INDIVIDUAL RISK WITHOUT AGGREGATE UNCERTAINTY:
A NONSTANDARD VIEW

Michael J. Stutzer

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Introduction

Equilibrium analysis in large economies whose agents face uncertainties is simplified when these uncertainties completely and rigorously disappear in the aggregate. Both Judd (1985) and Feldman & Gilles (1985) agree that numerous papers seeking to do so are beset by technical problems arising from the difficulty of extending laws of large numbers to continua of random variables.

Both Judd and Feldman & Gilles encourage the use of nonstandard, hyperfinite probability theory in resolving these economic modeling difficulties, but neither paper develops the idea further. This paper illustrates how the practitioner can use models with an infinite hyperreal number, rather than a continuum, of agents to simplify large scale economic analysis. The survey paper by Cutland (1983) is heavily relied on here for the machinery used, and should be read in conjunction with this paper. For more general background on nonstandard analysis, see the introductory text of Henle and Kleinberg (1980) and the more advanced treatment of Keisler (1976), Hurd and Loeb (1985), and Albeverio, et. al. (1986). For other applications of nonstandard analysis in large scale economies, see Emmons and Yannellis (1984).

Section 1: A Representative Problem.

The following problem in a model of Prescott and Townsend (1984) is representative of what we wish to solve. Suppose there are two types of agents: low risk agents and high risk ones. Each low risk agent receives a random endowment Z, which is Bernoulli distributed with probability $P(Z=Z_0) = \theta_1$. Each high risk agent receives an analogous random endowment with higher $P(Z=Z_0) = \theta_2$. The endowment pattern of low risk (or high risk) agents is thus specified as an independent Bernoulli process, with probability $P(Z_t=Z_0) = \theta_1$ (or θ_2), where t ranges over the set of all low risk (or high risk) agents. If there were a finite number n of low risk (or high risk) agents, then the proportion of low risk (or high risk) agents getting $Z = Z_0$ would be a binominal random variable with mean equal to θ_1 (or θ_2) and variance equal to $\theta_1(1-\theta_1)/n$ (or $\theta_2(1-\theta_2)/n$). Randomness does not rigorously and completely disappear no matter how large n is.

Prescott and Townsend want these proportions to be nonstochastic, however, equaling θ_1 and θ_2, respectively. To do so, they and others have hoped that the assumption of a continuum of agents would permit the application of an extended strong law of large numbers forcing these proportions to degenerate to distributions concentrated on θ_1 and θ_2, respectively.

As discussed in the previously cited articles by Judd and by Feldman & Gilles, there are two problems in extending laws of large numbers to a continuum of random variables. The first problem is that sets one would want to assign probability to may

not be measurable. In our representative problem, for some realizations assigning either Z_0 or Z_1 to each member of the continuum of low risk agents (represented, say, by the interval $(0,1]$), the subset of agents receiving Z_0 may not even be Lebesgue measurable, let alone have measure equal to θ_1 if it were. In fact, Judd's Thm. 1 implies that such realizations are contained in every Borel set of realizations of positive measure of this process. The second problem is that even when measurability is not an issue, the strong law of large numbers may fail. So even when we have a realization in which the subset of low risk agents on $(0,1]$ receiving Z_0 is Lebesgue measurable, its measure may not be θ_1. Because Judd's Thm. 2 implies that such realizations are contained in every Borel set of positive measure, this is also something worth worrying about.

In discussing various ways of avoiding these problems, both Judd and Feldman & Gilles encourage the application of non-standard economic models utilizing an infinite hyperreal number of agents, rather than a continuum. Neither article develops this notion. This one does. The following section presents an applications, practitioner-oriented development of the mathematical background necessary to develop this notion.

Section 2: Hyperreal Numbers

A <u>hyperreal number system</u>, denoted $^*\text{R}$, is an extension of the real number line R obtained by defining and including <u>infinitesimal numbers</u>. Infinitesimal numbers are defined to be numbers whose absolute value is less than <u>all</u> positive real num-

bers. Thus, the only real number which is also infinitesimal is zero. Because the extension produces an ordered field, sums, products and quotients are all defined and obey the usual laws of arithmetic and ordering relationships.

For example, let "dx" denote an infinitesimal number, and suppose $dx > 0$. Then, $1/dx$ is also a hyperreal number. By definition, for all real, positive ε, $dx < \varepsilon$. Dividing both sides by (positive) dx and ε yields $1/dx > 1/\varepsilon$, for all positive ε, no matter how small. Thus, the hyperreal number $1/dx$ is greater than all real numbers, and is termed infinite, albeit still smaller than $(1/dx) + 1$ or than $2/dx$, which are also infinite. Note that in making this argument, I've assumed that the order properties of quotients and the relations symbolized by "<" and ">" extend to the whole hyperreal number system. This is valid, for the relations symbolized by "<" and ">" and all other set theoretic real entities have extended counterparts on the hyperreals.

For another example, note that $h = r + dx$ is hyperreal, for any real number r. It turns out that any finite hyperreal has a unique representation as r plus an infinitesimal, with r being interpreted as the real number closest to it, or the real part of the finite hyperreal, denoted $\overset{\circ}{h}$. There is thus a haze of hyperreal numbers around each real number r, generated by taking r and adding all positive and negative infinitesimals to it, termed the monad of r. A hyperreal haze also surrounds each infinite hyperreal, positive or negative, as well. When two hyperreal numbers differ by an infinitesimal, like r and $r + dx$ do, we say that they are infinitely close.

Much of the usefulness of nonstandard analysis lies in the Transfer Principle, which provides a mechanism for transferring statements true in a real number system to statements true in a hyperreal number system, and vice-versa. Roughly speaking, the idea is to first write the statement known to be true in one number system as a bounded quantifier sentence in the first order predicate calculus, and then to rewrite it in the other number system, replacing symbols by their counterparts in the other system.

Somewhat more precisely, a first order formula is made up of symbols denoting variables, constants, relatives and functions, the usual logical connectives, parentheses and commas, and the quantifier symbols $\forall$ and $\exists$. A formula is said to be bounded when each quantifier restricts its quantified variable to lie in a specified set, and is said to be a sentence when there are no free (i.e., not quantified) variables within it. Sentences may contain constants, though. For example, the bounded quantifier sentence

(1) $\qquad (\forall\ x_1 \varepsilon R)(\forall\ x_2 \varepsilon R)((x_1 > 0 \& x_2 > 0 \& x_1 < x_2) \Rightarrow 1/x_1 > 1/x_2)$

composed of variables x_1 and x_2, the constant 0, the relations ε, $<$ and $>$ and the logical connectives $\&$ and $\Rightarrow$, is true. By application of the Transfer Principle (stated more compactly, "By transfer"), the bounded quantifier sentence

(1') $\qquad (\forall\ x_1{}^* \varepsilon\ {}^*R)(\forall\ x_2{}^* \varepsilon\ {}^*R)((x_1{}^* > 0 \& x_2{}^* > 0 \& x_1{}^* < x_2) \Rightarrow 1/x_1{}^* > 1/x_2)$

is also true, where the starred symbols denote the hyperreal counterparts. Because they are extensions, it is normal practice to suppress the starred notation on the symbols for the extended relations, functions, etc. Note in particular that any real constant r, like zero, has $^*r = r$. (1') justifies the operations previously used to "prove" that $1/dx$ is an infinite positive hyperreal when dx is a positive infinitesimal, by interpreting x_1 and x_2 to be the hyperreal numbers dx and ε (the latter is also real), respectively.

However, one must be careful in applying the Transfer Principle to transfer "everything." For example, the Archimedean property of the real numbers, which states that any particular real number is smaller than some real integer, is expressed as:

(2) $(\forall x_2 \varepsilon R)(\exists x_1 \varepsilon N) x_2 < x_1$

Where N denotes the real integers. In transferring this sentence to its hyperreal counterpart, both R and N must be replaced by their respective hyperreal counterparts $^*R$ and $^*N$. The resulting *Archimedean property of the hyperreals is that any particular hyperreal number is smaller than some hyperreal integer. Because there exist infinite hyperreal numbers (e.g., $1/dx$ as proven earlier), for each one there must exist a larger <u>infinite integer</u>. If one did not replace N by $^*N$ in transferring the statement, one would get the statement that any particular hyperreal number x_2 is smaller than some <u>real</u> integer x_1, which is obviously false (let $x_2 = 1/dx$, for example).

Hyperreal sets which are produced by transfer are said to be <u>internal</u>. So $^*N$ is internal, but $N \subset {}^*N$ is not internal. By transfer, any internal subset of $^*\mathbf{R}$ which is bounded above has a least upper bound. N is bounded above by any infinite integer, but has no least upper bound, and hence isn't internal, i.e., it is <u>external</u>. It turns out that N is not an element of any $^*A$, where A is a real set of sets, so it can't be produced by transfer.

Because there exists finite initial segments {1,2,3, ...,n}, of N, by transfer there must also exist internal initial segments of $^*N$, like {1,2,3, ..., M}, where M is a finite or infinite integer. Sets which can be put into one-to-one correspondence with these segments are called *finite, or <u>hyperfinite sets of cardinality M</u>. In our representative problem, we'll assume that the set of agents is a hyperfinite set, rather than a continuum, and use it to define a hyperfinite measure space, in section 4.

Section 3: Nonstandard Probability Theory

One might guess that hyperreal, internal σ-additive measure spaces could be easily produced by a straightforward application of the Transfer Principle to the defining axioms of real measure spaces. But this turns out to be wrong. Suppose that X is a real measurable space with a σ-field F and σ-additive (i.e., countably additive) measure μ on F. Application of the Transfer Principle yields a hyperreal measurable space $^*X$ with an internal $^*\sigma$-field $^*F$ and a $^*\sigma$-additive internal measure $^*\mu$. This means that a union of measurable sets indexed over all $^*N$, which includes the infinite integers, is in $^*F$ and that $^*\mu$ is additive over disjoint unions of sets (in $^*F$) indexed over all $^*N$. But it turns out that countable unions of measurable sets (i.e., those indexed over just N) won't be in $^*F$, i.e., they aren't internal unions, unless they can also be represented as finite unions. So $^*F$ is not normally a σ-field

The concept of Loeb measure gets around this problem by starting in the hyperreals, with an arbitrary hyperreal, internal measurable space X, an internal algebra (not σ-field) A of internal subsets of X, and an internal, finitely additive, nonnegative, hyperreal-valued measure ν: A $\to$ $^*R^+$. One can make an extended real-valued measure μ on A just by "rounding off" ν to the real part of its hyperreal value (when it is finite), denoted $^{\circ}\nu$, and by defining μ to be ∞ on sets where ν has an infinite hyperreal value.

In this notation, with a minor extension by Hensen (1979) Loeb showed that the Caratheodory Extension Theorem (see Ash (1972, p. 19) could be utilized to help prove:

Thm. 1 (Loeb (1975)) Loeb Measure

The extended real-valued function μ has a unique, σ-additive extension $\tilde{\mu}$ to the smallest (external) σ-algebra F containing A. For each S $\in$ F, the value of this extension is given by $\tilde{\mu}(S) = \inf_{\substack{B \in A \\ B \supset S}} \mu(B)$. When $\mu(X)$ is finite, it is also true that

$$\tilde{\mu}(S) = \sup_{\substack{B \in A \\ B \subset S}} \mu(B)$$

and there is a set C $\in$ A with $\tilde{\mu}((S-C) \cup (C-S)) = 0$.

Analogous to the construction of Lebesgue measure, one takes the completion of the σ-field with respect to $\tilde{\mu}$, producing the Loeb measurable sets F_L, and extends $\tilde{\mu}$ to F_L, denoted ν_L, the Loeb measure of the Loeb space (X, F_L).

Via Loeb's result, we now have a σ-additive, extended real valued measure ν_L on a σ-field F, extending $\overset{\circ}{\nu}$ and A, respectively. Further, one can approximate any measurable set S in F by a set in the smaller algebra A, whose symmetric difference with S has Loeb measure equal to zero.

Integration on Loeb spaces is facilitated by lifting theorems. A major result of Loeb stated, in Cutland (p. 552), is rewritten below:

Thm. 2: Lifting Theorem

Let f be an extended, real valued function on X. Then, f is Loeb measurable if and only if there exists an internal, A-measurable function, or random variable, $F: X \to {}^{*}R$ (i.e., all its upper and lower contour sets are in the algebra A), such that ${}^{\circ}F(x) = f(x)$ for ν_L almost every $x \, \varepsilon \, X$. F is called a lifting of f. Further, if both f and F are finitely bounded, $\int f d\nu_L = {}^{\circ *}\int F d\nu$, where "${}^{*}\int$" denotes *integration, the hyperreal counterpart of standard integration.

In the representative problem, which we solve in the next section, "${}^{*}\int$" is just summation over a hyperfinite set, which is an elementary calculation, illustrating the utility of Thm. 2.

Section 4: Application of Loeb Measure to the Problem

To apply the results of sec. 3 to our problem, assume that there is an infinite integer M of agents, so the hyperfinite set of agents can be represented by $\{1, 2, \ldots, M\}$. Following Prescott and Townsend, assume that a real fraction $\lambda_1 = N/L$ of them are the L low risk agents, while a real fraction $\lambda_2 = (1-\lambda_1) = N/H$ of them are the other H high risk agents. Each of the L (or H) low risk (or high risk) agents receives endowments of Z_0 or Z_1, identically and *independently[1]/ distributed among them with real $p(Z=Z_0) = \theta_1$ (or θ_2).

A suitable internal measure space of realizations for either of these two internal Bernoulli endowment processes is easily constructed. For the L low risk agents, the internal measurable space of realizations is the set X of all internal sequences of length L composed of zeroes and ones, where a one in the ith place, i.e., $x_i = 1$, means the i agent was endowed with Z_0 rather than Z_1. The infinite cardinality of X thus equals 2^L. The algebra A is the class of <u>internal</u> subsets of X, denoted *P(X), the hyperreal counterpart of the power set of X. The hyperreal-valued probability weighing function (see Cutland, Sec. 3.3) on X, denoted $\Delta\nu(x)$, is given by elementary probability theory:

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[1]/By transfer, a set of (hyperreal-valued) random variables $(F_i(x))$ is *<u>independent</u> if the joint cumulative distribution function of every hyperfinite internal subset of them is the product of their respective marginal distribution functions on *R.

$$(3) \qquad \Delta \nu(x) = \theta_1^{\sum_{i=1}^{L} x_i} (1-\theta_1)^{L-\sum_{i=1}^{L} x_i}$$

Then, for any set $B \in A$, the internal measure is

$$(4) \qquad \nu(B) = \sum_{x \in B} \Delta \nu(x)$$

In fact, because X is hyperfinite, any internal measure on A has a probability weighing function $\Delta \nu$ (Cutland, p. 556).

The fraction of Z_0 endowments is then an internal, finitely bounded, nonnegative, hyperreal-valued function $F: X \to {}^*[0,1]^+$ given by the sample average:

$$(5) \qquad F(x) = \sum_{i=1}^{L} x_i/L$$

It turns out that any internal function, such as (5), is A-measurable on a hyperfinite space (Cutland, p. 556-7), and thus is a hyperreal-valued random variable on X.

To aid in real world applications, we may demand that both probabilities and random variables be real-valued. To do so, by Thm. 1 we round off ν in (4) to its real part and extend to produce the Loeb measure ν_L associated with (3) and (4). We also round off F in (5) to its real part, producing the real, countable limit sample average

$$(5') \qquad f(x) \equiv {}^\circ F(x)$$

By the Lifting Thm. 2, f in (5') is a Loeb measurable random variable, and the expected value of it, taken with respect to the Loeb measure ν_L, is calculated as:

$$(6) \qquad E(f) \equiv \int_X f d\nu_L = {}^0{}^*\!\int F d\nu = {}^0\!\!\sum_{x\varepsilon X} F(x)\Delta\nu(x)$$

$$= {}^0\!\!\sum_{x\varepsilon X} \sum_{i=1}^{L} (x_i/L)\theta_1^{\sum_{i=1}^{L} x_i}(1-\theta_1)^{L-\sum_{i=1}^{L} x_i}$$

$$= {}^0(1/L)(L\theta_1) = \theta_1$$

where the next to last equality follows from the usual binomial distribution calculation. In fact, the answer would have been the same for any hyperreal θ_1 infinitely close to our hypothesized real θ_1.

As we have just seen, real integration on hyperfinite Loeb spaces is merely hyperfinite summation rounded off to its real part. But without finite boundedness of F, the sum may be an infinite hyperreal and thus will have no real part. It turns out that integration on Loeb spaces still follows the same calculation whenever $\sum_{\substack{x\varepsilon X \\ |F(x)|>N}} |F(x)|\Delta\nu(x)$ is infinitesimal for every infinite

integer N (see Cutland, p. 354). In this case, one says that F is S-integrable, in which case the hyperreal integral of F, on the set where F is infinite, is infinitesimal, so it won't affect the real part of the outcome.

The expected proportion of Z_0-endowed, low risk agents is thus θ_1. What about the variance? The same technique leads to:

$$(7) \qquad \text{var } (f) \equiv \int_X (f-E(f))^2 d\nu_L = \overset{0}{\underset{x \in X}{\sum}} (F(x)-\theta_1)^2 \Delta\nu(x)$$

$$= {}^{\circ}\theta_1(1-\theta_1)/L = 0$$

where the last equality in the above binomial calculation holds because θ_1 is finite (not because it is real) and L is infinite. So the proportion f appears to degenerate to a real constant θ_1, solving our problem. In fact, this can be directly proven using a hyperfinite Chebyshev inequality, proven below.

Thm. 3: <u>Hyperfinite Chebyshev Inequality</u>

Let F: $X \to {}^*R$ be an internal, finitely bounded, hyper-real-valued random variable on the internal, hyperfinite measure space X, equipped with the algebra A of all its internal subsets and the internal measure ν: $A \to {}^*R^+$. Then,

$$(7') \qquad \nu\{x: |F(x)-E(F)| > k\sqrt{\overline{\text{var } (F)}}\} \leqslant 1/k^2,$$

where k is any positive hyperreal, and the expectation and variance are defined by *integration.

Proof: Let G: $X \to {}^*R^+$ be internal and finitely bounded. For any positive hyperreal c,

$$(8) \qquad \overset{*}{\underset{X}{\int}} Gd\nu = \underset{x \in X}{\sum} G(x)\Delta\nu(x) \geqslant \underset{\substack{x \in X \\ G(x) \geqslant c}}{\sum} G(x)\Delta\nu(x) \geqslant c\underset{\substack{x \in X \\ G(x) \geqslant c}}{\sum} \Delta\nu(x)$$

$$= c\nu\{x: G(x) \geqslant c\},$$

where the last set is measurable because internal functions G are always A-measurable on hyperfinite spaces, as noted earlier. Let

$G(x) \equiv (F(x)-E(F))^2$ for some finitely bounded internal F, and let $c = k^2 E(G)$ for arbitrary positive hyperreal k. Substituting into (8), rearranging, and taking the hyperreal counterpart of the square root yields (7').

Of course, Thm. 3 is valid for arbitrary internal measure spaces, rather than just hyperfinite ones, just by replacing hyperfinite sums with ${}^*\!\int$ and requiring that F be A-measurable. But we would rarely need to use the more general theorem, for a large class of standard measure spaces can be constructed from Loeb spaces associated with hyperfinite spaces. For example, Lebesgue measure on $[0,1]$ can be constructed from the hyperfinite grid $X = \{0, \Delta t, 2\Delta t, 3\Delta t, \ldots, (1/\Delta t)\Delta t = 1\}$, where Δt is a positive infinitesimal which divides 1, yielding some positive infinite integer. A is the usual algebra of internal subsets of X, and $\Delta v(x) = \Delta t$, so v in (4) is just the counting measure. It turns out (Cutland, p. 358) that a set C $[0,1]$ is Lebesgue measurable if and only if $\{x: {}^{\circ}x \varepsilon C\}$ is a Loeb measurable subset of X, and the Lebesgue measure of C is the Loeb measure v_L of that set. More generally, Anderson (1982) has shown that arbitrary Radon measures on Hausdorff spaces have an analogous hyperfinite Loeb representation (also see Cutland, sec. 4.2).

In the representative problem, let $k = \text{var}(F)/m = \theta_1(1-\theta_1)/mL$, where m is any positive finite integer. Define $D_m = \{x: |F(x)-\theta_1| \geqslant 1/m\}$. Then, Thm. 3 implies that

$$(9) \qquad v(D_m) \leqslant \theta_1(1-\theta_1)m^2/L.$$

Because the Loeb space associated with X is countably additive, $D = \bigcup_{m=1}^{\infty} D_m$ is Loeb measurable. D is the set of realizations on which $F(x)$ and θ_1 differ in absolute value by no more than some real number. But using subadditivity and (9), compute:

$$(10) \qquad \nu_L(D) \leqslant \sum_{m=1}^{\infty} \nu_L(D_m) \leqslant \sum_{m=1}^{\infty} {}^o\theta_1(1-\theta_1)m^2/L = \sum_{m=1}^{\infty} 0 = 0.$$

Therefore, ${}^oF(x) \equiv f(x) = \theta_1 \quad$ a.e. (ν_L).

We have just established a strong law of large numbers for this hyperfinite Bernoulli case. Furthermore, a nonstandard central limit theorem of Anderson (1976, p. 27) implies that for any specified infinite integer $I \ll L$.

$$(11) \qquad D(\alpha) \equiv \nu\{x: \overline{\sqrt{\theta_1(1-\theta_1)/I}}^{-1} \, (\sum_{i=1}^{I} (x_i/I)-\theta_1)\leqslant\alpha\}$$

is infinitely close to the hyperreal counterpart ${}^*Z(\alpha)$ of the comulative standard normal distribution $Z(\alpha)$. So ${}^oD(\alpha) = Z(\alpha)$ for any real α. In other words, infinite, normalized partial sample averages of this process are essentially Gaussian, a result we could call the Hyperfinite Bernoulli Theorem.

Finally, note that the same derivation establishes that the proportion of high risk agents getting $Z = Z_0$ degenerates to θ_2. So, our problem is solved. Or is it? Remember that we chose to work with an infinite integer M of agents. Suppose that one desired to model an economy with a countable infinity of both low risk and high risk agents, so that all pieces of the model would be real. A countably infinite subset of our hyperfinite set of agents is <u>not</u> internal, nor is a subset of realizations (in X)

which agree on a countably infinite number of components an internal subset of X, i.e., it is not in the algebra A. The full power of Loeb measure can be brought to bear on this problem, for such a subset of X <u>is</u> in the Loeb σ-field containing A.

So suppose we only examine what happens to the Loeb measurable countable subset of low risk agents $N = \{1,2,3,\ldots\}$. Define the finite proportion $F_n(x) = \sum_{i=1}^{n} x_i/n$.

Define $D_n(m) = \{x: |F_n(x)-E(F)| \geqslant 1/m\}$. Note that $\liminf_n$ $D_n(m) \equiv \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} D_j(m) = \{x: |\lim_{n\to\infty} F_n(x)-E(F)| \geqslant 1/m\}$. As a countable union of countable intersections, it is Loeb measurable. As above, the set of realizations on which $\lim_{n\to\infty} F_n(x)$ differs from $E(F)$ by more than some real number is $D = \bigcup_{m=1}^{\infty} \liminf_n D_n(m)$. Relentlessly using subadditivity, compute:

$$(12) \qquad \nu_L(D) \leqslant \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \nu_L\left(\bigcap_{j=n}^{\infty} D_j(m)\right) \leqslant \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \inf_{j \geqslant n} \nu_L(D_j(m))$$

But $\inf_{j \geqslant n} \nu_L(D_j(m)) = \inf_{j \geqslant n} {}^{\circ}\nu(D_j(m)) \leqslant \inf_{j \geqslant n} m^2\theta_1(1-\theta_1)/j = 0$ via the

Chebyshev inequality. Therefore, we obtain,

$$\lim_{n\to\infty}^{\circ} F_n(x) = {}^{\circ}E(F) = \theta_1 \qquad \text{a.e. } (\nu_L).$$

Summarizing, on the Loeb space associated with X, we have shown that ${}^{\circ}F(x) = \lim_{n\to\infty} F_n(x) = {}^{\circ}E(F) = \theta_1$ a.e. (ν_L). We have thus produced a countably additive measure (i.e., ν_L) on a measure space X of realizations, with a countably additive σ-algebra (i.e., the Loeb algebra derived from the internal subsets of X), on which sample fractions are nonstochastic almost everywhere.

References

Albeverio S., Fenstad, J. E., Hoegh-Krohn, R., Lindstrom, T., "Nonstandard Methods in Stochastic Analysis and Mathematical Physics," Academic Press, (1986).

Anderson, R.M., "A Non-Standard Representation For Brownian Motion and Itô Integration," Israel Journal of Mathematics, 25 (1976), 15-46.

Anderson, R.M., "Star-Finite Representation of Measure Spaces," Transactions of the American Mathematical Society, 271 (1982), 667-687.

Ash, R.B., "Real Analysis and Probability," Academic Press, New York, (1972).

Cutland, N.J., "Nonstandard Measure Theory and Its Applications," Bull. London Math Society, 15 (1983), 529-589.

Emmons, D.W. and Yannelis, N.C., "On Perfectly Competitive Economies: Loeb Economies," IMA Preprint #109, Institution for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota, 55455, (Oct. 1984).

Feldman, M. and Gilles C., "An Expository Note on Individual Risk Without Aggregate Uncertainty," Journal of Economic Theory, 35 (1985), 26-32.

Henle, J.M. and Kleinberg, E.M., "Infinitesmal Calculus," MIT Press, Cambridge, Massachusetts, (1979).

Hurd, A. E. and Loeb, P. A., "An Introduction to Nonstandard Real Analysis," Academic Press, (1985).

Judd, K.B., "The Law of Large Numbers With A Continuum of IID Random Variables," Journal of Economic Theory, 35 (1985), 19-25.

Keisler, H.J., "Foundation of Infinitesimal Calculus," Prindle, Weber, and Schmidt, (1976).

Loeb, P., "Conversion from Nonstandard to Standard Measure Spaces and Applications in Probability Theory," Transactions of the American Mathematical Society, 211 (1975), 113-122.

Prescott, E.C., Townsend, R.M., "Pareto Optima and Competitive Equilibria With Adverse Selection and Moral Hazard," Econometrica 52 (1) (1984).