Historical

Federal Reserve Bank of Minneapolis
Research Department Working Paper

Solution of Linear-Quadratic-Gaussian Dynamic Games using Variational Methods

William Roberds

Working Paper 291
October 1985

I would like to thank Dennis Epple, Lars Hansen, and Ravi Jagannathan for their comments on earlier drafts. Remaining errors are my own.

Abstract

Methods are presented for solving a certain class of rational expectations models, principally those that arise from dynamic games. The methods allow for numerical solution using spectral factorization algorithms, and estimation of these models using maximum likelihood techniques.

Current Address of Author

William Roberds
Research Department
Federal Reserve Bank of Minneapolis
250 Marquette Avenue
Minneapolis, Minnesota 55431

612-340-7775

1. Introduction

The purpose of the present paper is to outline some procedures that are useful for solving particular types of rational expectations models. The principal application of the techniques described below is in solving discrete time dynamic games of the linear-quadratic-Gaussian (LQG) variety. These techniques are nonrecursive or "open loop" in character, and are derived from the variational methods presented in Sargent (1980) and Hansen and Sargent (1981). The approach taken in this paper closely follows that of Hansen, Epple, and Roberds (1985), so that the analysis below may be taken as a generalization of that paper.

Other techniques are available for solving the sort of models considered in this paper, such as those described in Levine and Currie (1985), Buiter (1983), and Whiteman (1985). However, the techniques presented below may be more useful to those researchers accustomed to formulating and solving models using the Hansen-Sargent notation and methodology.

The paper is presented in the following order: Section 2 lays out three types of two-player LQG games; Section 3 discusses their solution; Section 4 gives some examples of models that can be addressed by the methodology of this paper. Issues concerning numerical implementation are discussed in an Appendix.

2. Three Dynamic Games

Below I analyze dynamic games with two infinitely lived players, each having a time invariant, time additive[1], discounted quadratic objective functional. All stochastic forcing variables enter into the players' objective in a linear fashion, and are assumed to be normally distributed. The two player assumption can be relaxed, subject to computational constraints, but the other assumptions cannot. The purpose of the other assumptions is to facilitate econometric application by allowing linear least squares projections to be used in place of conditional means. In terms of notation, let

u_{1t}	be a column vector of decision variables of player 1 (abbreviated P1) at time t;
u_{2t}	be analagously defined for player 2 (P2);
f_{1t}	be a column vector of uncontrollable forcing variables influencing P1's payoff at time t;
f_{2t}	be analogously defined for P2;
β	$\in (0,1)$ be a discount factor common to both players.

P1's objective is given by:

$$J_1 \equiv E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ f_{1t}' u_{1t} - [A(L) u_{1t}]' \tfrac{1}{2} M_1 [A(L) u_{1t}]$$

$$- [B(L) u_{2t}]' \tfrac{1}{2} M_2 [B(L) u_{2t}]$$

$$- [A(L) u_{1t}]' M_3 [B(L) u_{2t}]$$

$$- u_{1t}' \tfrac{1}{2} N_1 u_{1t} - u_{2t}' \tfrac{1}{2} N_2 u_{2t} - u_{1t}' N_3 u_{2t} \Big\}$$

where $A(L)$ and $B(L)$ are matrix polynomials in the lag operator L, of finite dimension and degree; M_1, M_2, M_3, N_1, N_2, and N_3 are matrices of the appropriate dimension; M_1, M_2, N_1, and N_2 are symmetric, E_0 is the expectations operator , conditional on information available at time $t = 0$. P2's objective is given by

$$J_2 \equiv E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ f_{2t}' u_{2t} - [C(L) u_{1t}]' \tfrac{1}{2} P_1 [C(L) u_{1t}]$$

$$- [D(L) u_{2t}]' \tfrac{1}{2} P_2 [D(L) u_{2t}]$$

$$- [D(L) u_{2t}]' P_3 [C(L) u_{1t}]$$

$$- u_{1t}' \tfrac{1}{2} Q_1 u_{1t} - u_{2t}' \tfrac{1}{2} Q_2 u_{2t} - u_{2t}' Q_3 u_{1t} \Big\}$$

where $C(L)$ and $D(L)$ are finite dimensional, finite degree matrix polynomials in the lag operator L; P_1, P_2, P_3, Q_1, Q_2, and Q_3 are matrices of the appropriate dimension; P_1, P_2, Q_1, and Q_2 are symmetric, and E_0 is defined as before. The definiteness conditions

$$A(\beta^{\frac{1}{2}}e^{i\omega})'M_1A(\beta^{\frac{1}{2}}e^{-i\omega}) \ , \ D(\beta^{\frac{1}{2}}e^{i\omega})'P_1D(\beta^{\frac{1}{2}}e^{-i\omega}) > 0 \tag{2.1}$$

are assumed to be satisfied for $\omega \in [-\pi,\pi]$.[2]

The uncontrollable forcing vector process $X_t \equiv [f_{1t}'f_{2t}']'$ is assumed to be Gaussian and to have time invariant fundamental moving average representation

$$X_t = F(L)\ v_t + K \tag{2.2}$$

where v_t is vector white noise, and K is a constant. In the analysis that follows, K is normalized to equal to zero.

Each player i seeks to maximize his objective by choosing a sequence of strategies $\{g_{it}\}$. Each strategy maps the player's information (I_{it}) into a decision taken at time t, i.e. $u_{it} = g_{it}(I_{it})$. Below, the appropriate specifications of information sets are given for the three dynamic games considered.[3]

Game 1 (Open Loop Nash): Let I_0 represent the information set generated by the initial conditions for all variables in the model, and let Ω_t represent the information set generated by the shocks $v_t, v_{t-1}, \cdots$. Then for P1 (superscripts indicate the game number)

$$I^1_{1t} = \Omega_t \cup I_0 \cup \left\{\{g^e_{2t}\}^{\infty}_{t=0}\right\} \ , \tag{2.3}$$

while for P2

$$I^1_{2t} = \Omega_t \cup I_0 \cup \left\{ \{g^e_{1t}\}^{\infty}_{t=0} \right\}. \tag{2.4}$$

Here, g^e_1 and g^e_2 are anticipated strategy sequences. An *equilibrium* is a pair of strategy sequences (g^1_1, g^1_2) such that g^1_1 and g^1_2 are optimal for P1 and P2 respectively, when $g^e_1 = g^1_1$ and $g^e_2 = g^1_2$.

Notes on Game 1

The strategy sequences g_1 and g_2 are required to be optimal for almost every realization of $\{v_t\}$. They are also restricted to be affine in $\{v_t\}$, and the resulting sequence of equilibrium decisions must be "stable," i.e. of mean exponential order less than $\beta^{-\frac{1}{2}}$. The restriction to affine strategies allows for use of the certainty equivalence principle, while the mean exponential order assumption provides for a convenient resolution of some nonuniqueness problems.[4] These restrictions will apply in all games considered in this paper.

Because the information sets in Game 1 contain no state variables other than uncontrollable shocks, this sort of game is described by dynamic game theorists as "open loop."[5] Games in which controllable state variables appear in players' information sets are described as "closed loop" or "feedback" games. As emphasized by Kydland (1975) and others, the equilibria of open loop dynamic games will in general be different from the feedback or closed loop equilibria. The open loop approach taken in this paper is justified largely by computational considerations. Particularly for econometric applications, the open loop procedures discussed below may offer considerable gains in computational convenience over the procedures used to obtain closed loop and feedback equilibria.

It is also important to note that each player's information does not include knowledge of the other player's future decisions, but instead knowledge of the other player's future strategies. The distinction between decisions and strategies is an important one. The strategy sequences are determined once and for all at the beginning of the game. Decisions are taken simultaneously by both players in every period.

Game 2 (Open Loop Stackelberg): For P1 (the leader)

$$I_{1t}^{2} = \Omega_t \cup I_0 \tag{2.5}$$

while for P2 (the follower)

$$I_{2t}^{2} = I_{2t}^{1} \quad . \tag{2.6}$$

An equilibrium for this game is a pair of strategy sequences (g_1^2, g_2^2) such that g_2 is optimal for P2 when $g_1^e = g_1^2$, and g_1^2 is optimal for P1.

Notes on Game 2

In this game, P1 is not constrained to take P2's strategies as given, but is free to exploit the dependence of P2's choice of strategies on the choice of g_1. In equilibrium, the value of P1's objective is necessarily no less than in the Nash game.

One interesting feature of Game 2 is that the same equilibrium obtains if the information of the follower is changed to

$$I_{2t}^{\star} = \Psi_t \cup \left\{ \{g_{1t}^{e}\}_{t=0}^{\infty} \right\}, \qquad (2.7)$$

where Ψ_t represents the information set generated by the entire past history of all the processes in the model, including endogenous processes.[6]

In Section 3, it is shown that the equilibrium of Game 2 will in general be time inconsistent. That is, the original equilibrium strategy sequence g_1^2 will generally not remain optimal as time passes. Without some mechanism to guarantee that P1 will hold to the initial equilibrium strategy sequence, the equilibrium of Game 2 is not viable. For this reason, another sort of Stackelberg game is considered.

Game 3 (*Time Consistent Stackelberg Game*): P1's information is given by

$$I_{1t}^{3} = I_{1t}^{2} \quad , \qquad (2.8)$$

while P2's information is given by

$$I_{2t}^{3} = I_{2t}^{\star} \quad . \qquad (2.9)$$

Equilibrium is defined as in Game 2, except for an additional restriction on the strategies of P1. That is, in choosing a time t strategy, P1 is constrained to ignore the impact of of this choice on P2's choice of strategies dated before time t. In other words, in choosing g_{1t}^{3} , P1 must take as given g_{2s}^{3} for $s < t$.

Notes on Game 3

The distinctions between Game 2 and Game 3 will be clarified in the next section.

One distinction that deserves immediate mention is that in Game 3, P2 must be allowed access to the "larger" information sets $\{ I_{2t}^{\star} \}$. That is, if P2 were allowed access only to I_{2t}^{2} , the same equilibrium would no longer obtain in Game 3.

Because of the additional restrictions on the strategy sequence g_1 , the equilibrium value of P1's objective in Game 3 can be no larger than in Game 2. It will in general be quite difficult to compare Games 1 and 3 in this fashion, since both players' information sets differ across the two games.

3. Solution Procedures

By "solving" the models described in Section 2 is meant the following: for each of the games, the first order conditions of the two players will be reduced to a set of finite order expectational difference equations. These equations, in turn, can be solved for equilibrium laws of motion in the variables u_{1t} and u_{2t} using known methods for solving linear rational expectations models. Explicit formulas for the equilibrium strategy sequences are not derived.

The solution procedures make heavy use of the techniques developed by Hansen and Sargent (1981). Especially useful are the following differentiation rules. Suppose that $\{x_t\}$ and $\{y_t\}$ are sequences such that

$$S_1 \equiv \sum_{t=0}^{\infty} \beta^t [a(L)y_t]' B [c(L)x_t]$$

and

$$S_2 \equiv \sum_{t=0}^{\infty} \beta^t [d(L)y_t]' \tfrac{1}{2}F [d(L)y_t]$$

are finite, where a(L), c(L), and d(L) are matrix polynomials in the lag operator, and B and F are appropriately dimensioned matrices. Then

(D1) $\partial S_1 / \partial y_t = \beta^t a(\beta L^{-1})' B c(L) x_t$;

(D2) $\partial S_2 / \partial y_t = \beta^t d(\beta L^{-1})' F d(L) y_t$.

Certainty equivalence is also exploited, in that the models are first solved for conditional means. Terms involving expectations are then evaluated using Wiener-Kolmogorov prediction formulas.

Solution of Game 1

To initiate the solution procedure suppose that P1 knows the sequence of equilibrium strategies $\{g^1_{2t}\}$ of P2. It follows that, as of time t, P1 knows the current and past decisions of P2, and that P1 can correctly forecast P2's future decisions. The necessary first order conditions for P1's optimization problem are then

$$[N_1 + A(\beta L^{-1})'M_1 A(L)]\, E_t u_{1t} + [N_3 + A(\beta L^{-1})'M_3 B(L)]\, E_t u_{2t}$$
$$= f_{1t}\,, \qquad t = 0, 1, 2, \cdots, \tag{3.1}$$

where again E_t represents the conditional expectations operator. The operators L and L^{-1} are defined as follows for the sequence of conditional means $E_t u_{1t}$: $L\,[E_t u_{1t}] \equiv E_{t-1} u_{t-1}$, and $L^{-1}\,[E_t u_{1t}] \equiv E_t u_{t+1}$, i.e. negative powers of L do not shift forward information sets. The first order condition for P2 is similarly given by

$$[Q_3 + D(\beta L^{-1})'P_3\, C(L)]\, E_t u_{1t} + [Q_2 + D(\beta L^{-1})'P_2 D(L)]\, E_t u_{2t}$$
$$= f_{2t}\,, \quad t = 0, 1, 2, \cdots. \tag{3.2}$$

Now stack equations (3.1) and (3.2) to obtain the system

$$H(L)\, E_t\, U_t = X_t \qquad , \tag{3.3}$$

where

$$U_t \equiv [\, u_{1t}{}'\, u_{2t}{}'\,]' \qquad , \quad \text{and}$$

$$H(L) \equiv \begin{bmatrix} N_1 + A(\beta L^{-1})' M_1 A(L) & N_3 + A(\beta L^{-1})' M_3 B(L) \\ Q_3 + D(\beta L^{-1})' P_1 C(L) & Q_2 + D(\beta L^{-1})' P_2 D(L) \end{bmatrix} .$$

Equation (3.3) is an expectational difference equation of the type analyzed by Hansen and Sargent (1981), Whiteman (1983, chapter 4), and Watson (1985) , among others. What follows is a brief outline of the Hansen-Sargent-Whiteman approach to solving systems such as (3.3).

First, suppose that $H(L)$ can be factored as

$$H(L) = S(\beta L^{-1})' T(L) \tag{3.4}$$

where $S(z)$ and $T(z)$ are appropriately dimensioned one sided matrix polynomials of degree n, n being the largest degree of the matrix polynomials $A(z)$, $B(z)$, $C(z)$, and $D(z)$. It is further assumed that the roots of det $T(z)$ are distinct and outside the circle $|z| = \beta^{\frac{1}{2}}$, and that the roots of det $S(\beta z^{-1})$ are distinct and inside this circle. One can then write $S(\beta L^{-1})'^{-1}$ in partial fractions form as

$$S(\beta L^{-1})'^{-1} = \sum_j \frac{N_j}{L - z_j} \quad . \tag{3.5}$$

where the N_j are matrices of the appropriate dimension, and the z_j are the roots of det $S(\beta z^{-1})$. Since, in equilibrium, both players' decisions must be of mean exponential order less than $\beta^{-\frac{1}{2}}$, operating on both sides of (3.3) with $S(\beta L^{-1})'^{-1}$ yields

$$T(L)\, U_t = S(\beta L^{-1})'^{-1}\, E_t\, f_t \quad . \tag{3.6}$$

Finally, using (3.5) and the Wiener-Kolmogorov prediction formula, (3.6) can be expressed as

$$T(L)\, U_t = \sum_j \frac{N_j}{L - z_j} \left[L^n F(L) - z_j{}^n F(z_j) \right] v_t \quad . \tag{3.7}$$

Again the summation is over the roots of det $S(\beta z^{-1})$. Equation (3.7) is a "feedforward-feedback" representation of U_t that, together with initial conditions, gives the unique stable solution to equation (3.3). Methods by which systems such as (3.7) can be estimated are described in Hansen and Sargent (1980).

Solution of Game 2[9]

To initiate the solution procedure for the Stackelberg game, suppose that P2 knows the sequence of equilibrium strategies of P1. Then, as in the Nash game, P1's current and past decisions will be known to P2, and P2 will be able to correctly forecast future decisions of P1. P2's first order condition will be the same as in the Nash game, i.e. equation (3.2). Since P2 (the follower) now takes P1's strategies parametrically, it will be convenient to rewrite (3.2) as

$$\left[Q_2 + D(\beta L^{-1})' P_2 D(L) \right] E_t\, u_{2t}$$
$$= - \left[Q_3 + D(\beta L^{-1})' P_3\, C(L) \right] E_t\, u_{1t} + f_{2t}\,. \tag{3.8}$$

The characteristic polynomial of equation (3.8) has factorization

$$D(\beta z^{-1})' P_2\, D(z) + Q_2 = G(\beta z^{-1})' G(z) \quad , \tag{3.9}$$

where $G(z)$ is a polynomial having degree equal to that of $D(z)$, and the roots of det $G(z)$ exceed $\beta^{\frac{1}{2}}$ in modulus. Again requiring $\{u_{2t}\}$ to be stable allows equation (3.9) to be solved forward, yielding

$$G(L)u_{2t} = G(\beta L^{-1})'^{-1} \left\{ - \left[Q_3 + D(\beta L^{-1})' P_3 C(L) \right] E_t\, u_{1t} + f_{2t} \right\} \quad ,$$

$$t = 0, 1, 2, \cdots . \tag{3.10}$$

Equation (3.10) can be thought of as a "closed loop" representation of the sequence of optimal decisions $\{u_{2t}\}$. The members of this sequence are expressed in (3.10)as a function of lagged values of u_{2t}, and current and lagged values of u_{1t} and f_{2t} (after making the appropriate substitutions for terms involving expectations of future variables). Using this representation, one could go one step further and derive the sequence of optimal open loop strategies for P2 by operating on (3.10) with $G(L)^{-1}$. However, for the present purpose of deriving the equilibrium law of motion for u_{1t} and u_{2t}, this extra step is not necessary.[7]

The next step in solving Game 2 is to formulate P1's problem as a constrained maximization problem

$$\max_{\{g_{1t}\}\{g_{2t}\}} J_1 \qquad \text{s. t.} \quad (3.10) \ .$$

The Stackelberg leader P1 in effect chooses a strategy sequence for both players. However, the strategy sequence chosen for P2 must be chosen so that it is optimal for P2, taking P1's strategies as given, i.e. the resulting sequence of decisions $\{u_{2t}\}$ must satisfy (3.10).

To solve the leader's problem, form the Lagrangian expression

$$\mathcal{L}_1 \equiv J_1 + C_0$$

where

$$C_0 \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \lambda_t' \left[- G(L)\, u_{2t} \right.\right.$$

$$\left.\left. + G(\beta L^{-1})'^{-1} \{ f_{2t} - [Q_3 + D(\beta L^{-1})' P_3 C(L)]\, E_t\, u_{1t} \} \right] \right\}.$$

Here $\{\lambda_t\}$ is a vector Lagrange multiplier process, of the same dimension as u_{2t}. For $t < 0$, λ_t is defined to take on a value of zero. First order conditions for the leader's maximization problem are obtained by differentiating $\mathcal{L}_1$ with respect to u_{1t}, and u_{2t}, and are given by

$$[N_1 + A(\beta L^{-1})' M_1 A(L)]\, E_t u_{1t} + [N_3 + A(\beta L^{-1})' M_3 B(L)]\, E_t u_{2t}$$

$$+ [Q_3' + C(\beta L^{-1})' P_3' D(L)]\, G(L)^{-1} \lambda_t = f_{1t} , \tag{3.11}$$

$$[N_3' + B(\beta L^{-1})' M_3' A(L)]\, E_t u_{1t} + [N_2 + B(\beta L^{-1})' M_2 B(L)]\, E_t u_{2t}$$

$$+ G(\beta L^{-1})'\, E_t\, \lambda_t = 0 , \tag{3.12}$$

and the constraint (3.10), for $t = 0, 1, 2, \cdots$. Making the substitution $\ell_t \equiv G(L)^{-1} \lambda_t$, operating on (3.10) with $G(\beta L^{-1})'$, and stacking (3.11), (3.10), and (3.12), one obtains the system

$$H^\star(L)\, E_t\, U^\star_t = X^\star_t , \tag{3.13}$$

where

$$U^{\star}_t \equiv [\, U_t' \;\; \ell_t' \,]' \,,$$

$$X^{\star}_t \equiv [\, X_t' \;\; 0' \,]' \,,$$

and $H^{\star}(L) \equiv$

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{H(L)} & \begin{matrix} Q_3' + C(\beta L^{-1})' P_3' D(L) \\ 0 \end{matrix} \\ \hline N_3' + B(\beta L^{-1})' M_3' A(L) & N_2 + B(\beta L^{-1})' M_2 B(L) & Q_2 + D(\beta L^{-1})' P_2 D(L) \end{array}\right]$$

Equation (3.13) can also be derived by taking P2's first order condition (3.8) as the constraint in P1's optimization problem.

As with the Nash game, equation (3.13) can be solved by factoring $H^{\star}(L)$ when $H^{\star}(L)$ posesses the "right" factorization. The result is a feedforward-feedback representation for the augmented decision vector $U^{\star}_t$, which together with initial conditions yields the solution for $U^{\star}_t$ and hence for U_t.

Of particular interest are the initial conditions for the vector ℓ_t of Lagrange multipliers. At the beginning of Game 2, note that the correct initial conditions for ℓ_t are given by $\ell_t = 0$ for $t < 0$. However, as time evolves, ℓ_t will in general take on nonzero values. Now consider a dynamic subgame of Game 2, beginning in period $\tau > 0$. For any such subgame, the solution of P1's problem would require that ℓ_t be initialized to zero for $t < \tau$. Hence the equilibrium for the subgame will be different from the original equilibrium, and the optimal strategy sequence for the leader is said to be *time inconsistent.*

Comparison with Whiteman's Technique

Whiteman (1985) has proposed an alternative technique for solving for the equilibria of games such as Game 2. Whiteman's method differs principally from the one presented above in that (1) the leader's problem is formulated in the frequency domain; and (2) rather than using Lagrangian methods, Whiteman in effect substitutes equation (3.10) into the leader's objective.

To compare the two techniques, it is useful to rewrite $H^{*}(L)$ in the form

$$H^{*}(L) = \begin{bmatrix} H_{11}(L) & H_{12}(L) & H_{12}(L)'' \\ H_{21}(L) & H_{22}(L) & 0 \\ H_{12}(L)'' & H_{32}(L) & H_{22}(L) \end{bmatrix}, \qquad (3.14)$$

where the double prime indicates transposition and "β-conjugation." Now use (3.12), i.e. the last component of (3.13), to eliminate $E_t\ \ell_t$ from (3.13), yielding

$$\begin{bmatrix} H_{11}(L)-H_{21}(L)''H_{22}(L)^{-1}H_{12}(L)'' & H_{12}(L)-H_{21}(L)''H_{22}(L)^{-1}H_{32}(L) \\ H_{21}(L) & H_{22}(L) \end{bmatrix}$$

$$\times \begin{bmatrix} E_t\, u_{1t} \\ E_t\, u_{2t} \end{bmatrix} = \begin{bmatrix} f_{1t} \\ f_{2t} \end{bmatrix} , \qquad (3.15)$$

which I abbreviate as $\Theta(L)\; E_t\; U_t = X_t$. Alternatively, equation (3.15) could be derived by using equation (3.10) to substitute out for $E_t\; u_{2t}$ in P1's objective J_1, and differentiating J_1 with respect to $E_t\; u_{1t}$.

Essentially, the alternative technique proposed by Whiteman involves factoring $\Theta(L)$ and applying the Hansen-Sargent solution algorithm. Since $\Theta(L)$ can always be obtained from $H^{\star}(L)$, this approach could also be used with the methods presented above.

Some care must be exercised with this approach, however. In obtaining equation (3.15), equation (3.12) was operated on with $H_{22}(L)^{-1}$, which is in general a matrix rational function (or two-sided infinite order matrix polynomial) in the lag operator L. Since (3.12) is only guaranteed to hold for nonnegative time, this operation will only be justified under special circumstances. For example, this operation will be justified when both players' objectives have been normalized so that all variables take on a value of zero for negative time. This operation is also justified if one is only interested in the steady state of the particular game under consideration.

Solution of Game 3

The time inconsistency of P1's strategy sequence in Game 2 results because P1's choice of strategy g_{1t} for $t > 0$ has an effect on P2's choice of strategy g_{2s} for $0 \leq s < t$. At time t, if P1 were to recalculate his optimal policy sequence, these effects would no longer matter, causing P1 to change his choice of strategies.

These effects enter into the first order conditions for P1's problem only through the presence of lagged values of λ_t in equation (3.11). Since, in Game 3, P1 is required to ignore these effects, (3.11) must be replaced by the following first order condition:

$$[N_1 + A(\beta L^{-1})'M_1 A(L)]\, E_t u_{1t} + [N_3 + A(\beta L^{-1})'M_3 B(L)]\, E_t u_{2t}$$

$$+ \left\{ [Q_3' + C(\beta L^{-1})'P_3' D(L)]\, G(L)^{-1} \right\}_- \lambda_t = f_{1t} , \qquad (3.16)$$

where the notation $\{\ \}_-$ means to ignore positive powers of L. Equations (3.16), (3.10), and (3.12), which correspond to the new first order conditions for P1's problem, can now be stacked to yield the system[8]

$$H^c(L)\, E_t\, U^c{}_t = X^{\star}{}_t \qquad (3.17)$$

where

$$U^c_t \equiv [\, U_t' \ \lambda_t' \,] \ ,$$

$$H^c(L) \equiv \begin{bmatrix} H_{11}(L) & H_{12}(L) & H^c_{13}(L) \\ H_{21}(L) & H_{22}(L) & 0 \\ H_{12}(L)'' & H_{32}(L) & H^c_{33}(L) \end{bmatrix} ,$$

$$H^c_{13}(L) \equiv \left\{ [\, Q_3' + C(\beta L^{-1})'P_3' \, D(L) \,] \, G(L)^{-1} \right\}_- \quad , \text{and}$$

$$H_{33}(L) \equiv G(\beta L^{-1})' \ .$$

A characteristic feature of $H^c(L)$ is that its rightmost "column" involves no positive powers of L. This means that if $H^c(L)$ has canonical factorization $H^c(L) = S^c(\beta L^{-1})'T^c(L)$, and $T^c(0)$ is normalized to be a diagonal matrix, then the last "column" of $T^c(L)$ must be all zeroes. This last fact in turn implies that in equilibrium, the current value of U^c_t does not depend on past values of λ_t . Hence the time path of U_t will be independent of initial conditions for λ_t , and the strategy sequence for P1 will be time consistent.

Another interpretation of Game 3 would be as a game played by a follower P2 and an infinite sequence of Stackelberg leaders. The time t leader has an objective given by a time t version of J_1 , and chooses u_{1t} so as to maximize this objective. The time t leader cannot commit to future values of u_{1t} , although he can correctly forecast these decisions in equilibrium. For an example of how the solution of Game 3 can be derived under this interpretation, the reader is referred to Hansen, Epple, and Roberds (1985).

Whiteman (1985) has also proposed a method for solving Game 3. As with Game 2 , one can essentially replicate Whiteman's method by eliminating the Lagrange multiplier process from equation (3.17), and solving the resulting expectational difference equations.

4. Examples

Below are presented examples of models where the methods of Section 3 may be applied.

Example 1: Whiteman's "Generic Example"

Whiteman (1985) considers a very simple two player game between a hypothetical policymaker and a player representing "the public." Although Whiteman's model differs slightly from the class of models considered in Section 2, the solution methods presented above are still applicable.

In this model, the policymaker plays the role of P1 and the public the role of P2. The scalar forcing process f_t is first order autoregressive, and u_{1t} and u_{2t} are both scalars. P1's objective is to minimize the discounted weighted sum of expected fluctuations in the decisions of both players, i.e.

$$J_1 \equiv -\tfrac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left[u_{2t}^2+\eta\, u_{1t}^2\right] .$$

Here, η is a positive weight. P1's objective is thus to stabilize fluctuations in u_{2t}, subject to a quadratic cost associated with policy interventions. P2 has an essentially static objective: in each period, choose a minimum mean squared error forecast of

$$y_t \equiv -\rho^{-1} \sum_{j=0}^{\infty} \rho^{-j} \left[u_{1,t+j} + f_{t+j} \right] ,$$

where $|\rho| > \beta^{-1}$. The optimal estimate of y_t will be given by

$$u_{2t} = -\rho^{-1} (1 - \rho^{-1} L^{-1})^{-1} (E_t u_{1t} + E_t f_t) \quad , \tag{4.1}$$

which corresponds to equation (3.2).

To begin the analysis of this model, note that the optimal strategy for P1 in Game 1 is to set $u_{1t} = 0$ for all t. Since P1's objective is to stabilize u_{2t}, if P1 takes P2's strategies as given, then P1's optimal strategy sequence is the trivial one.

In Game 2, the Lagrangian for the leader's problem is

$$\mathcal{L}_1 = E_0 \tfrac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[u_{2t}^2 + \eta u_{1t}^2 \right]$$

$$+ E_0 \sum_{t=0}^{\infty} \beta^t \lambda_t \left[u_{2t} + \rho^{-1} (1-\rho^{-1}L^{-1})^{-1} (E_t u_{1t} + E_t f_t) \right] . \tag{4.2}$$

The first order conditions for the leader's problem will be (cf. equations (3.11) and (3.12))

$$\eta\, u_{1t} + \rho^{-1}(1 - (\beta\rho)^{-1}L)^{-1}\, \lambda_t = 0 \;; \tag{4.3}$$

$$u_{2t} + \lambda_t = 0 \,. \tag{4.4}$$

Note that equation (4.3) is valid when λ_t has been normalized to zero for negative t.

One approach to solution of equations (4.1), (4.3), and (4.4) would be to stack those equations and apply the method outlined in Section 3. Because of the very simple nature of the model, however, it is easy to solve by direct substitution. First, use equations (4.3) and (4.4) to solve for u_{2t} , which yields

$$u_{2t} = \eta\rho\, (1 - (\beta\rho)^{-1}L)\, u_{1t} \;. \tag{4.5}$$

Equation (4.5) holds for positive t; if $u_{1,-1}$ is normalized to zero, then it also holds for t = 0. Equation (4.5) can then be substituted into (4.1), and the resulting equation operated on with $(L^{-1} - \rho)$ to obtain

$$[\,-\eta\, (L^{-1} - \rho)\, (\beta^{-1}L - \rho\,) - 1\,]\, E_t\, u_{1t} = f_t \tag{4.6}$$

Hansen and Sargent (1980) show that when f_t follows the autoregressive law $f_t = \gamma f_{t-1} + \epsilon_t$, equation (4.6) has solution

$$u_{1t} = c_1 u_{1,t-1} + [\, c_0^{\;-1} / (1 - c_2\gamma)\,]\, f_t \;, \tag{4.7}$$

where $-\eta\, (z^{-1}-\rho)\, (\beta^{-1}z-\rho) + 1$ can be factored as $c_0\, (1-c_1 z)\, (1-c_2 z^{-1})$,

$c_0 < 0$, and c_1, $c_2 \in (0,1)$. The optimal strategy for P1 thus consists of partially offsetting the effect of the current shock f_t, subject to a "correction" of $c_1 u_{1,t-1}$. The time inconsistency of this strategy is manifested in the fact that equation (4.7) only holds for $t = 0$ if $u_{1,-1}$ has been normalized to zero. If P1 were to recalculate an optimal strategy sequence starting at some time $\tau > 0$, then $u_{1,\tau-1}$ would have to be set to zero, resulting in a different choice of strategies.

To find the optimal time consistent policy for P1, note that in Game 3, the leader's first order condition (4.3) must be replaced with

$$\eta u_{1t} + \rho^{-1}\lambda_t = 0 \ . \tag{4.8}$$

Using (4.8) and (4.4) to eliminate u_{2t} from equation (4.1) then implies that

$$[\eta\rho L^{-1} - (\eta\rho^2+1)]\, E_t\, u_{1t} = f_t \ . \tag{4.9}$$

Defining $d_0 \equiv -(1+\eta\rho^2)$, and $d_1 \equiv \eta\rho/(1+\eta\rho^2)$, equation (4.9) can be solved for u_{1t} to obtain

$$u_{1t} = d_0^{-1} f_t \,/\, (1-d_1\gamma) \ . \tag{4.10}$$

In Game 3, P1's equilibrium strategy sequence is by construction time consistent. This is reflected in the fact that, unlike equation (4.7), equation (4.9) will hold for all $t \geq 0$, and need not be modified for the initial period.

For Games 1-3, the equilibrium sequence of decisions for u_{2t} can be derived by substituting the appropriate expression for P1's equilibrium strategy sequence into equation (4.1), and evaluating expectations.

Example 2: Linear-Quadratic Duopoly Models

In Hansen, Epple, and Roberds (1985), the methods of Section 3 are applied to a model of a duopolistic industry that extracts a nonrenewable resource. These methods can also be applied to other linear-quadratic oligopoly models. As an example, consider Kydland's (1979) model of an industry where there are adjustment costs.

In this setup, there are two firms in the industry. Entry by other firms into the industry is not possible. Firm i produces output y_{it} and invests amount x_{it} over period t. Investment is determined as

$$x_{it} = y_{i,\,t+1} - (1-\delta)\,y_{it} \;, \tag{4.11}$$

where δ is the depreciation rate. The real cost of investment x_{it} to firm i at time t is given by

$$qx_{it} + c\,(x_{it} - \delta y_{it})^2 \;, \tag{4.12}$$

where q is the unit cost of capital and the term $c(x_{it}-\delta y_{it})^2$, $c > 0$, represents the adjustment cost associated with changing the firm's capital stock. Each firm seeks to maximize

$$J_i = E_0 \sum_{t=0}^{\infty} \beta^t \left[p_t y_{it} - q x_{it} - c(x_{it} - \delta y_{it})^2 \right] \quad , \tag{4.13}$$

where p_t is the real price of the firms' output at time t, net of any constant unit production cost. This price is determined by a linear inverse demand function

$$p_t = a_t - \alpha[y_{1t} + y_{2t}] \quad , \tag{4.14}$$

where a_t is a random shock to demand and α is a positive constant. To map Kydland's model into the notation of Section 2, set

$$u_{it} = y_{i,t+1} \; ; \tag{4.15}$$

$$A(L) = B(L) = C(L) = D(L) = (1\text{-}L) \; ; \tag{4.16}$$

$$f_{it} = \beta a_{t+1} - (1-\beta(1-\delta)q) \; ; \tag{4.17}$$

$$M_1 = P_2 = 2c \; ; \tag{4.18}$$

$$M_2 = M_3 = N_2 = P_1 = P_3 = Q_1 = 0 \quad ; \tag{4.19}$$

$$N_1 = 2N_3 = Q_2 = 2Q_3 = \beta\alpha \quad . \tag{4.20}$$

Similar substitutions can be used with other duopoly models. It is also easy to modify the objective of the second player so that P2 represents a "competitive fringe" of small firms that see themselves as having no impact on the price of the industry output.[10]

Example 3: Optimal Growth with a Public Good

As a final example, I consider a simple model of macroeconomic growth.[11] In this model there are two consumption goods and two capital goods. There are two representative agents, P1 being the "government" and P2 being a representative nongovernmental agent, called "the private sector." One type of capital good (call this m) can only be accumulated by the government. However, the government makes this capital good freely available to the private sector. The other capital good (call this k) can only be accumulated by the private sector. The stock of governmental or public capital as of time t evolves as

$$m_t = \gamma \, m_{t-1} + z_t \, , \tag{4.21}$$

where z_t is current governmental investment, and γ equals one minus the depreciation rate δ. Governmental investment z_t must be financed by lump sum subtractions from the stock of private capital at time t , k_t. Conversion of private capital into public capital incurs an adjustment cost $\frac{1}{2}b_1(z_t-\delta m_t)^2$. When z_t is negative, this is interpreted as a governmental subsidy of private investment. Such subsidies also incur adjustment costs. The government is not allowed to borrow or lend, and governmental (dis)investment must equal the amount of lump sum taxes (subsidies) in every period.

Private capital k_t is assumed to accumulate according to the law

$$k_t = \gamma k_{t-1} + i_t - z_t \,, \tag{4.22}$$

where i_t is private investment. Associated with a level of investment i_t are adjustment costs $\frac{1}{2}b_2(i_t-\delta k_t)^2$.

The capital goods m_t and k_t are used to produce consumption goods g_t and c_t. Neither g_t nor c_t are storable, and these goods are produced according to the linear technology

$$\begin{bmatrix} g_t \\ c_t \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} m_t \\ k_t \end{bmatrix} . \tag{4.23}$$

The additional restrictions are imposed that $A_{11} > A_{12} \geq 0$, that $A_{22} > A_{21} \geq 0$, and that the A matrix is nonsingular. If one thinks of g_t as a "public" consumption good, and c_t as a "private" consumption good, these assumptions imply that both kinds of capital may be used to produce both kinds of consumption goods. Governmental capital is more productive than private capital in the production of the public consumption good g_t, and vice versa for the production of the private consumption good c_t.

The utility associated with consumption (g_t, c_t) is assumed to be of the quadratic, additively separable variety:

$$U_{it}(g_t, c_t) = -\tfrac{1}{2}\theta_i \,(g_t-g^\star)^2 - \tfrac{1}{2}(1-\theta_i)\,(c_t-c^\star)^2 \tag{4.24}$$

for $i = 1, 2$, where $g^\star$ and $c^\star$ are bliss points and $1>\theta_i>0$. Both the government and the private sector seek to maximize the discounted sum of their respective utilities, minus the discounted costs of governmental and private investment.[12] After some substitutions, these problems can be shown to be equivalent to the following:

$$\max \ -J_i \ , \text{ where}$$

$$J_i = \tfrac{1}{2} \sum_{t=0}^{\infty} \beta^t \ \Big\{ u_i (m_t - m^\star)^2 + w_i (k_t - k^\star)^2$$

$$+ 2 v_i (m_t - m^\star)(k_t - k^\star) + b_2 [(1-\gamma L) k_t]^2$$

$$+ (b_1 + b_2) [(1-\gamma L) m_t]^2 + 2 b_2 [(1-\gamma L) k_t] [(1-\gamma L) m_t] \Big\} .$$

In the expression above, $m^\star$ and $k^\star$ are the stocks of capital necessary to efficiently produce $g^\star$ and $c^\star$. The terms u_i, v_i, and w_i are defined as

$$\begin{bmatrix} u_i & v_i \\ v_i & w_i \end{bmatrix} = A' \begin{bmatrix} \theta_i & 0 \\ 0 & (1-\theta_i) \end{bmatrix} A \ . \tag{4.25}$$

The model described above may be directly mapped into the setup of section 2 using the following substitutions:

$$u_{1t} = m_t \;; \tag{4.26}$$

$$u_{2t} = k_t \;; \tag{4.27}$$

$$A(L) = B(L) = C(L) = D(L) = (1-\gamma L) \;; \tag{4.28}$$

$$-f_{1t} = u_1 m^{\star} + v_1 k^{\star} \;; \tag{4.29}$$

$$-f_{2t} = v_2 m^{\star} + w_2 k^{\star} \;; \tag{4.30}$$

$$M_1 = P_1 = b_1 + b_2 \;; \tag{4.31}$$

$$M_2 = P_2 = b_2 \;; \tag{4.32}$$

$$M_3 = P_3 = b_2 \;; \tag{4.33}$$

$$N_1 = u_1 \;;\; N_2 = w_1 \;;\; N_3 = v_1 \;; \tag{4.34}$$

$$Q_1 = u_2 \;;\; Q_2 = w_2 \;;\; Q_3 = v_2 \,. \tag{4.35}$$

In addition, the term $(v_1 m^{\star} + w_1 k^{\star})\, k_t$ must be subtracted to the government's utility function.

Simulation of Example 3

Two possible reasons why one might want to want to simulate this model are described below. First, in the case where $\theta_1 = \theta_2$ and the government and private objectives coincide, it might be a useful normative exercise to derive the equilibrium sequence of taxes z_t under various assumptions concerning the type of game played by the government and the private sector. Since the preferences of both players would coincide, this tax sequence would be optimal for both players.

It is well known that for this sort of policy problem, where there is only one private agent and the government can impose lump sum taxes, that the equilibria of the three games studied in this paper will coincide.[13] Hence to derive the optimal path of taxes, investment, capital stocks, and consumption goods, one need only solve Game 1 (Nash).

Another reason for simulating this model would be to investigate the effect of "perverse" governmental preferences on the equilibrium paths of the variables in the model. For example, it might be the case that those responsible for the setting of governmental policy prefer higher levels of consumption of the public good g_t than does the private sector, i.e. $\theta_1 > \theta_2$. In such cases, the three games studied in this paper are likely to result in different equilibrium outcomes.

As an illustration, three simulations of the model were run, using the following hypothetical parameter values. For the first simulation, the values $\beta = .926$, $A_{11} = A_{22} = 1.25$, $A_{12} = A_{21} = .75$, $b_1 = b_2 = 20$, and $\theta_1 = \theta_2 = .15$ were used. Bliss points for c_t and g_t were set to a value of 100. In the first simulation, Game 1 was simulated over 100 time periods, given initial conditions $k_{-1} = m_{-1} = 10$. In the second simulation, Game 1 was again simulated after θ_1 was reset to .5. In a third simulation, Game 2 (Stackelberg) was simulated using the same parameter values as in the second simulation. Game 3 was not simulated because numerical investigations indicated that the necessary polynomial matrix factorization did not exist for these parameter values.[14]

The outcomes of these three simulations are depicted in Figures 1-4, and are labelled respectively "Optimal" (i.e. optimal from the standpoint of the private sector), "Game 1," and "Game 2." Figure 1, which depicts the time path of m_t, shows that when governmental and private sector preferences coincide, the government will initially subsidize private investment by converting governmental capital into private capital. In contrast, when the government overvalues the public consumption good g_t, public capital increases monotonically over time. The steady state stock of governmental capital also increases dramatically in the second and third simulations. Figure 2 shows that governmental overvaluation of g_t causes the steady state level of private capital to fall dramatically. Figure 3 depicts the time path of g_t as a percentage of its bliss value. As might be expected, stronger preferences for g_t by policymakers lead to overconsumption of the public good. Figure 4 shows that the opposite holds true for the private good c_t.

In general, Figures 1-4 illustrate that the effects of governmental overvaluation of the public consumption good are what one would intuitively expect, i.e. overinvestment in public capital and underinvestment by the private sector, with corresponding shifts in consumption.

The negative effects of this overvaluation are somewhat less (in the sense that deviations from optimal values are smaller) in the Stackelberg game, where the government is (by assumption) able to credibly precommit itself to a sequence of tax policies.

Appendix: Notes on Numerical Implementation

There are a number of methods of obtaining the factorizations of the $H(L)$, $H^{\star}(L)$, and $H^{c}(L)$ matrix polynomials that are required by the solution procedures outlined above.

One simple method is to use the procedure suggested by Whittle (1983) for factoring the spectral density matrix of a vector moving average process. This method was used in the simulations of Example 3. In each simulation $T(0)$ or $T^{\star}(0)$ was normalized as the identity matrix. In the case of Game 2, using this normalization required that the second and third rows of $H^{\star}(L)$ be interchanged so as to render $T^{\star}(0)$ diagonalizable. This last step is recommended when using this algorithm for Games 2 and 3.

Dagli and Taylor (1984) have also devised an iterative algorithm for the purpose of obtaining such factorizations. For large systems, this algorithm is easier to implement than Whittle's algorithm. As an iterative algorithm, it would seem to be particularly well suited to econometric applications.

Notes

1. In the Stackelberg games considered below, the leader's objective will not be time additive after substitution for the follower's reactions.

2. These conditions are sufficient for each player's problem to be well defined in the Nash game (see Hansen and Sargent (1981)), and for the follower's problem to be well defined in the Stackelberg games. Sufficient conditions for the leader's problem can be derived using the results in Telser and Graves (1971, chapter 6).

3. The term "information set" is used here in the game theoretic sense, i.e. a player's information set as of time t is the domain of his strategy function as of time t. The notation used below for information sets is intentionally heuristic.

4. That is, under this assumption one can often show the existence of a unique homogeneous solution to the players' first order conditions. See Hansen and Sargent (1981) for a discussion.

5. The term "open loop" is usually applied to games under certainty, i.e., the case for which $v_t = 0$ for all t in the setup above. Buiter's (1981) definition of "open loop" for the stochastic case does not allow open loop strategies to depend on uncontrollable shocks. The above definition of open loop strategies for the stochastic case corresponds to that of Kydland (1975), except for the restriction that strategies be

affine. The reader is referred to articles by Kydland (1975,1977) for comparisons of open loop, feedback, and closed loop dynamic games.

6. See Basar and Olsder (1982, p. 309).

7. That is, the operator L as defined above is identical to the operator B defined in Sargent (1980, p.337).

8. Throughout this section the existence of such factorizations will be assumed.

9. Levine and Currie (1984) derive a solution procedure very similar to the one outlined above. Their procedure differs mainly from the one above in that the leader's problem is formulated using "state-space" notation.

10. For an example of this sort of modification, see Hansen, Epple and Roberds (1985).

11. All quantities in this example are per capita.

12. Alternatively, J_i could be viewed as the discounted sum of consumer surplus minus investment costs.

13. See Hillier and Malcomson (1984). When the government can only impose proportional taxes, this result will not hold. See Sargent (1984) for a comparison of Games 2 and 3 in a proportional taxation

environment.

14. Similar existence problems were encountered by Kydland and Prescott (1977) while attempting to simulate a linear-quadratic time consistent Stackelberg model. This suggests that existence problems are likely to be encountered with Game 3.

References.

Basar, T.; and Bagchi, A. 1981. Stackelberg Strategies in Linear-Quadratic Stochastic Differential Games. *Journal of Optimization Theory and Applications* 35(3): 443-463.

Basar, T.; and Olsder, G.J. 1982. *Dynamic Noncooperative Game Theory.* Academic Press: New York.

Buiter, Willem H. 1981. The Superiority of Contingent Rules over Fixed Rules in Models with Rational Expectations. *The Economic Journal* 91: 647-670.

Buiter, Willem H. 1983. Optimal and Time-Consistent Policies in Continuous Time Rational Expectations Models. Technical Working Paper No. 29. National Bureau of Economic Research.

Chow, Gregory C. 1981. *Econometric Analysis by Control Methods.* John Wiley & Sons: New York.

Dagli, C. Ates; and Taylor, John B. 1984. Estimation and Solution of Linear Rational Expectations Models using a Polynomial Matrix Factorization. *Journal of Economic Dynamics and Control* 8: 341-348.

Hansen, Lars P.; Epple, Dennis.; and Roberds, William. 1985. Linear-Quadratic Duopoly Models of Resource Depletion. In

Energy, Foresight, and Strategy. Thomas Sargent, editor. Resources for the Future: Washington.

Hansen, Lars P.; and Sargent, Thomas J. 1980. Formulating and Estimating Dynamic Linear Rational Expectations Models. *Journal of Economic Dynamics and Control* (2): 7-46.

Hansen, Lars P.; and Sargent, Thomas J. 1981. Linear Rational Expectations Models for Dynamically Interrelated Variables. In *Rational Expectations and Econometric Practice.* Thomas Sargent, editor. University of Minnesota Press: Minneapolis.

Hillier, Brian; and Malcomson, James M. 1984. Dynamic Inconsistency, Rational Expectations, and Optimal Government Policy. *Econometrica* 52(6): 1437-1451.

Kydland, Finn. 1975. Noncooperative and Dominant Player Solutions in Discrete Dynamic Games. *International Economic Review* 16(2): 321-335.

Kydland, Finn. 1977. Equilibrium Solutions in Dynamic Dominant-Player Models. *Journal of Economic Theory* 15(2): 307-324.

Kydland, Finn. 1979. A Dynamic Dominant Firm Model of Industry Structure. *Scandinavian Journal of Economics*: 355-366.

Kydland, Finn; and Prescott, Edward. 1977. Rules Rather than

Discretion: the Inconsistency of Optimal Plans. *Journal of Political Economy* 85(3): 473-491.

Levine, Paul; and Currie, David. 1984. The Design of Feedback Rules in Linear Stochastic Rational Expectations Models. University of London. Working Paper.

Sargent, Thomas J. 1980. *Macroeconomic Theory.* Academic Press: New York.

Sargent, Thomas J. 1984. Dynamic Optimal Taxation. Unpublished Manuscript.

Telser, L.; and Graves, P. 1971. Functional Analysis in Mathematical Economics. University of Chicago Press: Chicago.

Watson, Mark W. 1985. Recursive Solution Methods for Dynamic Linear Rational Expectations Models. Harvard University. Working Paper.

Whiteman, Charles H. 1983. *Linear Rational Expectations Models: A User's Guide.* University of Minnesota Press: Minneapolis.

Whiteman, Charles H. 1985. Analytical Policy Design Under Rational Expectations. University of Iowa. Working Paper.

Whittle, P. 1983. *Prediction and Regulation by Linear Least Square Methods.* 2nd edition, University of Minnesota Press: Minneapolis.

Witsenhausen, Hans. 1971. Separation of Estimation and Control for Discrete Time Systems. *Proceedings of the IEEE* 59(11): 1557-1566.

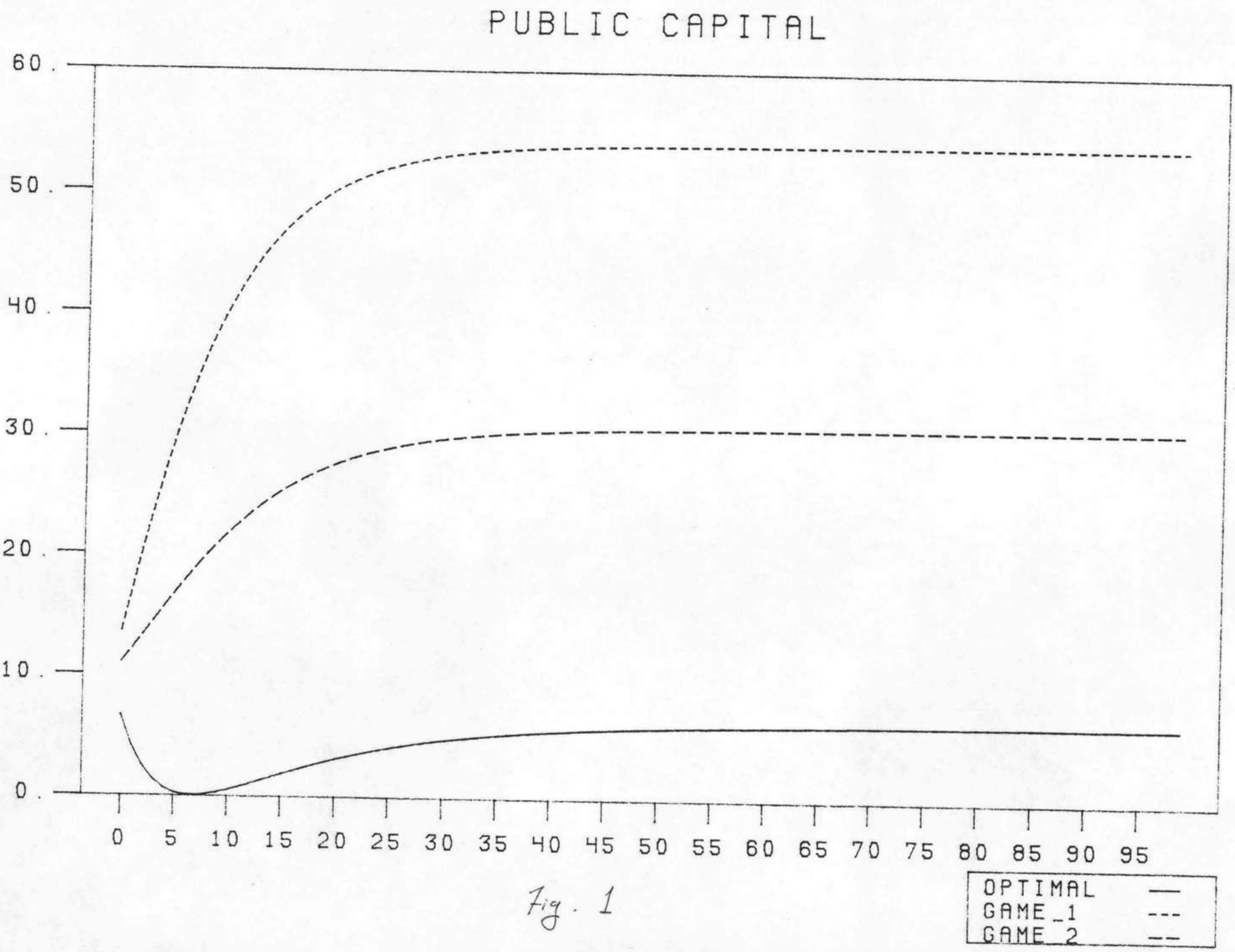
PUBLIC CAPITAL
60.
50.
40.
30.
20.
10.
0.
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95
OPTIMAL
GAME_1
GAME_2

Fig. 1

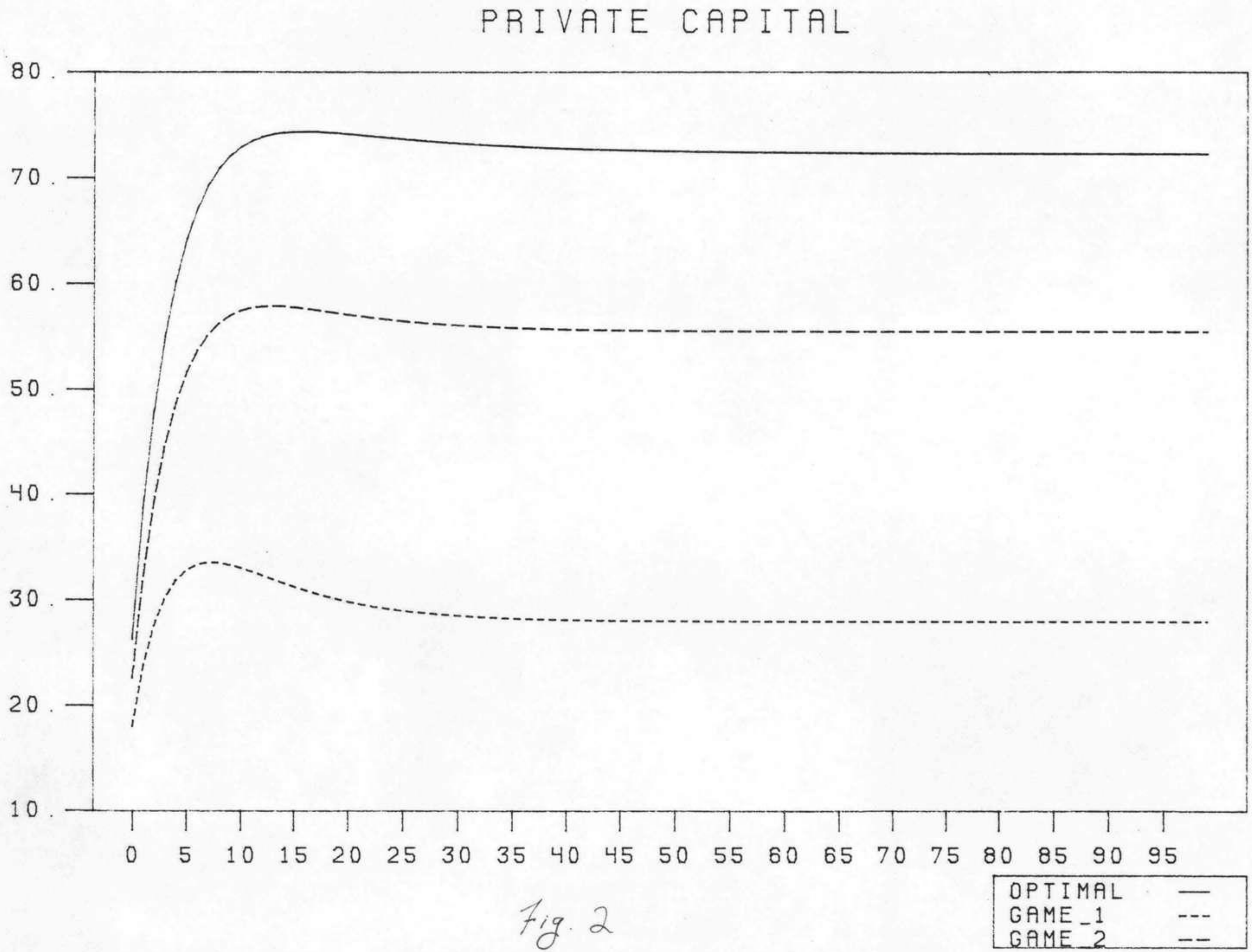
PRIVATE CAPITAL
80.
70.
60.
50.
40.
30.
20.
10.
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
90
95
OPTIMAL
GAME_1
GAME_2

Fig. 2

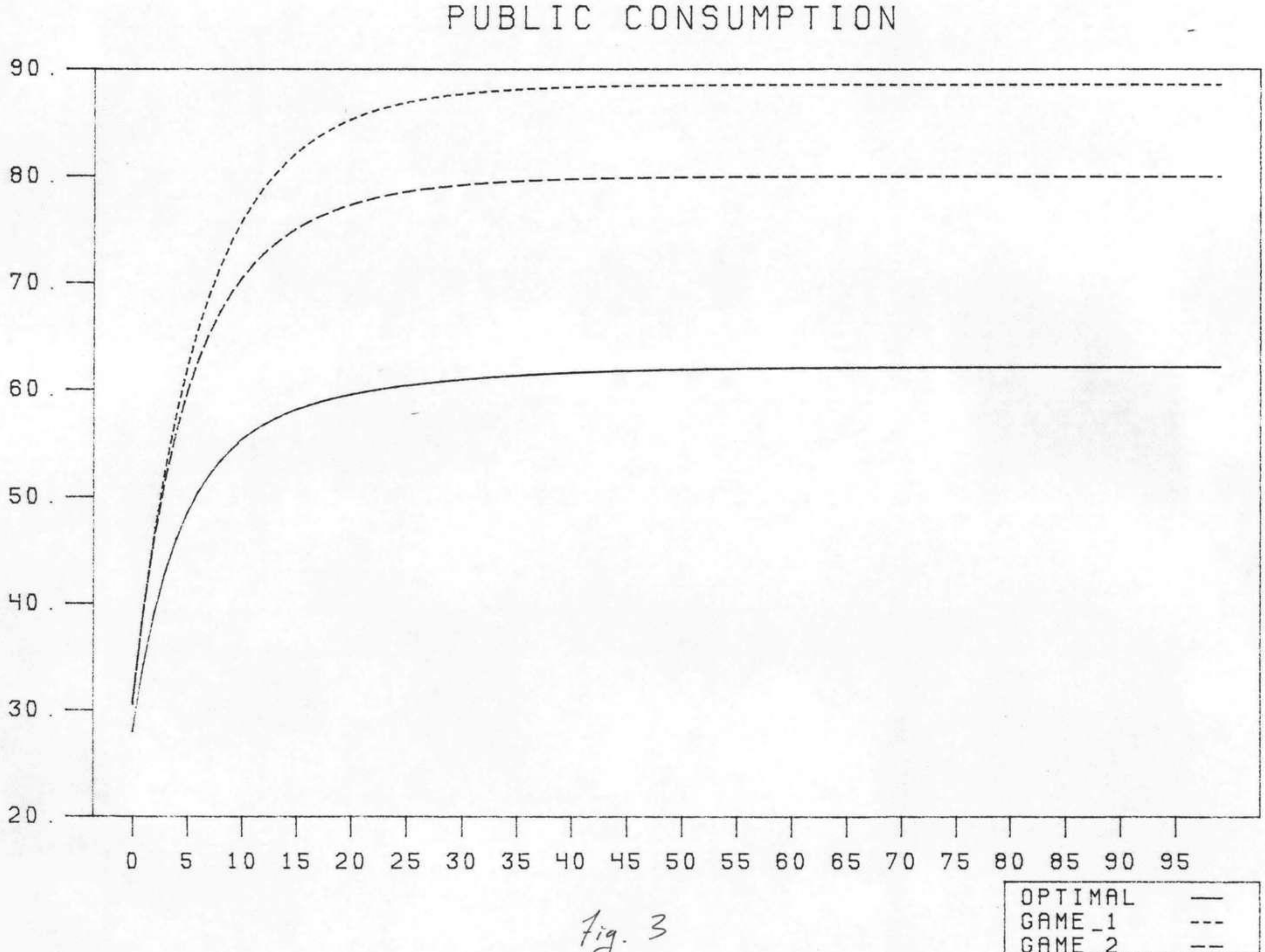
PUBLIC CONSUMPTION
90.
80.
70.
60.
50.
40.
30.
20.
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95
OPTIMAL
GAME_1
GAME_2

Fig. 3

PRIVATE CONSUMPTION

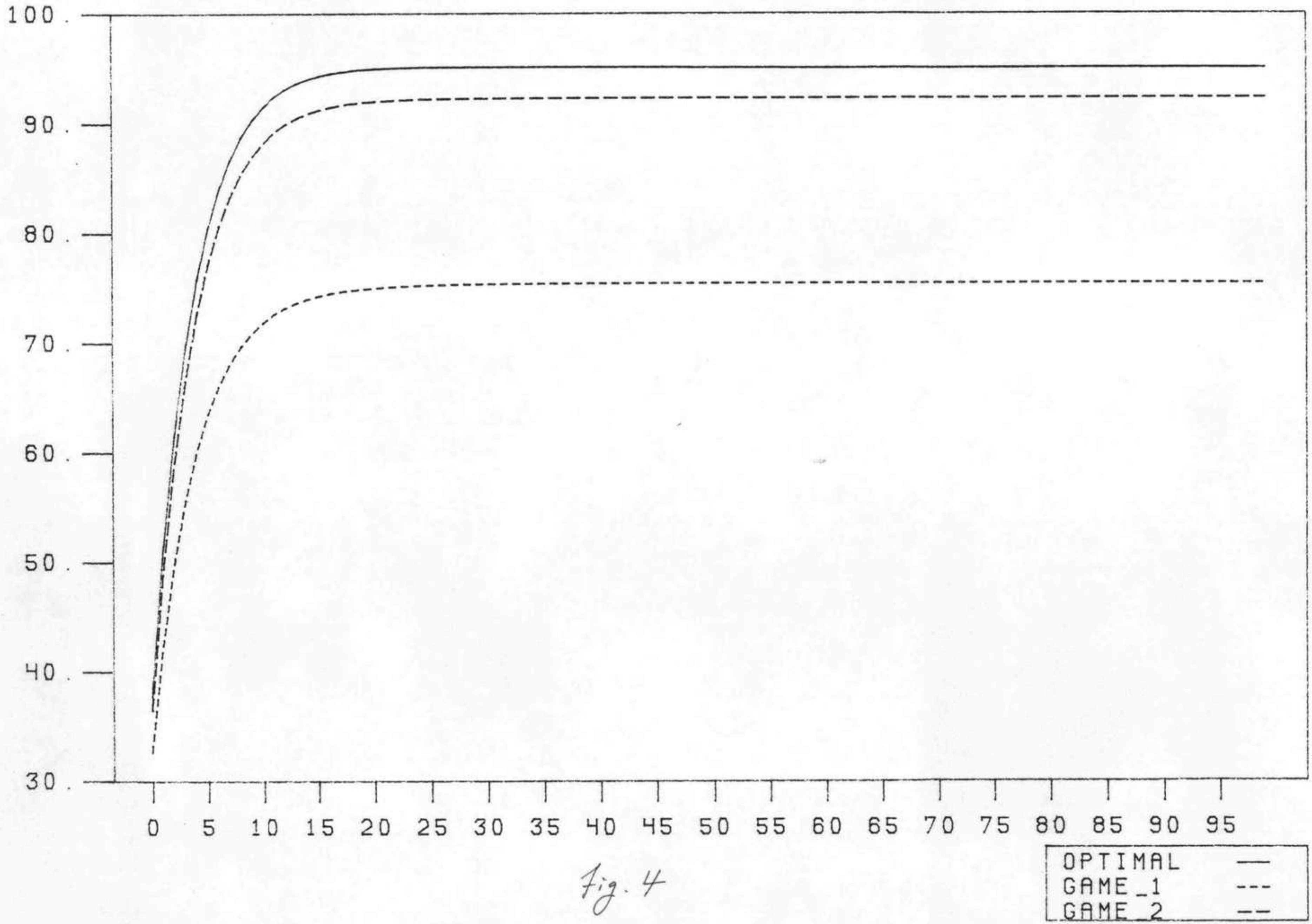

Fig. 4