\hat{\theta}_{k2}) = (\hat{\theta}_{k1},0)$; $\hat{\theta}_{k1}, > 0$. Thus the blocking coalition consists only of type 1 agents. But then clearly

$$(27) \qquad p_1U(\hat{c}_{k11}) + (1-p_1)U(\hat{c}_{k12}) \leq \overline{U}_1,$$

with $\overline{U}_1$ defined by (15). However,

$$(28) \qquad p_1U[c^*_{11}(\mu)] + (1-p_1)U[c^*_{12}(\mu)] \geq \overline{U}_1$$

by (13.a). Together (27) and (28) contradict (10), so that a blocking coalition cannot have $\hat{\theta}_{k1} > 0$, $\hat{\theta}_{k2} = 0$.

Case 2. $\mu_2 = \hat{\theta}_{k2}$, $0 \leq \hat{\theta}_{k1} < \mu_1$. Then the blocking coalition consists of positive measures of agents of each type, but the incumbent coalition retains some type 1 agents. Then clearly

$$(29) \qquad p_1 U[c^*_{11}(\mu-\theta_k;\theta_k,c_k)] + (1-p_1)U[c^*_{12}(\mu-\theta_k;\theta_k,c_k)] = \overline{U}_1.$$

Therefore, since coalition membership is voluntary,

$$(30) \qquad p_1 U(\hat{c}_{k11}) + (1-p_1)U(\hat{c}_{k12}) = \overline{U}_1.$$

But then

$$(31) \qquad p_2 U(\hat{c}_{k21}) + (1-p_2)U(\hat{c}_{k22}) \leq \overline{\overline{U}}_2,$$

where $\overline{\overline{U}}_{22}$ is the solution to the following problem:

$$\max\ p_2 U(c_{21}) + (1-p_2)U(c_{22})$$

subject to

$$(32) \qquad p_i U(c_{i1}) + (1-p_i)U(c_{i2}) \geq p_i U(c_{j1}) + (1-p_i)U(c_{j2});\ i,\ j = 1,\ 2.$$

$$(33) \qquad p_1 U(c_{11}) + (1-p_i)U(c_{12}) = \overline{U}_1$$

$$(34) \qquad \mu_2\{p_2(c_{12}-e_1)+(1-p_2)(c_{22}-e_2)\} + \theta_{k1}\{p_1(c_{11}-e_1) + (1-p_1)(c_{12}-e_2)\} \leq 0.$$

Since (33) implies that

$$p_1(c_{11}-e_1) + (1-p_1)(c_{12}-e_2) = 0$$

[Spence (1978)], clearly

$$(35) \qquad p_2U[c^*_{21}(\mu)] + (1-p_2)U[c^*_{22}(\mu)] \geq \bar{\bar{U}}_2,$$

as the problem defining $\bar{\bar{U}}_2$ is more heavily constrained than the problem (11)-(19). But (35) and (31), along with (30) and (13.a) contradict (9) and (10). Hence a blocking coalition cannot take this form.

Case 3. $(\theta_{k1}, \theta_{k2}) = (\mu_1, \mu_2)$, so the blocking coalition is the grand coalition. Now by the definition of $c^*_{is}(-)$,

$$(36) \qquad p_2U(\hat{c}_{k21}) + (1-p_2)U(\hat{c}_{k22}) \leq p_2U[c^*_{21}(\mu)] + (1-p_2)U[c^*_{22}(\mu)].$$

Therefore, by (9), (36) holds with equality. Thus, by condition (10),

$$(37) \qquad p_1U(\hat{c}_{k11}) + (1-p_1)U(\hat{c}_{k12}) > p_1U[c^*_{11}(\mu)] + (1-p_1)U[c^*_{12}(\mu)].$$

Together, (36) and (37) imply that the values $\hat{c}_{kis}$ also solve the problem (11)-(19), and do so in a way which yields higher expected utility to type 1 agents. But this contradicts the definition of $c^*_{is}(\mu)$. Hence this also cannot constitute a blocking coalition.

Case 4. $\mu_2 > \theta_{k2} > 0$, $\theta_{k1} \in [0,\mu_1]$. Now since $\theta_{k2}(\mu-\theta_{k2}) > 0$,

$$(38) \qquad p_2U(\hat{c}_{k21}) + (1-p_2)U(\hat{c}_{k22}) = p_2U[c^*_{21}(\mu-\theta_k;\theta_k,c_k)]$$

$$+ (1-p_2)U[c^*_{22}(\mu-\theta_k;\theta_k,c_k)] \geq p_2U[c^*_{21}(\mu)] + (1-p_2)U[c^*_{22}(\mu)],$$

where the first equality holds since coalition membership must be voluntary, and the second inequality holds by (9). Also, if the inequality in (38) is not strict,

$$(39) \qquad p_1U(\hat{c}_{k11}) + (1-p_1)U(\hat{c}_{k12}) > p_1U[c^*_{11}(\mu)] + (1-p_1)U[c^*_{12}(\mu)]$$

by (10). Then consider a consumption lottery which allocates type i agents in state s the consumption $\hat{c}_{kis}$ with probability $\theta_{ki}/\mu_i$, and $c^*_{is}(\mu-\theta;\theta_k,c_k)$ with probability $1 - (\theta_{ki}/\mu_i)$. This lottery is feasible (since each allocation is individually), and by (39) and (13.a) it yields type 1 agents expected utility at least $\overline{U}$. Also, from (38) (and (39) if the inequality in (38) is not strict), type 2 agents are at least as well off under this lottery as they are receiving $c^*_{2s}(\mu)$ with certainty in state s (and if they are not strictly better off, then type 1 agents are). Since $c^*_{2s}(\mu)$ is alleged to solve (11)-(19), this contradicts lemma 1. Hence no such lottery exists, and there is no blocking coalition with $\mu_2 > \theta_{k2} > 0$, $\theta_{k1} \in [0,\mu_1]$.

Thus all possible configurations of a blocking coalition have been ruled out. Therefore no blocking coalition exists, and $c^*_{is}(-)$ is a core allocation rule as claimed.

We now prove proposition 1 for $n > 2$. For this proof we require additional structure on the values $\theta_{ki}$. Hence we now employ assumption (3.d): a blocking coalition either has $\theta_{ki} = 0$ or $\theta_{ki} = \mu_i \; \forall \; i$.

Proof of Proposition 1 ($n \geq 2$). As in the proof for $n = 2$, we assume the existence of a blocking coalition announcing an allocation vector $\hat{c}_k$ and derive a contradiction. Again there are several possible configurations of $\hat{\theta}_k$ for a potential blocking coalition, which we consider case by case.

Case 1. $\theta_{ki} > 0 \; \forall \; i$ such that $j \leq i \leq \ell$, $\theta_{ki} = 0$, otherwise. Then, by (3.d), $\theta_{ki} = \mu_i \; \forall \; i$; $j \leq i \leq \ell$, i.e., the blocking coalition attracts all agents of types j through $\ell$.

(a) Suppose $j = 1$, $\ell = n$. Then the grand coalition constitutes a blocking coalition. Therefore, by (9)

(40) $$p_n U(\hat{c}_{kn1}) + (1-p_n)U(\hat{c}_{kn2}) \geq p_n U[c^*_{n1}(\mu)] + (1-p_n)U[c^*_{n2}(\mu)].$$

If (40) holds with equality, then

(41) $$p_{n-1}\, U(\hat{c}_{k,n-1,1}) + (1-p_{n-1})U(\hat{c}_{k,n-1,2}) \geq p_{n-1}U[c^*_{n-1,1}(\mu)] + (1-p_{n-1})U[c^*_{n-1,2}(\mu)],$$

etc. Of course for some type i

(42) $$p_i U(\hat{c}_{ki1}) + (1-p_i)U(\hat{c}_{ki2}) > p_i U[c^*_{i1}(\mu)] + (1-p_i)U[c^*_{i2}(\mu)]$$

where if $i \neq n$,

(43) $$p_{i+s}U(\hat{c}_{k,i+s,1}) + (1-p_{i+s})U(\hat{c}_{k,i+s,2}) \geq p_{i+s}U[c^*_{i+s,1}(\mu)] + (1-p_{i+s})U[c^*_{i+s,2}(\mu)]$$

$\forall$ $s = 1, \ldots, n - i$. However, (42) and (43) contradict the definition of $c^*_{is}(\mu)$, so that a blocking coalition cannot be of this form.

(b) Then suppose $\ell = n$, $j > 1$. Since $\theta_{ki} = \mu_i \;\forall\; i \geq j$, all types with index $q < j$ receive an allocation such that

(44) $$p_{j-1}U[c^*_{j-1,1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_{j-1})U[c^*_{j-1,2}(\mu-\theta_k;\theta_k,c_k)] = \overline{U}_{j-1}(\mu-\theta_k),$$

with $\overline{U}_{j-1}(\mu-\theta_k)$ defined by (15)-(19). Then define $\tilde{c}_{is}$; $i = j, \ldots, n$; $s = 1, 2$, to be the allocation solving

$$\max\ p_n U(c_{n1}) + (1-p_n)U(c_{n2})$$

subject to

$$p_iU(c_{i1}) + (1-p_i)U(c_{i2}) \geq p_iU(c_{j1}) + (1-p_j)U(c_{j2}) \;\forall\; i,\, j,$$

$$p_iU(c_{i1}) + (1-p_i)U(c_{i2}) \geq \overline{U}_i(\mu) \;\forall\; i = j, \ldots, n-1$$

$$p_{j-1}U(c_{j-1,1}) + (1-p_{j-1})U(c_{j-1,2}) = \overline{U}_{j-1}(\mu)$$

$$\sum_{i=1}^{n} \mu_i \{p_i(c_{i1}-e_1)+(1-p_i)(c_{i2}-e_2)\} \leq 0.$$

Clearly

$$(45) \qquad p_nU(\tilde{c}_{n1}) + (1-p_n)U(\tilde{c}_{n2}) \leq p_nU[c^*_{n1}(\mu)] + (1-p_n)U[c^*_{n2}(\mu)]$$

since the problem above is more heavily constrained than (11)-(19). If (45) holds with equality, then

$$(46) \qquad p_{n-1}U(\tilde{c}_{n-1,1}) + (1-p_{n-1})U(\tilde{c}_{n-1,2}) \leq p_{n-1}U[c^*_{n-1,1}(\mu)] + (1-p_{n-1})U[c^*_{n-1,2}(\mu)],$$

etc. Therefore, such a blocking coalition with $n = \ell$ is impossible.

(c) Then $\theta_{ki} = \mu_i$; $1 = j \leq i \leq \ell < n$, and $\theta_{ki} = 0$ otherwise. By (10), then,

$$(47) \qquad p_\ell U(\hat{c}_{k\ell 1}) + (1-p_\ell)U(\hat{c}_{k\ell 2}) \geq p_\ell U[c^*_{\ell 1}(\mu)] + (1-p_\ell)U[c^*_{\ell 2}(\mu)],$$

and if (47) holds with equality, there is an analogous expression for type $\ell - 1$, etc. Moreover, obviously

$$(48) \qquad \sum_{i=1}^{\ell} (\mu_i-\theta_{ki})[(c_{i1}-e_1)p_i+(c_{i2}-e_2)(1-p_i)] = 0.$$

Therefore,

$$(49) \qquad p_nU[c^*_{n1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_n)U[c^*_{n2}(\mu-\theta_k;\theta_k,c_k)] \geq p_nU[c^*_{n1}(\mu)] + (1-p_n)U[c^*_{n2}(\mu)]$$

since by (47) and (48) an incentive constraint in the problem (11)-(19) (and (20)) is (weakly) relaxed with no expenditure of resources. Also, if (49) holds with equality, there is an analogous expression for type $n - 1$ (if $n - 1 > \ell$), etc. Finally, in the initial problem (11)-(19) with $\theta = \mu$ it was feasible to assign types $i = \ell + 1, \ldots, n$ the allocation $c^*_{is}(\mu-\theta_k;\theta_k,c_k)$, and types $i = 1, \ldots, \ell$ the allocation $\hat{c}_{kis}$ in state s, since these are feasible allocations individually, and since it is easy to show that the allocations $\hat{c}_{kis}$ satisfy (13). Therefore, the existence of the posited allocations, along with (47) and (48), contradict the definition of $c^*_{is}(\mu)$, so this case is also impossible.

(d) Then $\theta_{ki} = \mu_i$; $1 < j \leq i \leq \ell < n$, and $\theta_{ki} = 0$ otherwise. Now by lemma 2

$$p_{j-1}U[c^*_{j-1,1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_{j-1})U[c^*_{j-1,2}(\mu-\theta_k;\theta_k,c_k)] = \overline{U}_{j-1}(\mu),$$

and therefore, by the corollary to lemma 2,

$$\sum_{i=1}^{\ell} (\mu_i-\theta_{ki})\{p_i(c_{i1}-e_1)+(1-p_i)(c_{i2}-e_2)\} \leq 0.$$

Also, by condition (10), (47) holds. Therefore, for the same reason as in case (1.c) above, (49) holds, and its analog for type $n - 1$ holds if (49) is an equality (and if $n - 1 > \ell$). Thus we can construct the same contradiction as in case (c). Having exhausted all possibilities under case 1, then, we have shown that this case is impossible.

<u>Case 2</u>. There exist indices i, j, and $\ell$ satisfying $j < i < \ell$ such that $\theta_{kj}\theta_{k\ell} > 0$ and $\theta_{ki} = 0$. Also, without loss of generality, let $\ell$ be the largest index with $0 < \theta_{k\ell}$ $(=\mu_\ell)$. Finally, let q be the largest index such that $\theta_{ki} = 0$ and $\theta_{k,i+1} > 0$.

(a) Let $\ell = n$. Then by lemma 2

$$(50) \qquad p_n U(\hat{c}_{n1}) + (1-p_n)U(\hat{c}_{n2}) \leq p_n U(\tilde{c}_{n1}) + (1-p_n)U(\tilde{c}_{n2})$$

where $\tilde{c}_{is}$ solves the problem

$$\max \ p_n U(c_{n1}) + (1-p_n)U(c_{n2})$$

subject to

$$p_i U(c_{i1}) + (1-p_i)U(c_{i2}) \geq p_i U(c_{j1}) + (1-p_i)U(c_{j2}) \ \forall \ i, j$$

$$p_i U(c_{i1}) + (1-p_i)U(c_{i2}) \geq \overline{U}_i(\mu); \ i = 1, \ldots, n-1; \ i \neq q$$

$$p_q U(c_{q1}) + (1-p_q)U(c_{q2}) = \overline{U}_q(\mu)$$

$$\sum_{i=1}^{n} \mu_i [p_i(c_{i1}-e_1)+(1-p_i)(c_{i2}-e_2)] \leq 0.$$

However,

$$(51) \qquad p_n U(\tilde{c}_{n1}) + (1-p_n)U(\tilde{c}_{n2}) \leq p_n U[c^*_{n1}(\mu)] + (1-p_n)U[c^*_{n2}(\mu)]$$

since the problem defining $\tilde{c}_{is}$ is more heavily constrained than (11)-(19). If either (50) or (51) is a strict inequality, then the above contradicts (9). If (50) and (51) both hold with equality, then similarly

$$p_{n-1}U(\hat{c}_{n-1,1}) + (1-p_{n-1})U(\hat{c}_{n-1,2}) \leq p_{n-1}U(\tilde{c}_{n-1,1}) + (1-p_{n-1})U(\tilde{c}_{n-1,2})$$

and

$$p_{n-1}U(\tilde{c}_{n-1,1}) + (1-p_{n-1})U(\tilde{c}_{n-1,2}) \leq p_{n-1}U[c^*_{n-1,1}(\mu)] + (1-p_{n-1})U[c^*_{n-1,2}(\mu)],$$

etc. Thus, if (50) and (51) hold with equality (10) is contradicted. Therefore, this case is impossible.

(b) Then $\ell < n$. By lemma 2

$$p_q U[c^*_{q1}(\mu-\theta_k;\theta_k,c_k)] + (1-p_q)U[c^*_{q2}(\mu-\theta_k;\theta_k,c_k)] = \overline{U}_q(\mu-\theta_k), \tag{52}$$

with $\overline{U}(\mu-\theta_k)$ defined by (15)-(19) and (20). Therefore, by (3.d), (13), lemma 2, and the fact that (52) holds with equality,

$$p_\ell U(\hat{c}_{\ell 1}) + (1-p_\ell)U(\hat{c}_{\ell 2}) \leqslant p_\ell U[c^*_{\ell 1}(\mu)] + (1-p_\ell)\, U[c^*_{\ell 2}(\mu)]. \tag{53}$$

Again, if (53) holds with equality, an analogous expression holds for $\ell - 1$ (if $\ell - 1 > q$, or holds for the largest index r less than $\ell$ such that $\theta_{kr} > 0$), etc. Thus either (53) (with strict inequality) contradicts (10), or an analogous expression for a lower index contradicts (10). Therefore, this case is also impossible, so there is no blocking coalition of the type posited.

Case 3. Then there exists just one index $\ell$ with $\theta_{k\ell}$ $(=\mu_\ell) > 0$. But then (invoking lemma 2 if $\ell \neq n$ or $\ell \neq 1$), clearly

$$p_\ell U(\hat{c}_{\ell 1}) + (1-p_\ell)U(\hat{c}_{\ell 2}) \leqslant p_\ell U[c^*_{\ell 1}(\mu)] + (1-p_\ell)U[c^*_{\ell 2}(\mu)].$$

As this contradicts (10) (or (9) if $\ell = n$), there is no blocking coalition with $\theta_{ki} = \mu_i$; $i = \ell$, $\theta_{ki} = 0$ otherwise.

Thus we have ruled out all possible configurations of a blocking coalition, proving the proposition.

Having shown that the allocation rule $c^*_{is}(\theta;\theta_k,c_k)$ is a core allocation rule, we now turn to proving that this is the unique core allocation rule in this setting.

Proposition 2. $c^*_{is}(\theta;\theta_k,c_k)$ is the unique core allocation rule.

As we have shown that $c^*_{is}(-)$ is a core allocation rule, we now show by construction that any other allocation rule is blocked by the grand coalition.

<u>Proof of Proposition 2</u>. Consider any allocation rule $\tilde{c}_{is}(\theta;\theta_k,c_k)$ announced by the incumbent coalition such that $\tilde{c}_{is}(\mu;0,0) \neq c^*_{is}(\mu;0,0)(= c^*_{is}(\mu))$ for some i for some s. We then construct a blocking coalition k with associated vector of measures $\theta_k = \mu$ as follows. The blocking coalition announces allocations $\hat{c}_{kis} = c^*_{is}(\mu)$. Then we show that $\theta_{ki} = \mu_i \; \forall \; i$, and that all conditions for k to be a blocking coalition are satisfied. There are two cases to consider:

<u>Case 1.</u> $\tilde{c}_{is}(\mu)$ does not maximize $p_nU(c_{n1}) + (1-p_n)U(c_{n2})$ subject to (12)-(14). Then obviously

$$p_nU[\tilde{c}_{n1}(\mu)] + (1-p_n)U[\tilde{c}_{n2}(\mu)] < p_nU(\hat{c}_{kn1}) + (1-p_n)U(\hat{c}_{kn2}).$$

Therefore, (9) is satisfied, and $\theta_{kn} = \mu_n$. Moreover, clearly if $\theta = (\mu_1, \ldots, \mu_{n-1}, 0)$, then

$$p_{n-1}U[\tilde{c}_{n-1,1}(\theta;\theta_k,\hat{c}_k)] + (1-p_{n-1})U[\tilde{c}_{n-1,2}(\theta;\theta_k,\hat{c}_k)] \leq \overline{U}_{n-1}(\mu)$$
$$\leq p_{n-1}U(\hat{c}_{k,n-1,1}) + (1-p_{n-1})U(\hat{c}_{k,n-1,2}).$$

Therefore, $\theta_{k,n-1} = \mu_{n-1}$. Repeating this argument we derive $\theta_{ki} = \mu_i$ for all i. Finally, $\hat{c}_{kis}$ is feasible by construction, so that all the conditions required for k to be a blocking coalition (and for $\hat{c}_{kis}$ to be a blocking allocation) are satisfied. Hence if $\tilde{c}_{is}(\mu)$ does not maximize $p_nU(c_{n1}) + (1-p_n)U(c_{n2})$ subject to (12)-(14) it is blocked, and, therefore, is not a core allocation.

Case 2. Suppose that the problem of maximizing $p_nU(c_{n1}) + (1-p_n)U(c_{n2})$ subject to (11)-(14) has more than one solution and that $\tilde{c}_{is}(\mu)$ is one such solution, but not the one giving greatest expected utility to type n - 1 agents (or type n - 2 agents, etc.). Then obviously the grand coalition blocks $\tilde{c}_{is}(-)$.

Thus any allocation rule $\tilde{c}_{is}(\mu) \neq c^*_{is}(\mu)$ for some i, for some s is blocked, establishing the proposition.

The logic of proposition 2 is straightforward. Under any feasible allocation rule other than $c^*_{is}(\theta;\theta_k,c_k)$ which is Pareto optimal $\forall$ $(\theta,\theta_k) \in \Delta^n \times \Delta^n$ (and hence a candidate for a core allocation rule), type n agents subsidize agents of other types. Hence, type n agents can always form a blocking coalition by defecting, and offering agents of other types an allocation weakly preferred by them to any allocation they could attain on their own. Since this is at least as good as what these agents can attain in the absence of type n agents, they are in essence "forced" to join the blocking coalition. This intuition also suggests why $c^*_{is}(\mu)$ is an unblocked allocation rule.

A Two Stage, Noncooperative Game

We now turn our attention to the description of a game which has a Nash equilibrium which produces the same allocation as does our cooperative equilibrium. In addition to the set of agents described in the previous section, then, let us introduce a set of insurance firms $F = \{1, \ldots, K\}$. We further partition the set of firms into a group of incumbent firms and a group of potential entrants. The interpretation of this partition is as follows. All potential insurance customers are initially assigned (in their population proportions) to the incumbent firms, i.e., customers are initially divided evenly among the set of incumbent firms. There is also a set of potential entrants who may attempt to attract customers from the incumbent firms. This division of firms serves two purposes. One is purely technical and can best be discussed after an exposition of the game. The other is to render the game similar in form to the cooperative game discussed previously. In particular, the position of incumbent firms is symmetric with that of the incumbent (grand) coalition in our previous discussion. Finally, without loss of generality, we may take there to be a single incumbent firm (say firm 1).

As an overview, our game evolves as follows. At stage one the incumbent firm announces an allocation rule analogous to that discussed previously. This rule specifies the state contingent allocations received by each customer of the incumbent as a function of (potentially) three things. One is the population experiences of the firm's customers (which depends solely on the measure of agents of each type purchasing policies from the incumbent). This is in keeping with our earlier discussion of mutual insurance companies. Second, this allocation rule specifies how the allocations received by the incumbent's customers depend on (i) the number of agents of each type who purchase policies from new entrants (who defect), and (ii) the allocations obtained by these defectors elsewhere. This is, of course, analogous to our earlier discussion.

After the incumbent announces an allocation rule, potential entrants decide which types of agents they wish to attract in positive measure, the measure they wish to attract of each type, and the state contingent allocation to be received by each agent purchasing a policy from them. Then the game ends.

Let $f \in F$ index firms. Then for $f \neq 1$, the decision variables of the firm are as follows. First, let $\theta_f$ denote the vector of measures of each type of agent "attracted by" firm f, i.e., $\theta_f \equiv (\theta_{f1}, \ldots, \theta_{fn}) \in \Delta^n$. Also, the firm must select, for all i such that $\theta_{fi} > 0$, an allocation $c_{fis}$. Let $c_{Iis}$ denote the allocation received by customers of the incumbent firm. Then there are several conditions which the choices $\theta_f$ and $c_{fis}$ must satisfy. One is that the choices $c_{fis}$ must be resource feasible for firm f, i.e., must satisfy

$$(54) \qquad \sum_{i=1}^{n} \theta_{fi}\,[p_i(c_{fi1}-e_1)+(1-p_i)(c_{fi2}-e_2)] \leq 0.$$

In addition, they must be incentive feasible within firm f, i.e., $\forall$ i, j such that $\theta_{fi}\theta_{fj} > 0$.

$$(55) \qquad p_iU(c_{fi1}) + (1-p_i)U(c_{fi2}) \geq p_iU(c_{fj1}) + (1-p_i)U(c_{fj2}).$$

An allocation is feasible if it is resource and incentive feasible.

Moreover, in order for $\theta_f$ and $c_f = [(c_{f11},c_{f12}),\ldots,(c_{fn1},c_{fn2})]$ to be consistent, we require that

$$(56) \qquad p_iU(c_{fi1}) + (1-p_i)U(c_{fi2}) \geq p_iU(c_{gi1}) + (1-p_i)U(c_{gi2}) \quad \forall\, g \in F$$

if $\theta_{fi} > 0$, and

$$(57) \qquad p_iU(c_{fi1}) + (1-p_i)U(c_{fi2}) \leq p_iU(c_{gi1}) + (1-p_i)U(c_{gi2})$$

for some $g \in F$ if $\theta_{fi} = 0$. More specifically, if firm f desires to set $\theta_{fi} = 0$, it must announce values $c_{fis}$ satisfying (57). Finally, if firm f wishes to set $\theta_{fi} > 0$ and $\theta_{fj} = 0$, it must select values $c_{fis}$ such that type j agents do not wish to obtain the allocations meant for type i agents at firm f. Therefore, firm f's allocations must satisfy

$$(58) \qquad p_jU(c_{fi1}) + (1-p_j)U(c_{fi2}) \leq p_jU(c_{gj1}) + (1-p_j)U(c_{gj2})$$

$\forall\, g \in F$, for all j such that $\theta_{fj} = 0$, for all i such that $\theta_{fi} > 0$.

As in the cooperative case, there is a great deal of ambiguity regarding values $\theta_{fi}$ in the case when type i agents might be indifferent between the allocation they can obtain from the incumbent firm, and from firm $f \neq 1$. We alternately employ the same assumptions as above:

(59.a) $\quad \theta_{fi} \in [0, \mu_i]$

if (56) holds as an equality for f = 1, or

(59.b) $\quad \theta_{fi} \in \{0, \mu_i\}$

if (56) holds as an equality for f = 1.

At stage 2 of our game, then, firms with $f \neq 1$ make choices of feasible allocations, along with consistent values of $\theta_f$, so that the strategy spaces of these firms are subsets of $\Delta^n \times R^{2n}$. At stage 1 of the game, the incumbent firm announces an allocation rule which specifies $c_{1is}$ as a function of $\theta_1, \theta_{\tilde{F}} \equiv (\theta_2, \ldots, \theta_\mu)$, and of $c_{\tilde{F}} \equiv (c_2, \ldots, c_m)$. We say that an allocation rule is feasible if it specifies a feasible set of allocations $\forall\ \theta_1$, $\forall\ \theta_{\tilde{F}} \in \Delta^{(K-1)n}\ \forall\ \tilde{c}_F \in R^{2(K-1)n}$ such that $c_f$ is feasible; $f \neq 1$. In addition to requiring that an announced allocation rule be feasible, we require that any announced allocation rule satisfy an additional condition. We begin with a definition.

<u>Definition</u>. An allocation $c_{is}$ is $\theta$-Pareto optimal if it is feasible, and if given $\theta$ there does not exist another feasible allocation $\tilde{c}_{is}$ such that

(60) $$p_i U(\tilde{c}_{i1}) + (1-p_i)U(\tilde{c}_{i2}) \geq p_i U(c_{i1}) + (1-p_i)U(c_{i2})$$

$\forall$ i such that $\theta_i > 0$, and if

(61) $$\sum_{i=1}^{n} \theta_i [p_i(\tilde{c}_{i1} - c_{i1}) + (1-p_i)(\tilde{c}_{i2} - c_{i2})] \leq 0,$$

with either (60) (for some i) or (61) holding with strict inequality.

In short, an allocation is θ-Pareto optimal here if, for a fixed group of agents, there is no feasible, Pareto noninferior allocation which consumes no more resources. As we will see momentarily, (61) simply says that $\tilde{c}_{is}$ cannot be (strictly) more profitable for the relevant firm than $c_{is}$ for a given (fixed) set of customers. Hence an allocation is θ-Pareto optimal if there is no feasible Pareto superior allocation which consumes no more resources, or if there is no feasible allocation satisfying (60) which earns greater profits for the firm. Then we require that any allocation announced by the incumbent be $\theta_1$-Pareto optimal $\forall\ \theta_1 \in \Delta^n$, $\forall\ \theta_{\tilde{F}}$, $\forall\ c_{\tilde{F}}$. This restriction prevents the incumbent from threatening to offer its customers allocations in certain contingencies which do not (a) maximize its profits given the set of customers summarized in $\theta_1$, and (b) for given profits are Pareto dominated for the set of customers specified by θ. Hence the requirement that announced allocation rules must specify $\theta_1$-Pareto optimal allocations merely rules out "threats" made by the incumbent which it would not wish to carry out should the relevant contingency arise. The strategy space of the incumbent, then, is the set of maps of the form $c_{1is}(\theta_1;\theta_{\tilde{F}},\tilde{c}_F)$ which are feasible and specify $\theta_1$-Pareto optimal allocations. Let C* denote the space of admissible allocation rules.

Having said all this, we may now specify firm profits. Let $\pi_f$ denote the profits of firm f. Then

$$(62)\qquad \pi_f \equiv \pi(\theta_f,c_f;\theta_{-f},c_{-f}) \equiv \sum_{i=1}^{n} \theta_{fi}\,[p_i(e_1-c_{fi1})+(1-p_i)(e_2-c_{fi2})],$$

where $\theta_{-f} \equiv (\theta_1, \ldots, \theta_{f-1}, \theta_{f+1}, \ldots, \theta_M)$, etc.

It should be clear from our discussion (and our terminology) that the incumbent firm will wish to deter entry (i.e., to prevent $\theta_{fi} > 0$; $f \neq 1$). This follows from the fact that if an entrant can earn a profit the incumbent should have been able to choose an allocation rule which prevented entry, and which increased the incumbent's profits (since obviously the attraction of some type i such that $\theta_{fi} > 0$, $f \neq 1$, is profitable). Hence we couch our definition of a Nash equilibrium in these terms. In particular,

Definition. A Nash equilibrium is an announced allocation rule $\tilde{c} \in C^*$ such that

(i) given $c_1 = [(c_{111}, c_{112}), \ldots, (c_{1n1}, c_{1n2})]$,

$$\pi_f[\theta_f, c_f; \mu-\theta_f, c_1(\mu-\theta_f; \theta_f, c_f)] \leq 0 \tag{63}$$

for all feasible $c_f$ and consistent values of $\theta_f$.

(ii) for all admissible allocation rules $c \in C^*$ that satisfy (63),

$$\pi_1(\mu, \tilde{c}; 0,0) \geq \pi_1(\pi, \hat{c}; 0,0). \tag{64}$$

The notation in (63) simply means that given that all firms other than the incumbent and f itself have chosen not to enter the activity of selling insurance, f must also choose not to enter this activity. Hence an equilibrium allocation rule is the most profitable (admissible) rule which eliminates rents for any potential entrant.

Existence of a Nash Equilibrium

Consider the allocation rule $c^*_{1is}(\theta_1; \theta_f, c_f)$ solving (11)-(19) (and (20)), where now $\theta_1$ is the vector $(\theta_{11}, \ldots, \theta_{1n})$, and $\theta_f$ and $c_f$ are interpreted as the vector of measures $(\theta_{f1}, \ldots, \theta_{fn})$ and $c_f = [(c_{f11}, c_{f12}), \ldots, (c_{fn1}, c_{fn2})]$ of any potential (single) entrant. It suffices to focus on this since the incumbent need only deter entry of each firm $f = 2, \ldots, K$ singly, i.e., when all firms other than f have not entered the insurance market. Then we have the following result.

Proposition 3. The announcement of the allocation rule $c^*_{1is}$ constitutes a Nash equilibrium of our two stage game.

Proof. Obviously $c^* \varepsilon C^*$. Therefore the proof may proceed in two steps. First we show that the choice $c^*_1$ deters entry. Then we show that any other choice of allocation rule which results in at least the same level of profits as $c^*_1$ fails to deter entry. Hence $c^*$ satisfies conditions (i) and (ii) of our equilibrium definition.

(i) It is a straightforward implication of proposition 1 that $c^*_1$ deters entry. To see this, suppose some potential entrant $f \neq 1$ could earn nonzero profits. Then there would exist some set of indices i with $\theta_{fi} > 0$. Let $q = \max\{i:\theta_{fi} > 0\}$. Then

$$p_q U(c_{fq1}) + (1-p_q)U(c_{fq2}) \geq p_q U[c^*_{1q1}(\mu)] + (1-p_q)U[c^*_{1q2}(\mu)],$$

condition (56) holds for all other i such that $\theta_{fi} > 0$, and since (63) fails,

$$\sum_{i=1}^{n} \theta_{fi} [p_i(e_1-c_{fi1})+(1-p_i)(e_2-c_{fi2})] > 0.$$

But all of the above implies that there exists a blocking coalition for $c^*_{is}$ in the cooperative game. This contradicts proposition 1 so there exists no such potential entrant.

(ii) Then suppose there existed an allocation rule $\hat{c}_1 \varepsilon C^*$ with $\hat{c}_1(\mu;0,0) \neq c^*_1(\mu;0,0)$ such that (64) failed. We may consider two cases.

Case 1. $\hat{c}_1(\mu;\ 0,\ 0)$ does not solve (11)-(19). We now show that entry is not deterred. In particular, let firm $f \neq 1$ offer the allocations

$$c_{fis} = c^*_{is}(\mu);\ i \neq n$$

$$c_{fns} = c^*_{ns}(\mu) - \varepsilon$$

with $\varepsilon > 0$ satisfying

$$p_n U[c^*_{ni}(\mu)-\varepsilon] + (1-p_n)U[c^*_{n2}(\mu)-\varepsilon] > p_n U[\hat{c}_{n1}(\mu)] + (1-p_n)U[\hat{c}_{n2}(\mu)]$$

which is possible since $\hat{c}(\mu)$ does not solve (11)-(19). Then all agents purchase policies from firm f and firm f earns profits $\mu_n\varepsilon > 0$. Hence entry is not deterred.

Case 2. $\hat{c}_1(\mu;0,0)$ solves (11)-(19), but is not the solution giving highest utility to type n - 1 agents, etc. Then let firm f ≠ 1 offer the allocations

$$c_{fis} = c^*_{is}(\mu);\ i \neq n-1$$

$$c_{f,n-1,s} = c^*_{n-1,s}(\mu) - \varepsilon,$$

with $\varepsilon > 0$ satisfying

$$p_{n-1}U[c^*_{n-1,1}(\mu)-\varepsilon] + (1-p_{n-1})U[c^*_{n-1,2}(\mu)-\varepsilon] > p_{n-1}\ U[\hat{c}_{1,n-1,1}(\mu)]$$
$$+ (1-p_{n-1})U[\hat{c}_{1,n-1,2}(\mu)],$$

which is possible by hypothesis. But then all agents are willing to purchase policies from firm f, and thus f earn profits $\varepsilon\mu_{n-1} > 0$. Hence entry is not deterred in this case either.

We have shown, then, that the allocation rule $c^*_{1is}$ is a Nash equilibrium allocation rule, as it satisfies (i) and (ii) in the definition. Moreover, from the argument above, it is the only allocation rule the incumbent can announce which deters entry. Hence we have

Proposition 4. All equilibrium allocation rules $\tilde{c}_1(-)$ have $\tilde{c}_1(\mu;0,0) = c_1^*(\mu,0,0)$.

Finally, we should note that, since we have used proposition 1 in our argument, for n = 2 we can use the weak assumption (59.a) on the values $\theta_{fi}$ for the proof above, while if n > 2 the proof requires (59.b).

It remains to discuss two issues. One is the intuition behind proposition 3, and the other is the asymmetry between the incumbent firm (which specifies an allocation rule as its strategy) and potential entrants (whose strategies may be viewed as allocations). With regard to the intuition of proposition 3, we might simply note that a standard method of producing existence in this setting is to use a reactive equilibrium concept of the form used by Wilson (1977) (or its adaptation by Miyazaki (1977)). Our approach augments the strategy space of the incumbent firm to permit it to announce how it will react to any potential form of entry (and consequent encroachment on its customers). Potential entrants do not need to conjecture how incumbent firms will respond to their actions, as they do in the Wilson setting, however, since the form that the incumbents' reactions will take is specified in advance. What our results show is that it is not necessary to have firms "drop" policies which become unprofitable after entry. Rather, it is sufficient to have them reoptimize in the manner specified by our rule.

With regard to why the strategy space of the incumbent firm is specified differently from that of potential entrants, this is done for the following technical reason. We could have all firms announce allocation rules, which would depend (at least potentially) on the allocations (and hence allocation rules) of other firms. Since each firm's rule might depend on the rules of other firms, it would be necessary to check that each set of rules produced a well-defined allocation for each possible division of customers among firms. To avoid this purely technical issue, we have adopted the asymmetric specification implicit in our two-stage game.

## A Related Signalling Environment

We now briefly discuss a version of the Spence (1973) signalling environment due to Prescott and Townsend (1984), and which is also used by Rosenthal and Weiss (1984). In this environment there is a continuum of "workers" who can be divided into a finite set of types, indexed by $i = 1, \ldots, n$. Each worker is endowed with a single unit of labor, which is supplied inelastically. In addition, there is a technology for converting labor into a single consumption good. In this technology a group of workers with positive measure is required to produce any output, and a group of type i workers of positive measure can produce $\pi_i$ units of the good per worker. In addition, the contribution of any individual to output is not directly observable.

There are two alternate interpretations which can be placed on this economy. One is that workers cooperate to produce output, i.e., form coalitions. Another is that there is a set of firms with access to this technology. The two cases correspond to the cooperative and noncooperative equilibrium concepts (respectively) employed above.

As is standard in the signalling literature, each worker knows his own type. This type is private information, ex ante. However, there also exists a signalling technology for transmitting information about type. Let $S \varepsilon R_+$ be a set of feasible signals with $0 \varepsilon S$. Then an agent may emit any signal $s \varepsilon S$ to convey information regarding his type.

Let c denote consumption, with $R_+$ being the set of feasible consumptions for each agent. Then a type i agent has preferences over (c, s) pairs given by

$$U_i(c,s) = \psi_i c_i - s_i.$$

Indices are ordered so that $\psi_1 < \psi_2 < \dots < \psi_n$, and following Spence (1973), the technological parameters then obey $\pi_1 < \pi_2 < \dots < \pi_n$. Finally, as above, $\mu_i$ denotes the measure of type i agents in the population with $\sum_{i=1}^{n} \mu_i = 1$.

As before, we can (depending on whether we are considering the cooperative or noncooperative case) define the behavior of coalitions (or firms) and allocation rules exactly as previously. The allocation rule of interest here solves the problem (for vector of measures $\theta$)

$$\max \psi_n c_n - s_n$$

subject to

$$\psi_i c_i - s_i \geq \psi_i c_j - s_j \;\forall\; i, j,$$

$$\psi_i c_i - s_i \geq \overline{U}_i \;\forall\; i = 1, \dots, n-1,$$

$$\sum_{i=1}^{n} \theta_i (c_i - \pi_i) \leq 0,$$

where the $\overline{U}_i$ are defined recursively by

$$\psi_1 \pi_1 = \overline{U}_1,$$

$$\overline{U}_2 = \max \psi_2 c_2 - s_2$$

subject to

$$\psi_2 c_2 - s_2 \geq \psi_2 c_1 - s_1$$

$$\psi_1 c_1 - s_1 \geq \psi_1 c_2 - s_2$$

$$\sum_{i=1}^{2} \theta_i (c_i - \pi_i) \leq 0$$

$$\psi_1 c_1 - s_1 \geq \overline{U}_1,$$

etc. If $\theta_i = 0$ for some i, an analog of (20) is appended to the problem.

It is not hard to see that the model has a formal structure essentially identical to that of the adverse selection environment considered above. Hence it should not be surprising that exactly the same argument can be applied here to prove propositions 1-4 for this economy. As this would be an exact repetition of earlier arguments, we will simply leave this as an assertion. It deserves to be emphasized, though, that propositions 2 and 4 apply, i.e., our equilibrium concepts applied to the Spence economy produce a <u>unique</u> equilibrium (both cooperative and noncooperative).

Notes

[1] Our specific version is due to Prescott and Townsend (1984).

[2] See, e.g., Myerson (1983).

[3] See, e.g., Foley (1970), Richter (1974), Starrett (1973).

[4] (3.d) should be interpreted as holding after the possible defection of some other groups of agents.

[5] See, e.g., Myerson (1983).

[6] Foley (1970), Richter (1974), Starrett (1973).

[7] With an obvious modification to account for the presence of lotteries.

[8] See Spence (1978).

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