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OPTIMALITY AND MONETARY EQUILIBRIA IN STATIONARY OVERLAPPING GENERATIONS MODELS WITH LONG LIVED AGENTS: GROWTH VERSUS DISCOUNTING

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GROWTH VS. DISCOUNTING IN OLG MODELS

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Abstract

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This paper studies the relationship between the existence and optimality of a monetary steady-state and the nonoptimality of nonmonetary steady-states. We construct a sequence of stationary overlapping generations economies with longer and longer lived generations in which all agents maximize a discounted sum of utilities with a common discount rate. Under some assumptions the following result is established: If the discount rate is greater (less) than the population growth rate, then eventually every nonmonetary steady-state is optimal (non-optimal) and a monetary steady-state does not exist (exists and is optimal).

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List of Symbols

O = capital "oh", same as zero

o = lower case "oh"

ℓ = script "el", l = type "el", 1 = one

K (k) = upper (lower) case "kay"

α = greek lower case "alpha"

ω = greek lower case "omega"

β = greek lower case "beta"

τ = greek lower case "tau"

γ = greek lower case "gamma"

μ = greek lower case "mu"

λ = greek lower case "lambda"

θ = greek lower case "theta"

$\sum$ = greek large "sigma"

I. Introduction

This paper extends previous work on the relationship between the optimality of nonmonetary equilibria and the existence of monetary equilibria in pure exchange stationary overlapping generations (OLG) models. In simple OLG environments (e.g., Wallace [17]) the following relationship exists: A fixed supply valued fiat money equilibrium exists (and there is one that is optimal) if, and only if, the nonmonetary equilibrium is nonoptimal.[1/] Focusing on steady states, a nonmonetary equilibrium is optimal or not depending on whether the interest rate exceeds or falls short of the growth rate. The interest rate in a nonmonetary steady state depends on all aspects of the environment; i.e., preferences and lifetime endowment patterns of members of each generation and also technology in a model with production (e.g., Diamond [7], Cass and Yaari [6]). Further, there may be multiple nonmonetary steady states, some of which may be optimal and others nonoptimal.[2/] In such a case, existence of a monetary steady state depends on aggregate desired assets of the population at an interest rate equal to the growth rate (Gale [9], Cass and Yaari [6]). If desired assets are positive, a monetary steady state will exist, otherwise it will not. As before, this too depends on all aspects of the environment. Consequently, no simple answer can be found in terms of the primitives of an OLG model regarding the optimality of nonmonetary steady states and the existence of a monetary steady state, and the relationship between them. Furthermore, it is difficult to assess whether situations in which nonmonetary steady states are nonoptimal and a monetary steady state exists (and is optimal), or the converse situations arise due to special patterns of endowments and preferences or may arise more generally. This is especially true when agents live many periods and there is heterogeneity within generations.[3/]

The above situation is in marked contrast to a popular class of models of a fixed number of infinitely lived agents (or equivalently, a growing number of families connected by bequest motives) who maximize a discounted sum of utilities with a common, fixed and positive discount rate. The classic one-sector growth model is one in which the steady state interest rate is entirely determined by the discount rate and exceeds the growth rate. Furthermore, competitive equilibria are optimal and monetary equilibria do not exist.

The analysis in the present paper is motivated by the above observations. We consider OLG models with long (but finitely) lived agents who maximize a discounted sum of utilities with a fixed, positive and common discount rate (d). Heterogeneity within generations is permitted. The intuition is that if agents have sufficiently long lives then the interest rate in a nonmonetary steady state ought to be (almost) entirely determined by the discount rate independently of period utility functions and lifetime endowment patterns. Therefore, the optimality of such steady states ought to depend entirely on whether the discount rate exceeds or falls short of the growth rate; i.e., on whether agents exhibit sufficient impatience relative to the economy's growth rate. The same condition may then determine whether or not a monetary steady state exists; i.e., we should expect it to exist when the discount rate is less than the growth rate but not in the contrary case.

The above intuition is formalized in the following manner. We construct a sequence of OLG economies indexed by T, the length of life of each generation. Population growth rate is fixed at n and each generation is taken to consist of H types of agents. Types are distinguished by their one-period utility functions and lifetime endowment patterns which are generated as follows. We take H infinite sequences of nonnegative numbers. The lifetime endowment vector of a type h agent in a T-period lived agent OLG economy (henceforth, OLG(T)) is then taken to be given by the first T elements of the corresponding infinite sequence. We then examine nonmonetary steady state interest rates and the existence of monetary equilibria and establish the following results.

Let r(T) by any nonmonetary steady state interest rate for the OLG(T) economy and let A(T) be the desired aggregate assets of the population at an interest rate equal to the growth rate, n. Then, subject to some additional assumptions, we show the following. As T becomes large, any such sequence $\{r(T)\}$ will exceed n if d is greater than n. Conversely, if d is less than n, then eventually any sequence $\{r(T)\}$ will remain below n. Further, the sequence $\{A(T)\}$ will eventually be bounded away from and above (below) zero if d is less than (greater than) n. The following conclusions emerge from the above results.

(i) With sufficiently long lived agents, if d is greater than n, every nonmonetary steady state is optimal and monetary steady states do not exist. Conversely, if d is less than n, every nonmonetary steady state is nonoptimal and a monetary steady state exists and is optimal. This result suggests the following generalization of Wallace's [17] result: A monetary steady state exists and is optimal if, and only if, every nonmonetary steady state is nonoptimal.

(ii) Therefore, nonmonetary steady states may be nonoptimal generally (as well as optimal, generally). Consequently monetary steady states may exist generally (as well as not exist, generally) in this class of OLG economies with long lived agents. In a sense, neither of these situations seems special, at least in the sense of requiring particular types of endowment patterns. Asymptotically, an important factor is the degree of impatience relative to the growth rate.

We believe that implication (ii) should answer the criticism that Tobin [16] and others have made of OLG models of money. Based on two-period lived agent models, they point out that for a valued fiat money equilibrium to exist, endowments in youth should be relatively larger than in old age and that the life cycle pattern of savings should display savings followed by dissavings. This, they suggest, is unrealistic. Our results show that one does not need lifetime endowment patterns to be tilted in any particular way for monetary equilibria to exist. Rather, what matters is the relationship between the discount rate for consumers and the economy's growth rate.

Our results have obvious implications for the analysis of asset bubbles in OLG economies, as in Tirole [15]. He shows that the existence of such "bubbly" equilibria is intimately connected to the nonoptimality of the equilibria without bubbles and rents. In terms of our characterization, we may say that with long lived agents (and subject to some other assumptions), if the utility discount rate exceeds the growth rate then asset bubbles cannot exist; whereas, in the converse case, they always will exist.

The particular specification of intertemporal preferences adopted here, namely, time-separable with a fixed, positive and common (across agents) rate of time preference, is quite strong. The results from models of infinitely lived agents suggest that time preference plays an important role in intertemporal models (Lucas and Stokey [12]). Since we are considering OLG models with longer and longer lived agents, the specification adopted here seems, as a first step, natural. It permits us to obtain a simple and economically interesting characterization of situations in which a monetary steady state does or does not exist and whether barter steady states are or are not optimal. This characterization is in terms of agents' common rate of time preference and growth rate and is valid for a reasonable and wide class of endowment patterns and heterogeneity within generations when agents are sufficiently long lived.

We suspect that the results established here will carry over for more general preference structures. What we have in mind are recursive (but not necessarily time separable) preferences over consumption streams with a well defined notion of a rate of time preference as in Epstein [8] or Lucas and Stokey [12]. It would seem that for the results described earlier, the important thing is the relationship between a notion of time preference and the growth rate. This need not be restricted to preferences of the discounted sum of utilities type with a common, fixed discount rate.

The particular choice of preferences is also motivated by a desire to compare the determination of interest rates and consumption profiles in OLG economies and a class of models of a fixed number of infinitely lived agents referred to earlier.4/ This topic was pursued in greater detail in an earlier paper (Aiyagari [1]). In that paper, attention was restricted to the case of zero growth and even stronger results were obtained. It was shown that every sequence of nonmonetary steady state interest rates $\{r(T)\}$, in fact, converges to d, and that consumptions at any fixed age converge to permanent income evaluated using d.5/ It has not been possible to obtain such sharp results in the present case even though the intuition for them is strong.

Some justification for focusing on nonmonetary steady states is perhaps desirable here. Each OLG economy is a pure consumption loans economy with no outside assets and is viewed as having a given starting date at which aggregate assets of the population are zero. As Gale [9] shows, for given initial conditions, such an economy can only converge (if it ever does) to a balanced or nonmonetary steady state. We omit consideration of the difficult issue of convergence to a steady state from given initial conditions and focus directly on the behavior of steady states. [6/] Alternatively, and to avoid dealing with initial conditions, each OLG economy could be viewed as having neither beginning nor end. Equilibria are defined by requiring goods markets as well as asset markets to clear and optimality is restricted to be forward looking. Given the underlying stationarity in preferences and endowments, we may focus directly on steady state equilibria (see, Benveniste and Cass [4]).

The rest of this paper is organized as follows: Section II describes the sequence of OLG economies we consider, the assumptions imposed on preferences and endowment patterns, and characterizes the steady states. In Section III, we prove the main results of the paper described before. Section IV concludes. Proofs of some propositions are relegated to Appendix A.

II. Sequence of OLG Economies

The model described here is similar to that in Cass and Yaari [6]. At each date t (t=1,2,...), $(1+n)^t$ agents are born, each of whom lives T periods. At a given date t, agents of different generations are indexed by their current age s, which runs from 0 through (T-1); s = 0 describes the newly born agents. Heterogeneity is introduced by indexing agents also by their type h which runs from 1 through H. The fraction of each generation which is type h is given by γ^h and these fractions sum to one. At the initial date (t=1), there also exist agents born at dates 0, -1, -2, ..., -(T-2).

Endowments are described as follows. For fixed h, let $\{\alpha_s^h\}_{s=0}^{\infty}$ be an infinite sequence of nonnegative numbers. Assumptions on these sequences will be described shortly. Let $\omega_s^h(t,T)$ be the (nonstorable) endowment of agent type h of age s at time t in an OLG model with T period lived agents. Then, we put

$$(2.1) \qquad \omega_s^h(t,T) = \alpha_s^h,$$

h = 1, 2, ..., H and s = 0, 1, ..., T - 1. The sequence $\{\alpha_s^h\}$ for s = 0, 1, ..., (T-1) and h = 1, 2, ..., H describes the distribution of endowments among members of a given generation and across members of different generations. The model is stationary since the population characteristics are time independent, except for growth.

Let $c_s^h(t,T)$ be the consumption of agent type h of age s at time t. Preferences of such an agent are described by

$$(2.2) \qquad \sum_{\tau=0}^{T-s-1} \beta^{\tau} U_h(c_{s+\tau}^h(t+\tau,T))$$

where β is the discount factor and is positive and less than one. The discount rate, d, is given by $(1-\beta)/\beta$ and is positive. Note that the one-period utility function may be different for different agents in a given generation though it is the same for all type h individuals who differ only by dates of birth.

A steady state nonmonetary competitive equilibrium for the above economy is described by the following. Let r_T be the interest rate and let $\ell_s^h(T)$ be claims to consumption (loans due) held by agent (h,s) at any date. Then an agent faces the following sequence of budget constraints.

$$(2.3) \qquad \ell_s^h(T) + \alpha_s^h(T) = c_s^h(T) + \frac{\ell_{s+1}^h(T)}{1 + r_T}$$

$s = 0, 1, \ldots, T - 1$ and $h = 1, 2, \ldots, H$.

$$(2.4) \qquad \ell_0^h(T) = \ell_T^h(T) = 0.$$

Note that all loans are taken to be one-period loans. Further, all loans are inside loans; there are no outside assets in this economy. In addition, preferences are strictly selfish and therefore no bequests are allowed in the budget constraints.

The market clearing condition may be expressed as

$$(2.5) \qquad a_T = \sum_{s=0}^{T-1} \sum_{h=1}^{H} \gamma^h \ell_s^h(T)(1+n)^{-s} = 0$$

where a_T is total assets of the population relative to the size of the young ($s=0$) generation. The first order necessary conditions for a utility maximum subject to the budget constraints are

$$(2.6) \qquad \frac{\beta U_h'(c_{s+1}^h(T))}{U_h'(c_s^h(T))} = \frac{1}{1 + r_T}$$

$s = 0, 1, \ldots, T-2$ and $h = 1, 2, \ldots, H$. A steady state equilibrium consists of r_T^*, $c_s^{*h}(T)$, $\ell_s^{*h}(T)$ for $s = 0, 1, \ldots, T - 1$ and $h = 1, 2, \ldots, H$ which satisfy (2.3)-(2.6).

The above economy is well defined for every T. This is because the sequences of endowments in (2.1) are taken to be the truncations of the H infinite sequences $\{\alpha_s^h\}_{s=0}^{\infty}$, truncated at (T-1). Similarly, the preferences defined in (2.2) can be extended naturally as T is increased.

By virtue of (2.4), the sequence of budget constraints in (2.3) can be collapsed into the following single lifetime budget constraint:

$$(2.7)\qquad \sum_{s=0}^{T-1} \frac{1}{(1+r_T)^s} c_s^h(T) = \sum_{s=0}^{T-1} \frac{\alpha_s^h}{(1+r_T)^s}$$

$h = 1, 2, \ldots, H$.

We now develop an alternative version of the market clearing condition (2.5). From (2.3) and (2.4) we have

$$\ell_{T-1}^h(T) = c_{T-1}^h(T) - \alpha_{T-1}^h$$

$$\ell_{T-2}^h(T) = c_{T-2}^h(T) - \alpha_{T-2}^h + \frac{\ell_{T-1}^h(T)}{1+r_T}$$

$$= c_{T-2}^h(T) - \alpha_{T-2}^h + \frac{c_{T-1}^h(T)-\alpha_{T-1}^h}{1+r_T}.$$

Proceeding backwards in this way we get

$$\ell_1^h(T) = c_1^h(T) - \alpha_1^h + \frac{c_2^h(T)-\alpha_2^h}{1+r_T} + \ldots + \frac{c_{T-1}^h(T)-\alpha_{T-1}^h}{(1+r_T)^{T-2}}.$$

Substituting the above expressions in (2.5), rearranging terms and noting that $\ell_0^h(T) = 0$, we obtain

$$(2.8)\qquad a_T = \sum_{s=1}^{T-1} (1+n)^{-s} \sum_{j=0}^{s-1} \left(\frac{1+n}{1+r_T}\right)^j \sum_{h=1}^{H} \gamma^h \left(c_s^h(T)-\alpha_s^h\right).$$

If $r_T \neq n$, the above can be simplified by noting that

$$(1+n)^{-s}\sum_{j=0}^{s-1}\left(\frac{1+n}{1+r_T}\right)^j = \frac{\{(1+r_T)^{-s}-(1+n)^{-s}\}(1+r_T)}{n-r_T}.$$

Hence, we have

$$(2.9) \qquad \left(\frac{n-r_T}{1+r_T}\right)a_T(r_T \neq n) = \sum_{s=1}^{T-1}\{(1+r_T)^{-s}-(1+n)^{-s}\}\sum_{h=1}^{H}\gamma^h(c_s^h(T)-\alpha_s^h)$$

$$= -\sum_{s=0}^{T-1}(1+n)^{-s}\sum_{h=1}^{H}\gamma^h(c_s^h(T)-\alpha_s^h)$$

in view of the budget constraints (2.7) multiplied by γ^h and then summed over h. It follows that for $r_T \neq n$ to be an equilibrium, the market clearing condition (2.5) may be replaced by

$$(2.10) \qquad \sum_{s=0}^{T-1}(1+n)^{-s}\sum_{h=1}^{H}\gamma^h(c_s^h(T)-\alpha_s^h) = 0$$

which is the economy's resource constraint.

When $r_T = n$, the budget constraints (2.7) automatically imply the resource constraint (2.10). Aggregate assets a_T from (2.8) may be expressed as

$$(2.11) \qquad a_T(r_T=n) = \sum_{s=0}^{T-1}s(1+n)^{-s} \sum_{h=1}^{H}\gamma^h(c_s^h(T)-\alpha_s^h)$$

which may or may not equal zero. The above considerations lead to the following:

Definition 1 (nonmonetary steady state). A nonmonetary steady state for an OLG(T) economy consists of r_T and $\{c_s^h(T)\}$ for s = 0, 1, ..., T - 1 and h = 1, 2, ..., H which satisfy either:

(i) equations (2.6), (2.7), and (2.10) with $r_T \neq n$, or

(ii) equations (2.6), (2.7), and (2.11) at zero with $r_T = n$.

Correspondingly, a monetary steady state may be defined as follows:

Definition 2 (monetary steady state). A monetary steady state for an OLG(T) economy consists of $\{c_s^h(T)\}$ for s = 0, 1, ..., T - 1, and h = 1, 2, ..., H that satisfy (2.6) and (2.7) with $r_T = n$ and such that $a_T(r_T = n)$ from (2.11) is positive.

Such a steady state can be supported as an equilibrium by a fixed positive quantity of fiat money distributed (appropriately) among the initial old (s=1,2,...,T-1) generations. If M represents the fixed stock of money and p(t) is the price level at time t, then in such an equilibrium

$$\frac{M}{p(t)(1+n)^t} = a_T(r_T = n) > 0$$

and obviously, $p(t)/p(t+1) = 1 + r_T = 1 + n$.

We now describe the assumptions imposed on endowment sequences and preferences.

Assumption 1. The sequence $\{\sum_{h=1}^{H} \gamma^h \alpha_s^h\}$ is bounded and bounded away from zero; i.e.,

$$0 < a \leq \sum_{h=1}^{H} \gamma^h \alpha_s^h \leq A < \infty.$$

Even though we assume a single good at each date, its endowment could be thought of as income from a bundle of different goods whose relative prices are fixed (across the sequence of economies), possibly via a within period fixed coefficients technology for transforming goods. With this interpretation, the above assumption on endowments is not unreasonable.

Assumption 2. $U_h(c)$ is twice continuously differentiable, strictly increasing and strictly concave and satisfies:

$$\lim_{c\to 0} U_h'(c) = \infty, \lim_{c\to\infty} U_h'(c) = 0.$$

With this assumption, we may define consumption as a function of marginal utility (denoted by p) implicitly as follows:

$$p = U_h'(c^h(p)).$$

It is then apparent that $c^h(p)$ is continuously differentiable, strictly decreasing and satisfies:

$$\lim_{p\to 0} c^h(p) = \infty, \lim_{p\to 0} c^h(p) = 0.$$

The elasticity of marginal utility, denoted $\mu^h(p)$, can be expressed as

$$\mu^h(p) = \frac{-c^h(p)}{p(dc^h/dp)} > 0.$$

In a context with uncertainty, $\mu^h(p)$ would be the measure of relative risk aversion. The next assumption puts upper bounds on the H functions, $\mu^h(p)$.

Assumption 3. $\mu^h(p) < \mu < \infty$. Further, μ is "small."

How large we can allow μ to be depends on the other parameters of the model; namely, β, n, and the bounds a, A on the endowment sequence. While specific formulas can be given, they do not appear to be easily interpretable and hence are avoided. This assumption also permits us to put bounds on the growth rate of an agent's consumption relative to the population growth rate and thereby on per-capita consumption demand relative to per-capita endowment. When μ is small we can show that (see the discussion after proposition 1), at an interest rate equal to the population growth rate, the growth rate of an agent's consumption is greater (less) than the population population growth rate if the discount rate is less (greater) than the population growth rate.

III. Nonmonetary Steady states

To recapitulate, let r_T be any nonmonetary steady state interest rate. We will show that if the discount rate d $(d=(1-\beta)/\beta)$ is less than n, then eventually the sequence $\{r_T\}$ stays below n; whereas if d is greater than n, then the sequence $\{r_T\}$ eventually stays above n. We will also examine the sequence $\{a_T(r_T=n)\}$ given by (2.11) and show that eventually it is bounded away from and stays above (below) zero if d is less than (greater than) n. It follows that for all sufficiently large T, every nonmonetary steady state is optimal if d exceeds n and nonoptimal if d is less than n. Further, a monetary steady state exists for all T sufficiently large if, and only if, d is less than n; i.e., if, and only if, every nonmonetary steady state is nonoptimal.

In what follows, we do not worry about existence of a steady state r_T for each OLG(T). This can easily be guaranteed if, in each such economy, there is some agent type h for whom α_s^h

is positive in at least two periods, s (see Gale [9], p. 34). Therefore, we proceed as if a balanced steady state r_T exists for each OLG(T).

We now make some changes in notation and establish a preliminary result which is used repeatedly in the main proofs. Let

$$(3.1) \qquad \lambda_T = 1/\beta(1+r_T) = \frac{1+d}{1+r_T}$$

$$(3.2) \qquad \lambda^* = 1/\beta(1+n) = \frac{1+d}{1+n}$$

$$(3.3) \qquad p_s^h(T) = U_h'(c_s^h(T)).$$

From equation (2.6) and assumption 2, we can then write

$$(3.4) \qquad p_{s+1}^h(T) = \lambda_T p_s^h(T)$$

$$(3.5) \qquad c_s^h(T) = c^h(p_s^h(T)).$$

<u>Proposition 1.</u>

(i) If $\lambda_T < 1$, then

$$(3.6) \qquad c_{s+1}^h(T)/c_s^h(T) > \lambda_T^{-1/\mu}$$

whereas,

(ii) if $\lambda_T > 1$, then

$$(3.7) \qquad c_{s+1}^h(T)/c_s^h(T) < \lambda_T^{-1/\mu}.$$

<u>Proof.</u> We will omit the indexes h and T in this proof as these are not relevant and should not cause any confusion.

$$\ln c(\lambda p) = \ln c(p) + \ln \lambda \left. \frac{d \ln c(\lambda p)}{d \ln \lambda} \right|_{\hat{\lambda}}$$

$$= \ln c(p) + \ln \lambda \left(\frac{\lambda p c'(\lambda p)}{c(\lambda p)}\right)_{\hat{\lambda}}$$

$$= \ln c(p) - \ln \lambda / \mu(\hat{\lambda} p)$$

where $\hat{\lambda}$ is between λ and one. The inequalities (3.6) and (3.7) follow from this and assumption 3 because,

$$\frac{c_{s+1}}{c_s} = \frac{c(\lambda p_s)}{c(p_s)}.$$

The impact of assumption 3 can now be explained as follows. When the interest rate r_T equals the growth rate n, λ_T equals λ^* which will be less than one if d is less than n. In this case, having a small μ makes lifetime consumptions increase more rapidly than the economy's growth rate. This is because,

$$c_{s+1}/c_s > (\lambda^*)^{-1/\mu} = \left(\frac{1 + n}{1 + d}\right)^{1/\mu} > 1 + n.$$

Conversely, if d is greater than n, λ^* will be greater than unity and having a small μ makes consumptions decrease faster than the economy's growth rate. This happens because,

$$c_{s+1}/c_s < (\lambda^*)^{-1/\mu} = \left(\frac{1 + n}{1 + d}\right)^{1/\mu} < 1 + n.$$

These facts will turn out to be decisive in evaluating the behavior of $a_T(r_T=n)$ and in analyzing the behavior of interest rates.

<u>Proposition 2.</u>

(i) If $d < n$, then $a_T(r_T=n)$ diverges to plus infinity.

(ii) If $d > n$, then $\limsup_{T\to\infty} a_T(r_T=n) < 0$.

Proof. In Appendix A.

Proposition 3.

(i) If $d < n$, a monetary steady state exists for all T sufficiently large.

(ii) If $d > n$, a monetary steady state does not exist for any T sufficiently large.

Proof. Follows from the previous proposition and the definition of a monetary steady state.

We will now show that if $d < n$, then eventually every nonmonetary steady state is nonoptimal. Conversely, if $d > n$, then eventually every nonmonetary steady state will be optimal.

Proposition 4. Let $d < n$ and $\{r_T\}$ be any sequence of equilibrium interest rates. Then

(i) the sequence $\{r_T\}$ is bounded.

(ii) If $\{r_{T_k}\}$ is any convergent subsequence converging to r, then $r < n$.

Proof. In Appendix A.

Proposition 5. If $d < n$, then eventually every nonmonetary steady state is nonoptimal.

Proof. Proposition 4 shows that if $\{r_T\}$ is any sequence of equilibrium interest rates then $\limsup r_T < n$, which proves the proposition.

We now consider what happens when the utility discount rate d exceeds the growth rate n. We wish to show that eventually every steady state is optimal.

Proposition 6. Suppose that $d > n$. Then eventually every nonmonetary steady state is optimal. That is, if $\{r_T\}$ is any sequence of such equilibrium interest rates, then for all T sufficiently large, $r_T > n$.

Proof. In Appendix A.

IV. Conclusion

In this paper, we have provided a simple characterization of the optimality of nonmonetary steady states and used this to extend earlier results on the relationship between nonoptimality of nonmonetary steady states and the existence of a monetary steady state, in the context of a class of OLG models with long lived agents and population growth. Preferences are restricted to be of the discounted sum of utilities type with a fixed and common discount rate but heterogeneity within generations regarding one-period utility functions and lifetime endowment patterns is permitted. We are able to establish under additional assumptions that when agents are sufficiently long lived:

(i) If the utility discount rate exceeds the growth rate, then every nonmonetary steady state will be optimal and a monetary steady state will not exist.

(ii) Conversely, if the utility discount rate is less than the growth rate, then every nonmonetary steady state will be nonoptimal and a monetary steady state always exists.

One implication of these results is that an optimal steady state always exists, either with or without a positively valued fiat money (see Benveniste and Cass [4]). Another, and to us more interesting, implication is that in this class of OLG economies with long lived agents, optimality and existence of a monetary steady state seem to hinge on the relationship between the rate of time preference exhibited by agents and the population growth rate. One does not need special patterns of lifetime endowments to generate nonoptimal steady states and valued fiat money equilibria. In this sense, our results dilute one of the criticisms of OLG models of money made by Tobin [16] and others. Based on an analysis of two-period lived generation models, they suggest that the existence of a monetary steady state requires that agents save in early periods of life and then dissave; or that lifetime endowments be heavily tilted towards earlier ages. Instead, our analysis suggests that when agents are long lived, the relevant consideration is the impatience of consumers relative to the economy's growth rate. In the absence of a priori knowledge about which way the relationship goes, one can only conclude that optimality and nonexistence of a monetary steady state are perhaps, as general (given our specification of intertemporal preferences) as nonoptimality and existence of a monetary steady state. That is, neither of these situations is, in some sense, special. Our results also suggest the following: A monetary steady state exists (and is optimal) if, and only if, every non-monetary steady state is nonoptimal. This is obviously reminiscent of earlier results (e.g., Wallace [17]).

Footnotes

1/A nonmonetary equilibrium is also variously referred to as "balanced" by Gale [9], "real" by Kehoe and Levine [10], and "barter" by Cass, Okuno, and Zilcha [5]. For us, a nonmonetary equilibrium is one in which aggregate assets of the population are zero at each date. A monetary equilibrium is one with a fixed positive supply of valued fiat (outside) money. Note that in our terminology, in contrast to Kehoe and Levine [10], golden rule steady states with "negative" money are not monetary equilibria.

2/Kehoe and Levine [11] show that, in general, there is an odd number of such steady states.

3/See, for instance, Tobin's [16] critique of OLG models of money. He criticizes the two period model because it requires saving followed by dissaving for a monetary equilibrium to exist; i.e., requires endowments to be tilted in favor of earlier ages.

4/This class of models of a fixed number of infinitely lived agents may be viewed as arising out of OLG models of the type considered here with bequest motives and operative bequests. However, this transformation is well-defined only when the discount rate d exceeds the population growth rate n. For this case and in light of the results in this paper, the contrast between the two frameworks noted previously is not as great.

5/It follows that eventually every nonmonetary steady state is optimal. It was also shown that eventually a monetary steady state never exists.

6/ Some discussion of these is contained in Kehoe and Levine [10].

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Appendix A

In the proofs that follow, sums over s are always taken over s = 0 through (T-1) and sums over h from 1 through H.

Proof of proposition 2

(i) Let $d < n$. Using the budget constraint (2.7) and the inequality (3.6), we have,

$$\sum_s (1+n)^{-s} \alpha_s^h = \sum_s (1+n)^{-s} c_s^h(T)$$
$$< c_{T-1}^h(T) \ (\lambda^*)^{(T-1)/\mu} \sum_s (1+n)^{-s} (\lambda^*)^{-s/\mu}.$$

Therefore, we have,

$$\sum_s s(1+n)^{-s} \sum_h \gamma^h c_s^h(T) > (T-1)(1+n)^{-(T-1)} \sum_h \gamma^h c_{T-1}^h(T)$$
$$> \frac{(T-1)(1+n)^{-(T-1)} (\lambda^*)^{-(T-1)/\mu} \sum_s (1+n)^{-s} \sum_h \gamma^h \alpha_s^h}{\sum_s (1+n)^{-s} (\lambda^*)^{-s/\mu}}$$
$$\xrightarrow{T} \infty$$

for small enough μ because $(1+n)(\lambda^*)^{1/\mu} = (1+d)^{1/\mu}(1+n)^{1-1/\mu}$ will be less than one.

Further, $\sum_s s(1+n)^{-s} \sum_h \gamma^h \alpha_s^h$ is bounded because $n > d > 0$. It follows from (2.11) that $a_T(r_T=n) \xrightarrow{T} \infty$.

(ii) Let $d > n$. Then,

$$a_T(r_T=n) = \sum_s s(1+n)^{-s} \sum_h \gamma^h (c_s^h(T) - \alpha_s^h)$$
$$< \sum_s s(1+n)^{-s} \sum_h \gamma^h c_0^h(T)(\lambda^*)^{-s/\mu} - a \sum_s s(1+n)^{-s}$$

(because of (3.7) and assumption 1)

$$< A(\textstyle\sum_s (1+n)^{-s}) \sum_s s(1+n)^{-s}(\lambda^*)^{-s/\mu} - a \sum_s s(1+n)^{-s}$$

(because of the budget constraint (2.7) and assumption 1.)

If $n = 0$, then the sequence on the right diverges to minus ∞. If $n > 0$, then the sequence converges to a negative limit for small enough μ. This is because $(1+n)(\lambda^*)^{1/\mu} = (1+d)^{1/\mu}(1+n)^{1-1/\mu}$ can be made quite large by making μ small, since $d > n$.

Lastly, suppose that $-1 < n < 0$. In this case,

$$\textstyle\sum_s s(1+n)^{-s} / \sum_s (1+n)^{-s} \to +\infty$$

and hence,

$$a_T(r_T = n) \to -\infty.$$

<u>Proof of proposition 4.</u>

(i) Suppose to the contrary that $\{r_T\}$ is unbounded above. Then it has a subsequence diverging to $+\infty$. To avoid clutter in notation, we suppose that $\{r_T\} \to +\infty$ and hence $\{\lambda_T\} \to 0$. From the budget constraint (2.7) and equation (3.6) we have,

$$\textstyle\sum_s (\beta\lambda_T)^s \alpha_s^h = \sum_s (\beta\lambda_T)^s c_s^h(T)$$

$$< c_{T-1}^h(T)(\lambda_T)^{(T-1)/\mu} \textstyle\sum_s (\beta\lambda_T)^s (\lambda_T)^{-s/\mu}.$$

It then follows from the resource constraint (2.10) that,

$$\text{(A.1)} \quad \textstyle\sum_s \sum_h (1+n)^{-s} \gamma^h \alpha_s^h = \sum_s \sum_h (1+n)^{-s} \gamma^h c_s^h(T)$$

$$> (1+n)^{-(T-1)} \sum_h \gamma^h c^h_{T-1}(T)$$

$$> \frac{(1+n)^{-(T-1)}(\lambda_T)^{-(T-1)/\mu} \sum_s \sum_h (\beta\lambda_T)^s \gamma^h \alpha^h_s}{\sum_s (\beta\lambda_T)^s (\lambda_T)^{-s/\mu}}$$

Now, let $\theta_T = \beta\lambda_T^{1-1/\mu}$. Since $\lambda_T \to 0$ either $\theta_T \to 0$ or $\theta_T \to \infty$ or $\theta_T \equiv 1$. If $\theta_T \to 0$ the relevant terms on the right hand side can be written as,

$$(\textstyle\sum_s \theta_T^s)^{-1}(\lambda_T^{1/\mu}(1+n))^{-(T-1)} \to \infty.$$

If $\theta_T \to \infty$ then the same expression can be written as,

$$\theta_T^{T-1}(\textstyle\sum_s \theta_T^s)^{-1}(\beta\lambda_T(1+n))^{-(T-1)} \to \infty.$$

Lastly, if $\theta_T \equiv 1$, we can rewrite the right hand side as,

$$T^{-1}(\beta\lambda_T(1+n))^{-(T-1)} \to \infty.$$

In all cases, we get a contradiction since $n > d > 0$ and therefore, the left side remains bounded; hence the sequence $\{r_T\}$ must be bounded above. It is obviously bounded below by minus one and hence bounded.

(ii) By (i), a convergent subsequence exists. Again, to avoid clutter in notation, we assume that $\{r_T\}$ itself converges to r. We will show separately that r cannot exceed n or equal n.

(a) <u>r cannot exceed n:</u> Assume the contrary. Then,

$$\lambda = \frac{1+d}{1+r} < \lambda^* = \frac{1+d}{1+n} < 1.$$

Since $\{\lambda_T\}$ converges to λ which is less than one, we can apply (3.6) in the budget constraint (2.7) and use it with the resource constraint (2.10) to obtain the inequality (A.1), just as in (i). Now, let θ_T be as defined in (i) above and,

$$\theta = \beta(\lambda)^{1-1/\mu}$$

$$\theta^* = \beta(\lambda^*)^{1-1/\mu} > 1$$

by assumption (3). If $\theta > 1$, then eventually $\theta_T > 1$ and the relevant terms on the right side of (A.1) can be written

$$\frac{(1+n)^{-(T-1)}\theta_T^{T-1}}{(\beta\lambda_T)^{T-1}\sum_s \theta_T^s} \to +\infty$$

because, $\{\lambda_T\} \to \lambda < \lambda^*$ and $\theta_T \to \theta > 1$. If $\theta \leq 1$, then $\theta < \theta^*$ and hence eventually $\theta_T < \theta^*$. It follows that the right side of (A.1) can be rewritten as,

$$\frac{(\theta^*)^{T-1}(\lambda_T)^{-(T-1)/\mu}}{(\lambda^*)^{-(T-1)/\mu}\sum_s \theta_T^s} > \frac{(\theta^*)^{T-1}}{\sum_s (\theta^*)^s}\left(\frac{\lambda^*}{\lambda_T}\right)^{(T-1)/\mu}$$

$$\to +\infty$$

because $\lambda_T \to \lambda < \lambda^*$ and $\theta^* > 1$.

In all cases, we arrive at a contradiction which shows that r cannot exceed n.

(b) <u>$r \neq n$</u>. Suppose to the contrary that $r = n$. Then, $\{\lambda_T\} \to \lambda^* < 1$. Inequality (A.1) is therefore again applicable and using the notation from part (a), we have

$$(\beta\lambda_T)^{T-1}(1+n)^{T-1} > \frac{\theta_T^{T-1} \sum_s \sum_h (\beta\lambda_T)^s \gamma^h \alpha_s^h}{\sum_s \theta_T^s \sum_s \sum_h (1+n)^{-s} \gamma^h \alpha_s^h}.$$

Since, $\theta_T + \theta^* > 1$, it follows that $(\beta\lambda_T(1+n))^T$ is bounded below and away from zero by some $m > 0$.

Next, using (3.6) in the resource constraint (2.10) we have

$$\text{(A.2)} \qquad \sum_s (1+n)^{-s} \sum_h \gamma^h \alpha_s^h = \sum_s (1+n)^{-s} \sum_h \gamma^h c_s^h(T)$$

$$< \left(\sum_h \gamma^h c_{T-1}^h(T)\right)(\lambda_T)^{\frac{T-1}{\mu}} \sum_s (1+n)^{-s} (\lambda_T)^{-s/\mu}.$$

Now, multiply the budget constraint (2.7) by γ^h and sum over h to obtain

$$\sum_s (\beta\lambda_T)^s \sum_h \gamma^h \alpha_s^h = \sum_s (\beta\lambda_T)^s \sum_h \gamma^h c_s^h(T)$$

$$> (\beta\lambda_T)^{T-1} \sum_h \gamma^h c_{T-1}^h(T)$$

$$> \frac{\theta_T^{T-1} \sum_s (1+n)^{-s} \sum_h \gamma^h \alpha^h}{\sum_s (1+n)^{-s} (\lambda_T)^{-s/\mu}}.$$

Therefore, we have

$$(\beta\lambda_T)^{T-1}(1+n)^{T-1} < \frac{\sum_s (1+n)^{-s} (\lambda_T)^{-s/\mu} \sum_s (\beta\lambda_T)^s \sum_h \gamma^h \alpha_s^h}{(1+n)^{-(T-1)} (\lambda_T)^{\frac{-(T-1)}{\mu}} \sum_s (1+n)^{-s} \sum_h \gamma^h \alpha_s^h}.$$

Since $(1+n)^{-1}(\lambda_T)^{-1/\mu} + \theta^* > 1$, it follows that $[\beta\lambda_T(1+n)]^{T-1}$ is bounded above by some $M < \infty$. Therefore, we have,

(A.3) $$0 < m \leq [\beta\lambda_T(1+n)]^{T-1} \leq M < \infty.$$

Now, if $\lambda_T < \lambda^*$, then for any j such that $0 \leq j \leq T-1$ we have

$$1 \geq [\beta\lambda_T(1+n)]^j \geq [\beta\lambda_T(1+n)]^{T-1} \geq m > 0.$$

Therefore, for any s such that $0 < s \leq T-1$ we have

(A.4) $$s \geq \sum_{j=0}^{s-1} [\beta\lambda_T(1+n)]^j \geq s\, m.$$

Conversely, if $\lambda_T > \lambda^*$, then for $0 \leq j \leq T-1$, we have

$$M \geq [\beta\lambda_T(1+n)]^{T-1} \geq [\beta\lambda_T(1+n)]^j \geq 1$$

and hence, for s such that $0 < s \leq T-1$, we obtain,

(A.5) $$s\, M \geq \sum_{j=0}^{s-1} [\beta\lambda_T(1+n)]^j \geq s.$$

Using, (A.2), (A.4), and (A.5) in the alternative form of the loan market clearing condition given by (2.8) and (2.5), we have

(A.6) $$M \sum_s s(1+n)^{-s} \sum_h \gamma^h \alpha_s^h \geq \sum_s (1+n)^{-s} \sum_{j=0}^{s-1} [\beta\lambda_T(1+n)]^j \sum_h \gamma^h \alpha_s^h$$

$$= \sum_s (1+n)^{-s} \sum_{j=0}^{s-1} [\beta\lambda_T(1+n)]^j \sum_h \gamma^h c_s^h(T)$$

$$\geq m \sum_s s(1+n)^{-s} \sum_h \gamma^h c_s^h(T)$$

$$> m(T-1)(1+n)^{-(T-1)} \sum_h \gamma^h c^h_{T-1}(T)$$

$$> \frac{m(T-1)(1+n)^{-(T-1)}(\lambda_T)^{\frac{-(T-1)}{\mu}} \sum_s (1+n)^{-s} \sum_h \gamma^h \alpha^h_s}{\sum_s (1+n)^{-s}(\lambda_T)^{-s/\mu}}.$$

Since $(1+n)^{-1}(\lambda_T)^{-1/\mu} \to \theta^* > 1$, it follows that the right side of (A.6) diverges to $+\infty$ whereas the left side is bounded since $n > d > 0$. This contradiction establishes that $r \neq n$.

Putting the results of parts (a) and (b) together, we conclude that $r < n$, which proves (ii).

Proof of proposition 6.

Suppose to the contrary that there are infinitely many T's for which $r_T < n$. Then there is a subsequence $\{r_{T_k}\}$ converging to r less than or equal to n and, as before, to avoid notational clutter, we will proceed as if $\{r_T\}$ converges to r less than or equal to n. Therefore, we have

$$\lambda_T = \frac{1 + d}{1 + r_T} \to \lambda = \frac{1 + d}{1 + r} \geq \lambda^* = \frac{1 + d}{1 + n} > 1.$$

Eventually, therefore, λ_T exceeds one and we may use inequality (3.7) in the form,

$$\text{(A.7)} \qquad c^h_s(T) \leq c^h_0(T)(\lambda_T)^{-s/\mu}.$$

There are several cases to consider and we do so one by one.

(i) $r < n < 0$. Using (A.7) in the budget constraint (2.7) we have

$$\sum_s (\beta\lambda_T)^s \alpha_s^h = \sum_s (\beta\lambda_T)^s c_s^h(T)$$

$$< c_0^h(T) \sum_s (\beta\lambda_T)^s (\lambda_T)^{-s/\mu}.$$

Using the above in the resource constraint (2.10) we get,

$$\text{(A.8)} \qquad A > \frac{\sum_s (1+n)^{-s} \sum_h \gamma^h \alpha_s^h}{\sum_s (1+n)^{-s}} = \frac{\sum_s (1+n)^{-s} \sum_h \gamma^h c_s^h(T)}{\sum_s (1+n)^{-s}}$$

$$> \frac{\sum_h \gamma^h c_0^h(T)}{\sum_s (1+n)^{-s}}$$

$$> \frac{\sum_s (\beta\lambda_T)^s \sum_h \gamma^h \alpha_s^h}{\sum_s (1+n)^{-s} \sum_s (\beta\lambda_T)^s (\lambda_T)^{-s/\mu}}$$

$$> \frac{a \sum_s (\beta\lambda_T)^s}{\sum_s (1+n)^{-s} \sum_s (\beta\lambda_T)^s (\lambda_T)^{-s/\mu}}.$$

Since, $n < 0$, $\sum_s (1+n)^{-s} < T(1+n)^{-(T-1)}$. If $\beta(\lambda_T)^{1-1/\mu} < 1$ then $\sum_s (\beta\lambda_T)^s (\lambda_T)^{-s/\mu} < T$. In this case, (A.8) can be rewritten as,

$$\text{(A.9)} \qquad A > a(\beta\lambda_T)^{T-1}(1+n)^{T-1}/T^2.$$

Now, $\beta\lambda_T(1+n) \to \beta\lambda(1+n) = \frac{1+n}{1+r} > 1$ and hence the right hand side diverges to $+\infty$ resulting in a contradiction. On the other hand, if $\beta(\lambda_T)^{1-1/\mu} > 1$, then we have

$$\sum_s(\beta\lambda_T)^s(\lambda_T)^{-s/\mu} < T(\beta\lambda_T)^{T-1}(\lambda_T)^{-(T-1)/\mu}$$

and (A.8) becomes

$$\text{(A.10)}\quad A > \frac{a(1+n)^{T-1}}{T(\lambda_T)^{-(T-1)/\mu}} > \frac{a(1+n)^{T-1}}{T(\lambda^*)^{-(T-1)/\mu}} = \frac{a}{T}\left((1+n)(\lambda^*)^{1/\mu}\right)^{T-1}.$$

Again we arrive at a contradiction because the right side of (A.10) diverge to $+\infty$. This is because, by assumption 3, μ is small and hence $(1+n)(\lambda^*)^{1/\mu}$ will be greater than one.

(ii) $\underline{r < 0 < n}$. Since $n > 0$, $\sum_s(1+n)^{-s}$ converges. If $r < 0$, then $\beta\lambda_T + \beta\lambda > 1$. If $\beta(\lambda_T)^{1-1/\mu} < 1$, then (A.8) becomes,

$$A > \frac{a(\beta\lambda_T)^{T-1}}{T\sum_s(1+n)^{-s}}$$

whereas, if $\beta(\lambda_T)^{1-1/\mu} > 1$, then (A.8) becomes,

$$A > \frac{a(\beta\lambda_T)^{T-1}}{T(\beta\lambda_T)^{T-1}\lambda_T^{-(T-1)/\mu}\sum_s(1+n)^{-s}}.$$

In either case, the right side of (A.8) diverges to $+\infty$ resulting in a contradiction.

If $r = 0$ so that $\beta\lambda_T + \beta\lambda = 1$, then $\beta(\lambda_T)^{1-1/\mu} + \beta(\lambda)^{1-1/\mu} = \beta^{1/\mu} < 1$, so that $\sum_s(\beta\lambda_T)^s(\lambda_T)^{-s/\mu}$ converges. We therefore have to investigate the behavior of

$$\sum_s(\beta\lambda_T)^s = \frac{(\beta\lambda_T)^T-1}{\beta\lambda_T-1}.$$

If $(\beta\lambda_T)^T$ does not converge to one, then obviously the above expression diverges to $+\infty$. So, suppose that $(\beta\lambda_T)^T$ also converges to one. If $\beta\lambda_T > 1$ we have

$$\sum_s (\beta\lambda_T)^s > T$$

whereas if $\beta\lambda_T < 1$ we have

$$\sum_s (\beta\lambda_T)^s > T(\beta\lambda_T)^{T-1}.$$

In either case, the right side of (A.8) diverges to $+\infty$ which is a contradiction.

(iii) $\underline{0 < r < n}$. Since $r_T < n$, we may use (A.7) in the form

$$\text{(A.11)} \qquad c_s^h(T) < c_0^h(T)(\lambda_T)^{-s/\mu} < c_0^h(T)(\lambda^*)^{-s/\mu}.$$

From the resource constraint (2.10) we have

$$\begin{aligned}\text{(A.12)} \qquad \frac{(1+n)A}{n} &> \sum_s (1+n)^{-s} \sum_h \gamma^h \alpha_s^h \\ &= \sum_s (1+n)^{-s} \sum_h \gamma^h c_s^h(T) \\ &> \sum_h \gamma^h c_0^h(T).\end{aligned}$$

We now use the alternative form of the loan market clearing condition given by (2.5) and (2.8). Since $r_T < n$, we have

$$s\left(\frac{1+n}{1+r_T}\right)^s > \sum_{j=0}^{s-1}\left(\frac{1+n}{1+r_T}\right)^j > s.$$

Using this in (2.8) and (2.5), we obtain

$$\sum_s s(1+r_T)^{-s} \sum_h \gamma^h c_s^h(T) > \sum_s (1+n)^{-s} \sum_j \left(\frac{1+n}{1+r_T}\right)^j \sum_h \gamma^h c_s^h(T)$$

$$= \sum_s (1+n)^{-s} \sum_j \left(\frac{1+n}{1+r_T}\right)^j \sum_h \gamma^h \alpha_s^h$$

$$> \sum_s s(1+n)^{-s} \sum_h \gamma^h \alpha_s^h$$

$$> a \sum_s s(1+n)^{-s}.$$

Therefore, we can write

$$\text{(A.13)} \qquad a \sum_s s(1+n)^{-s} < \sum_s s(1+r_T)^{-s} \sum_h \gamma^h c_s^h(T)$$

$$< \sum_s s(\lambda^*)^{-s/\mu} \sum_h \gamma^h c_0^h(T)$$

$$< (1+n)\ A \sum_s s(\lambda^*)^{-s/\mu}/n.$$

The steps follow because $r_T + r > 0$ and hence $(1+r_T)^{-1} + (1+r)^{-1} < 1$ and then we use (A.11) and (A.12). However, the above inequality will result in a contradiction for small enough μ.

This proves that r cannot be between zero and n. Putting the results of (i), (ii), and (iii) together, we conclude that there cannot be infinitely many T's for which r_T is less than n. That is, for all T sufficiently large, r_T is greater than or equal to n and hence, every nonmonetary steady state is optimal as claimed.