Federal Reserve Bank of Minneapolis
Research Department

NOTES ON GAME THEORY:
I. EXTENSIVE FORM GAMES

Patrick J. Kehoe

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*Federal Reserve Bank of Minneapolis and University of Minnesota.

Extensive Form Games

I will use a variant of Van Damme/Selten's notation.

1. Extensive Form Game $\Gamma = (K, X^p, E, C, p_0, V)$

Game tree K

The game tree is a finite tree with two types of nodes decision nodes and terminal nodes: X = set of decision nodes and Z = set of terminal nodes. Let $<$ be a partial order as $X \cup Z$ that denotes precedence. If $x < y$ we say x comes before y and y comes after x. (Assume $<$ totally orders the predecessors of each member of X Z so that there are no cycles i.e., each node in tree can be reached by one and only one path from the initial node.) Let,

- $pd_1(x) = \max\{y | y < x\}$ = immediate predecessor of x.
- $pd_n(x) = n^{th}$ predecessor of x (defined recursively using above).
- $S(x) = \{y | x \in p_1(y)\}$ immediate successors of x.
- $Z(x) = \{y \in z | x < y\}$ = terminal successors of x.

Player Partition $X = \{X_0, X, \ldots, X_I\}$

The player partition divides decision nodes into I + 1 sets where X_0 is the set of decision nodes of nature, and for $i \neq 0$, X_i is the set of decision nodes of player i.

Information Partition $E = (E, \ldots, E_I)$

E_i is a partition of decision nodes of X_i of player 1 into so called information sets e_i of player i. $E_i = \{e_i\}$. Assume for every information set $e \in E$

(i) every path in tree intersects e at most once.

(ii) all nodes in the information set have the same number of immediate successors.

Choice Partition C

The choice partition C is a collection $C = \{C_e | e \in \bigcup_{i=1}^{I} E_i\}$ where C_e is a partition of $\cup\{S(x) | x \in e\}$ into so called choices at e such that every choice contains exactly one element of $S(x)$, for all $x \in e$. The interpretation is as follows: if player i reaches his information set $e \in E_i$ and takes choice $c \in C_e$ then if he is actually at $x \in e$, the next node reached by the play will be some immediate successor of x and thus lies in $S(x)$. We will identify a choice at an information set with the next set of nodes reached by play given that choice.

Prior on X_0

Each player believes nature chooses a node $x \in X_0$ with probability $p_0(x)$, so $p(\cdot)$ is, a probability distribution on immediate successors of x_0.

Payoff Function $V = (V_1, \ldots, V_I)$: $Z \to R^I$

If terminal node z is reached, player i receives utility $v_i(z)$ (see Example 1).

2. Strategies

Pure Strategies

A pure strategy π_i of player i is a function π_i which assigns a choice c from C to every $e \in E_i$

$$\Pi_i = \{\pi_i : \underset{e \in E_i}{X}\ e \to \underset{e \in E_i}{X}\ C\},$$

Mixed Strategies

A mixed strategy σ_i of player i is a probability distribution Π_i.

$$S_i = \{\sigma_i : \Pi_i \to R^{\#\Pi_i} | \sigma_i(\pi_i) \geq 0 \text{ all } \pi_i \text{ and } \sum_{\pi_i \in \Pi_i} \sigma_i(\pi_i) = 1\}.$$

Local Strategy (at an information set)

A local strategy b_{ie} of player i at the information set $e \in E_i$ is a probability distribution over the set of choices C_e at e where a probability $b_{ie}(C)$ is assigned to each choice $c \in C_e$. A local strategy is called a pure local strategy if it assigns 1 to some particular choice c at e and 0 to other choices at e.

Behavior Strategies

A behavior strategy b_i of player i is a function that assigns a local strategy b_{ie} to every $e \in E_i$. Let B_i = the collection of all behavior strategies of i and $B = B_i \times \ldots \times B_I$ so $b = (b,\ldots,b_I) \in B$.

Realization Probabilities

If players play behavior strategy $b \in B$, then for every node $x \in X \cup Z$ we can compute the realization probability $P^b(x)$ where

$P^b(x)$ = the probability that node x is reached in play if behavior strategy vector $b \in B$ is played.

In a similar manner, for any arbitrary set of nodes A which is a subset of $X \cup Z$

$P^b(A)$ = probability that at least one node of A is reached in play if b is played.

Expected Payoffs

With the help of these realization probabilities we can define the expected payoff to player i given that behavior strategy vector $b \in B$ is played.

$$V_i(b) = \sum_{z \in Z} P^b(z) V_i(z).$$

Completely Mixed Behavioral Strategies

A behavior strategy b_i of player i is said to be completely mixed if it assigns completely mixed local strategy b_{ie} to each information set $e \in E_i$, where a completely mixed local strategy b_{ie} assigns a strictly positive probability to every choice $c \in C_e$. Let B_i^0 denote set of completely mixed behavior strategies. Let $B^0 = B_1^0 \times \ldots \times B_I^0$.

Conditional Realization Probabilities (with complete mixing)

A completely mixed behavior strategy vector $b \in B^0$ induces a well-defined conditional probability distribution $p^b(\cdot|e)$ over all information sets $e \in E$ according to the formula

$$(1) \qquad \text{for } x \in e,\ p^b(x|e) = \frac{P^b(x)}{\sum_{y \in e} P^b(y)}$$

where $P^b(x|e)$ = the probability that x is reached in play when b is played given that information set e has been reached (confer Figure 2). Now if we try to define conditional probabilities over information sets (according to (1)) for arbitrary behavioral strategies we will have a problem. The reason is that under arbitrary behavioral strategies some information sets will be reached with probability zero and for such an information set (1) will not make sense (since we will be dividing by zero).

Notice in Example 2 that the strategy y vector b induces a probability distribution on the whole tree, for example:

$$P^b(x_3) = b_0(L)b_1(L|e_{11})$$

$$P^b(x_8) = b_0(L)b_1(L|e_{11})b_2(r|e_{21}).$$

It also induces a conditional probability distribution over each information set in the tree (if its completely mixed the conditional probabilities are always well-defined)

$$P^b(x_3|e_{21}) = \frac{P^b(x_3)}{P^b(x_3) + P^b(x_4)} = \frac{b_0(L)b_1(L|e_{11})}{b_0(L)b_1(L|e_{11}) + b_0(L)b_1(R|e_{11})}$$

$$= \frac{b_1(L|e_{11})}{b_1(L|e_{11}) + b_1(R|e_{11})}.$$

Using equation (1) and the figure we can compute $P^b(x_8|e_{13})$ to be

$$(2) \qquad P^b(x_8|e_{13}) = \frac{P^b(x_8)}{P^b(x_7) + P^b(x_8)}$$

$$= \frac{b_0(L)b_1(L|e_{11})b_2(r|e_{21})}{b_0(L)b_1(L|e_{11})b_2(l|e_{21}) + b_0(L)b_1(L|e_{11})b_2(r|e_{21})}.$$

So

$$(3) \qquad P^b(x_8|e_{13}) = \frac{b_2(r|e_{21})}{b_2(l|e_{21}) + b_2(r|e_{21})}.$$

Notice two things about this conditional probability. First, if either of the numbers $b_0(L)b_1(L|e_{11})$ were zero, then the above formula would be nonsense (of course they cannot be if b_{ie} is completely mixed).

Second, notice how the previous choices of player 1 (i.e., strategies $b_1(L|e_{11})$ and the rest) either cancel out or don't appear in the final formula (3). Well, this always happens and it is the key reason why the agent normal form "works." Selten proves this in his paper.

Selten's Lemma 4 (reworded)

Given a completely mixed behavior strategy vector $b = (b_1,\ldots,b_I)$ for an extensive form game with perfect recall, the conditional realization probabilities $P^b(x|e)$ for $e \in E_i$ do not depend on the strategy b_i of player i.

The lemma is proved by observing in a game with perfect recall the information sets e_i of player i have the property that the same choices of player i are in every path to a vertex $x \in e_i$.

Conditional Realization Probabilities (for arbitrary strategies)

For some arbitrary behavior strategy vector $b \in B$, some information sets in E will be reached with zero probability. Let E(b) denote the subset of information sets that are reached with strictly positive probability.

$$E(b) = \{e \in E | P^b(e) = \sum_{x \in e} P^b(x) > 0\}.$$

Let E/E(b) denote the information sets reached with zero probability under b. Now for any information set $e \in E(b)$ we still have a well-defined conditional probability distribution, it is only the unreached information sets in E/E(b) that give us trouble.

Conditional Expected Payoff (at an information set)

For every information set e of player i that is reached with positive probability under b, we have a well-defined notion of conditional expected payoff, denoted $v_i(b|e)$ which equals the conditional expected utility to player i if b is played and given information set $e \in E_i$ is reached. For each e in $E_i(b)$,

$$(4) \qquad V_i(b|e) = \sum_{x \in e} P^b(x|e)\Big[\sum_{z > x} P^b(z|x) V_i(z)\Big]$$

where $P^b(z|x) \equiv P^b(z)/P^b(x)$. To interpret this formula imagine: given that x has been reached what is i's conditional expected payoff? Clearly, if x has been reached, only those terminal nodes z that follow x will be reached, and some particular terminal node z after x will be reached with conditional probability $P^b(z|x)$ and at that node z, player i gets $v_i(z)$.

Now all that player i knows is that he is in some node x of information set e (recall definition of information set). Even though player i does not know what node he is sitting at, he can compute the probability of being at x if he knows what other players behavioral strategies are by using the formula $P^b(x|e) = P^b(x)/\sum_{y\varepsilon e} P^b(y)$. The expected utility condition on reaching e is then (4).

Notice if $e \varepsilon E_i$, but $e \varepsilon E/E(b)$ then player i does <u>not</u> have a well-defined conditional expected payoff at e. Without such a payoff it is not clear what "rational behavior" means at e. We will return to this point later.

<u>The Problem of Unreached Information Sets</u>

In Example 3, I claim: (L,R,R) is a Nash equilibrium in pure strategies (on equivalently, (1,0;0,1;0,1) is a Nash equation in behavioral strategies). In this equilibrium, player 2's behavior does not make much sense. In it 2 plays R with probability 1. However, if node x_2 were ever reached (since 2 knows that 3 is going right for sure) then 2 should play L with problem 1 and get 4 instead of playing R and getting 1. How can we formalize the idea that 2's behavior is nonsensical? First, the unconditional expected return to 2 under arbitrary b.

$$(2.1) \qquad V_2(b) = P^b(z_1)V_2(z_1) + P^b(z_2)V_2(z_2) + P^b(z_5)V_5(z_5)$$

$$= P^b(z_2) \cdot 2 + P^b(z_4) \cdot 4 + P^b(z_5) \cdot 1$$

where

$$P^b(z_2) = b_{1Lb}{}^{3R},\ P^b(z_4) = b_{1R}b_{2L}b_{3R},\ P^b(z_5) = b_{1R}b_{2R}.$$

In contemplating best responses to any (b_1,b_3) notice that if $b_{1L} = 1$, 2's behavior is irrelevant in determining his payoff since he will never be reached anyway he might as well play anything.

So clearly 2's behavior is <u>not</u> irrational in terms of <u>unconditional</u> expected utilities (since it does not even depend on what he chooses when $b_{1L} = 1$). How about in terms of <u>conditional</u> expected utility, (given that $e_2 = \{x_2\}$ is reached with positive probability)

$$(2.2) \qquad V_2(b|e_2) = P^b(x_2|e_2)[\sum_{z > x_2} P^b(z|x_2)V_i(z)].$$

Now $\{z|z > x_2\} = \{z_3,z_4,z_5\}$. So

$$V_2(b|e_2) = P^b(x_2|e_2)[P^b(z_3|x_2)V_2(z_3) + P^b(z_4|x_2)V_2(z_4)$$

$$+ P^b(z_5|x_2)V_2(z_5].$$

Now for any b such that e_2 is reached under b, $P^b(x_2|e_2) = 1$ so

$$V_2(b|e_2) = [b_{2L}b_{3L}V_2(z_3) + b_{2L}b_{3R}V_2(z_4) + b_{2R}V_2(z_5)]$$

$$= [4b_{2L}b_{3R} + b_{2R}].$$

So let $BR_2(b|e_2)$ = set of best responses by 2 to b given e_2 is reached be defined to be solution to

$$\max_{b_{2,e_2} \varepsilon B_{2,e_2}} V_2(b|e_2) \quad (\text{given } e_2 \ \varepsilon \ E_2(b)).$$

Clearly the conditional best response to 2 at e_2 is to play R with probability one. (For b such that e_2 is reached.) The problem is that for the particular b are interested in, namely $\hat{b} = (1,0;0,1;0;1)$, e_2 is <u>not</u> reached $(e_2 \in E_2/\hat{E}_2(b))$ so that (2.2) does not make sense since $P^b(x_2|e_2)$ does not make sense.

This then is the crux of the problem. Intuitively 2's behavior in the equilibrium $\hat{b}$ is unreasonable because if he were ever actually called upon to carry out his move he would have no incentive to. However, at the equilibrium strategy configuration he will never be called on to carry it out. In addition, at the equilibrium strategy $\hat{b}$ there is no well-defined conditional problem distribution our e_2 so we can't even define his conditional choice problem. Now Selten, Kreps-Wilson and others, have proposed a number of ways to solve this problem. One interpretation of their work is that they have various methods for imposing well-defined conditional probability distributions over information sets that are not reached under the behavior strategy $\hat{b}$ under consideration.

Briefly, Selten in defining <u>trembling hand perfect equilibrium</u> requires the conditional distribution be generated by a sequence of completely mixed strategy vectors $\{b_k\}$ such that:

1. $b^k \to \hat{b}$
2. Each b^k is an <u>equilibrium</u> of a certain game (which we will define momentarily).

In contrast, <u>Kreps and Wilson</u> (KW) require something weaker. Just as in Selten's definition, Kreps and Wilson require the conditional distribution over information sets not reached in equilibrium be generated by a sequence of completely mixed strategy vectors b^k such that $b^k \to \hat{b}$. However, KW do not require these b^k be an equilibrium to anything. It should be clear that under KW's criterion we can generate (to a large extent) almost any conditional probability distributions over unreached nodes that we want. (However, there are some subtle restrictions about how these conditional distributions must be related, that I will discuss later.)

For now, simply realize that both criterion are a way of generating well-defined conditional probability distributions over information sets that are reached with zero probability under the proposed equilibrium configuration of behavior strategies.

Note 1. Notice that I have been assuming that at information sets which are reached with positive probability it makes sense for an agent to maximize conditional expected utility. It may not be totally obvious that maximizing conditional expected utility node by node is equivalent to maximizing unconditional expected utility over paths through the tree. It is a theorem that in games with perfect recall these are the same and this gives a simple way to "decentralize" an agents decision making process into a sequence of small "conditional" problems. I will explain this in more detail.

First, some more definitions:

(Global) Best Reply

A behavior strategy $\tilde{b}_i$ of i is a best reply to b if

$$\tilde{b}_i \varepsilon \arg \max_{b_i' \varepsilon B_i} v_i(b_i', b_{-i}).$$

Local Best Reply (an an information set)

A local behavior strategy $\tilde{b}_{ie}$ at an information set e of player i is a local best reply to b if

$$\tilde{b}_{ie} \in \arg \max_{b''_i \in B_i} v_i(b''_i, b_{-i})$$

where $b''_i = (b_i/b'_{ie})$ is the behavior strategy b_i with b_{ie} replaced with b'_{ie} and rest of the strategy is the same.

<u>Conditional Local Best Reply</u> (at an information set)

$\tilde{b}_{ie}$ is a conditional local best reply to b at e if

$$\tilde{b}_{ie} \in \arg \max_{b'_i = (b_i/b'_{ie}) \in B_i} V_i(b'_i, b_{-i} | e).$$

The best reply concepts are not in conflict with each other.

<u>Proposition</u>:

1. $\tilde{b}_i$ is a best reply to b iff for every component $\tilde{b}_{ie}$ of $\tilde{b}_i$, $\tilde{b}_{ie}$ is a local best reply to b at e, for all $e \in E(b)$.

2. $\tilde{b}_i$ is a best reply to b iff for every component $\tilde{b}_{ie}$ of $\tilde{b}_i$, $\tilde{b}_{ie}$ is a conditional local best reply at e, for all $e \in E(b)$.

Now, so far we have been discussing conditional probability distributions over information sets that are generated by a behavioral strategy b and we have denoted them $P^b(x|e)$. We noted that these are well defined only at information sets that are reached, that is for $e \in E(b)$. Let us now define arbitrary conditional probability distributions over information sets (that may or may not come from a behavior strategy).

Let μ: $X \times E \to [0,1]$ the set of conditional probability distributions over all information sets. For $x \in e$ let $\mu(x|e)$ be the conditional probability of x given information set e which contains x is reached. For each e in E let the conditional probability distribution $\mu(\cdot|e)$ be such that $\sum_{x \in e} \mu(x|e) = 1$, $\mu(x|e) \geq 0$. So $\mu = \{\mu(\cdot|e): e \in E\}$ is actually a whole collection of conditional probability distributions each of which specify a conditional probability distribution over a particular information set. KW call μ a system of beliefs.

Now, we can use this notion of system of beliefs to mechanically solve our problem. Given a behavior strategy b:

1. For any information set that is reached under b, let all agents use the conditional probability distribution generated by b, when solving their problems (i.e., calculating their conditional local best replies).

2. For any information set that is not reached under b, let us simply make up some conditional probability distribution $\mu(\cdot|e)$ over the set of nodes $\{x|x \in e\}$.

Let agents believe the conditional probability distribution over e is (under b)

$$P^b(x|e) \text{ if } e \varepsilon E(b)$$

$$\mu(x|e) \text{ if } e \varepsilon E/E(b)$$

Or we could simply let agents always use μ but then impose that μ agree with the conditional distribution generated by b on information sets where that distribution is well-defined, that is

$$\text{Given b,} \quad \mu^b(x|e) = \begin{array}{c} P^b(x|e) \text{ for } e \varepsilon E(b) \\ \text{(some arbitrary)} \\ \mu(x|e) \text{ for } e \varepsilon E/E(b) \end{array} .$$

In this interpretation, we have "solved" our problem of not having well-defined probabilities distributions (under b) at certain information sets by simply making them up.

Now the goal of Kreps, Wilson, Selten and others is to find reasonable ways to make up beliefs for unreached information sets. Let μ^b denote the collection $\{\mu^b(x|e)|e \in E\}$. Call (μ^b, b) an assessment. Selten's criterion is that given a proposed equilibrium behavior strategy $\hat{b}$, a system of beliefs $\mu^{\hat{b}}$ is reasonable only if it can be generated as the limiting conditional probability distribution from some sequence b^k of completely mixed strategies such that:

1. $b^k \rightarrow \hat{b}$
2. b^k is an equilibrium of perturbed game (Γ, η^k).

Where η^k is a sequence of mistake probabilities at information sets such that $\eta^k(c|e) \rightarrow 0$ for all $c \in C_e$, for all $e \in E$. We call such a $(\mu^{\hat{b}}, \hat{b})$ a trembling hand perfect equilibrium.

Kreps-Wilson criterion is weaker. Given a proposed equilibrium behavior strategy vector $\hat{b}$, a system of beliefs $\mu^{\hat{b}}$ is reasonable only if it can be generated as the limiting conditional distribution of a sequence b^k of completely mixed strategies such that

1. $\hat{b}^k \rightarrow b$.

The crux of the difference is that:

<u>For Selten</u>. The beliefs must be generated from completely mixed behavior strategies that are equilibria to the perturbed games (Γ, η^t).

<u>For Kreps and Wilson</u>. The beliefs are again generated from completely mixed behavior strategies but these don't have to be equilibria to anything.

From this is should be clear:

Proposition. Every trembling hand perfect equilibrium is a sequential equilibrium but every sequential equilibrium is not trembling hand perfect.

The first part of the proposition follows from definition. The second part can be shown by Example 4.

In Example 4 I claim that (L,r) is a sequential equilibrium but not a trembling hand perfect equilibrium. Denote the behavior strategies of 1 by $b_1 = (b_{1L}, b_{1R})$ and of 2 by $b_2 = (b_{2\ell}, b_{2r})$. In this notation, the claim is that $\hat{b} = (1,0;0,1)$ is sequential but not THP. Under both sequential and THP we will have to check that our proposed equilibrium $(\hat{b}, \mu^{\hat{b}})$ is sequentially rational. What this means is that given beliefs $\mu^{\hat{b}}$ the behavior strategies $\hat{b}_1 + \hat{b}_2$ are conditional local best replies at each information set.

Since we will have to check it in a minute let us write out what this means for an arbitrary assessment (b,μ). In this example there are only two information sets e_1 and e_2. We need to check player 1's behavior at e_1 and players 2's at e_2. [I will write the x out using our notation so that you get used to seeing it, so bear with me.]

Player 1.

$$V_1(b|e_1) = \sum_{x \varepsilon e_1} \mu(x|e_1)\left[\sum_{z>x} P^b(z|x)V_1(z)\right].$$

With

$$e_1 = \{x_0\},\ \mu(x_0|e_1) = 1 \text{ and } \{z|z > x_0\} = \{z_1, z_2, z_3, z_4\}$$

this becomes

$$V_1(b|e_1) = 1 \cdot \left[P^b(z_1|x_0)V_1(z_1) + P^b(z_2|x_0)V_1(z_2)\right]$$

$$+ P^b(z_3|x_0)V_1(z_3)P^b(z_4|x_0)V_1(z_4)$$

$$= b_{1L}[1b_{2\ell} + 1 \cdot b_{2r}] + b_{1R}[2b_{21} - b_{2r}].$$

Thus 1 goes left with probability 1 if $[b_{2\ell} + b_{2r}] > [2b_{2\ell} - b_{2r}]$ which reduces to $b_{2\ell} < 2/3$. (Any beliefs μ will have $\mu(x_0|e_1) = 1$.) The best reply for 1 is

$$BR_1^{\mu}(b_2|e_1) = \begin{cases} (1,0) & \text{if } b_{21} < \frac{2}{3} \\ (0,1) & \text{if } \quad > \\ (\alpha,1-\alpha) & \text{if } \quad = \end{cases}$$

Player 2.

$$V_2(b|e_2) = \sum_{x \varepsilon e_2} \mu(x|e_2)\left(\sum_{z > x} P^b(z|x)V_2(z)\right).$$

With

$$e_2 = \{x_1,x_2\}, \ \{z|z > x_1\} = \{z_1,z_2\}, \ \{z|z > x_2\} = \{z_3,z_4\},$$

we have

$$V_2(b|e_2) = \mu(x_1|e_2)[P^b(z_3|x_2)V_2(z_3) + P^b(z_2|x_1)V_2(z_2)]$$

$$+ \mu(x_2|e_2)[P^b(z_3|x_2)V_2(z_3) + P^b(z_4|x_2)V_2(z_4)]$$

$$= \mu(x_1|e_2)[b_{2\ell} + b_{2r}] + \mu(x_2|e_2)[-b_{2r}]$$

$$= b_{2\ell}[\mu(x_1|e_2)] + b_{2r}[\mu(x_1|e_2) - \mu(x_2|e_2)].$$

So 2 goes left with probability 1 if $\mu(x_2|e_2) > 0$, and 2 mixes if $\mu(x_2|e_2) = 0$. Thus, (a) for any set of beliefs μ with $\mu(x_2|e_2) > 0$.

$$BR_2^{\mu}(b_1|e_2) = (1,0) \text{ all } b_1.$$

(Call the set $\{\mu:\mu(x_2|e_2) > 0\}$ type A beliefs. (b) for any set of beliefs μ with $\mu(x_2|e_2) = 0$

$$BR_2^{\mu}(b_1|e_2) = (\alpha,1-\alpha), \ \alpha \ \varepsilon \ [0,1] \text{ all } b_1.$$

(Call the set $\{\mu: \mu(x|e_2) = 0\}$ <u>type B beliefs</u>. So now it becomes clear that the only way we can support our proposed equilibrium in which 2 goes right with probability 1 is to have 2 believe that node x_2 is never reached $(\mu(x_2|e_2) = 0)$.

The Question is Are Such "Beliefs" by 2 Reasonable?

Kreps and Wilson say yes they are reasonable, while Selten says no they are not reasonable. Now in both sequential and THP the proposed beliefs $\mu^{\hat{b}}$ have to agree with the conditional probability distribution induced by the proposed equilibrium behavior strategy $\hat{b}$ at all nodes that are reached with strictly positive probability (since the conditional probability distribution is only defined at these). So

$$\mu^{\hat{b}}(x) = P^{\hat{b}}(x|e) \text{ all } x \in e \text{ for all } e \in E(\hat{b}).$$

Here there are only two information sets, $E = (e_1, e_2)$ and under the proposed equilibrium $\hat{b}$, both are reached. So

1. $\mu^{\hat{b}}(x_1|e_2) = P^{\hat{b}}(x_1|e_2) = \hat{b}_{1L} = 1.$
2. $\mu^{\hat{b}}(x_2|e_2) = P^{\hat{b}}(x_2|e_2) = \hat{b}_{2L} = 0.$
3. $[\mu^{\hat{b}}(x_2|e_0) = P^{\hat{b}}(x_1|e_0) = 1].$

So $\mu^{\hat{b}}$ is completely pinned down by $\hat{b}$.

Since there are no unreached information sets under $\hat{b}$, we have no leeway to play with the conditional probabilities at unreached nodes.

Again in <u>both</u> sequential and THP we need to find a sequence of assessments (b^k, μ^{b^k}) such that:

1. <u>Mixing</u>: b^k is completely mixed $\forall$ k.

2. Convergence: $(b^k, \mu^{b^k}) \to (\hat{b}, \mu^{\hat{b}})$ (our proposed equilibrium).

3. Sequential Best Replies: Given beliefs $\mu^{\hat{b}}$, the strategies $\hat{b}$ are sequential best replies for all players.

Conditions (1), (2), and (3) are all we need for a sequential equilibrium. (Note: The "consistency" condition of KW is subsumed by my definition of μ^b, which I repeat here

$$\text{Given } b, \quad \mu^b(x|e) = \begin{array}{ll} P^b(x|e) & \text{for } e \in E(b) \\ \text{any } {}^*\mu(x|e) & \text{for } e \in E/E(b) \end{array}$$

(*) where "any" $\mu(x|e)$ means some arbitrary $\mu(x|e)$ that does not contradict the information structure of the game--more on this point later.)

However THP requires not only (1), (2), and (3) but also (4) Perturbed Equilibrium: (b^k, μ^{b^k}) are equilibria to the perturbed games (Γ, η^k), and as $k \to \infty$, $\eta^k(c|e) \to 0$ $\forall$ $c \in C_e$, $\forall$ $e \in E$.

For our example condition (4) will not be met for any possible sequence (b^k, μ^{b^k}) that could possibly support $(\hat{b}, \mu^{\hat{b}})$.

I. Supporting $(\hat{b}, \mu^{\hat{b}})$ as a Sequential Equilibrium

I claim $b^k = \left(1-(\varepsilon_1)^k, \varepsilon_1^k; \varepsilon_2^k, 1-\varepsilon_2^k\right)$ works for any ε_1, $\varepsilon_2 > 0$. Let us Check (1), (2), and (3)

1. Mixing: clearly b^k is completely mixed for each k.
2. Convergence: (of b^k to $\hat{b}$ and μ^{b^k} to $\mu^{\hat{b}}$)

   (A) Clearly $b^k \to (1,0;0,1) = \hat{b}$

   (B) μ^{b^k} is given by

$$\mu^{b^k}(x_1|e_2) = P^{b^k}(x_1|e_2) = b_{1L}^k = 1-\varepsilon_1^k \to 1 = \hat{b}_{1L}$$

$$\mu^{b^k}(x_2|e_2) = P^{b^k}(x_2|e_2) = b^k_{1R} = \varepsilon^k_1 \to 0 = \hat{b}_{1R}$$

$$[\mu^{b^k}(x_1|e_1) = 1 = \mu^{\hat{b}}(x_1|e_1)]$$

so

$$\mu^{b^k} \to \mu^{\hat{b}}$$

3. Sequential Best Replies: We need to check that given beliefs $\mu^{\hat{b}}$, $\hat{b}_1 = (\hat{b}_{1L}, \hat{b}_{1R}) = (1,0)$ is player 1's (conditional) local best reply at e_1, and $\hat{b}_2 = (\hat{b}_{21}, \hat{b}_{2r}) = (0,1)$ is player 2's conditional best reply at e_2.

For 1: we know for *any* set of beliefs μ

$$BR^{\mu}_1(b_2|e_1) = \begin{matrix} (1,0) & b_{21} < \frac{2}{3} \\ (0,1) & > \\ (\alpha, 1-\alpha) & = \end{matrix} .$$

So for our $\hat{\mu}$ and our $\hat{b}$ which has $\hat{b}_{21} = 0$ ($< 2/3$) we know

$$BR^{\hat{\mu}}_1((0,1)|e_1) = (1,0) = \hat{b}_2.$$

For 2: we know for: type A beliefs (i.e., μ such that $\mu(x_2|e_2) > 0$)

$$BR^{\mu}_2(b_1|e_2) = (1,0) \text{ \underline{all} } b_1.$$

type B beliefs (i.e., μ such that $\mu(x_2|e_2) = 0$)

$$BR^{\mu}_2(b_1|e_2) = (\alpha, 1-\alpha),\ \alpha \in [\sigma, 1] \text{ all } b_1.$$

But $\mu^{\hat{b}}$, $\mu^{\hat{b}}(x_2|e_2) = 0$ B is type B so

$$BR^{\mu^{\hat{b}}}(\hat{b}_1|e_2) = [0,1] \times [0,1] \text{ which contains } (0,1) = \hat{b}_2.$$

So $(\hat{b}, \mu^{\hat{b}})$ is a sequential equilibrium.

II. Impossibility of Supporting $(\hat{b}, \mu^{\hat{b}})$ as a THP Equilibrium

(Recall that an equilibrium be THP requires there exist a sequence of minimum mistake probabilities $\eta^k = \{\eta^k(c|e), \forall c \in e, \forall e \in E\}$ that all converge to zero as k goes to infinity and a sequence of associated perturbed games (Γ, η^k) where the behavioral strategies $(b^{'k}, \mu^{b^k})$ are equilibrium of (Γ, η^k).)

Now to show $(\hat{b}, \mu^{\hat{b}})$ is not THP we must show for all η^k sequences there cannot be a sequence of (b^k, μ^{b^k}) which are equilibria of (Γ, η^k). Inspection of player 2's best response makes it clear. Since any such sequence of b^k is completely mixed then the associated "beliefs" μ^{b^k} will be type A beliefs (with $\mu^k(x_2|e_2) = b^k_{1R} > 0$ all k) so the equilibrium (b^k, μ^{b^k}) will always have 2 playing left at e_2 for all b_1 so b^k will look like $(b^k_{1L}, b^k_{2R}; 1, 0) \neq (\ldots; 0, 1)$ so $(\hat{b}, \mu^{\hat{b}})$ is not THP.

Now what is the intuitive story for what is going on? Basically, the only way 2 will play right is if he/she is <u>sure</u> 1 is going left. Selten thinks that such beliefs are unreasonable because if 1 trembles at all, 2 will always play left. That is, in any game that is close to our game (in terms of perturbed strategies) the equilibria will be far from our equilibria. So this equilibria is "unstable with respect to small perturbations in strategies."

Kreps and Wilson don't think (at least given their definition) that such beliefs are unreasonable since they can be induced as the limiting conditional probability distribution of some sequence of completely mixed behavior strategies (of course, these behavioral strategies will not necessarily be equilibrium of anything).

III. However, $(\hat{b}, \mu^{\hat{b}})$ is a Weak Trembling Hand Perfect Equilibrium

KW relax Selten's definition of THP to allow trembles on the payoff vector v along with trembles on the strategies.

Weak Trembling Hard Perfect (WTHP)

An assessment $(\hat{b}, \mu^{\hat{b}})$ is WTHP if there exists a sequence $\{b^k, \mu^{b^k}, V^k\}$ such that

1. Mixing: b^k is completely mixed.
2. Convergence: of strategies, beliefs and payoff vectors

$$(b^k, \mu^{b^k}, V^k) \to (\hat{b}, \mu^{\hat{b}}, V)$$

3. Sequential Best Reply: Given $\mu^{\hat{b}}$, strategies are sequential best replies.
4. Perturbed Equilibria: (b^k, μ^{b^k}) are equilibria to perturbed game $\hat{\Gamma} = (\Gamma, \eta^k, V^k)$,

K and W prove:

KW Prop 6. For any extensive form game, the sets of weak THP and sequential equilibria coincide.

Return to our example, this means *if* we allow ourselves to perturb the outcome functions (v_i) slightly along the sequence then we can support $(\hat{b}, \mu^{\hat{b}})$. Originally

$$v_1 = \left(v_1(z_1), v_1(z_2), v_1(z_3), v_1(z_4)\right) = (1,1,2,-1)$$

and

$$v_2 = \left(v_2(Z_1), v_2(Z_2), v_2(Z_3), v_2(z_4)\right) = (1,1,0,-1).$$

I claim the following $v^k = (v_1^k, v_2^k)$ will work. Leave v_1 alone (put $v_1^k \equiv v_1$) and put $v_2^k = (1-\varepsilon^k, 1+\varepsilon^k, 0, -1)$. To see why let us compute 2's conditional best response at e_2 given 2 has payoffs v_2^k instead of v_2.

$$v_2(b_1|e_2, v_2^k) = b_{21}[(1-\varepsilon^k)b_{1L}] + b_{2r}[(1+\varepsilon^k)b_{1L} - b_{1R}].$$

So 2 goes left with probability 1 if $(1-\varepsilon^k)b_{1L} > (1+\varepsilon^k)b_{1L} - b_{1R}$ which implies $b_{1R} > (2\varepsilon)b_{1L}$. Thus

$$BR_2(b_1|e_2, v_2^k) = \begin{cases} (1,0) & \text{if } b_{1R} > (2\varepsilon^k)b_{1L} \\ (0,1) & \text{if } b_{1R} < (2\varepsilon^k)b_{1L}. \\ (\alpha_1,\alpha_2) & \text{if } b_{1R} = (2\varepsilon^k)b_{1L} \end{cases}$$

Now consider the perturbed game (Γ, η^k, v^k). The best responses with mistake probabilities $\eta^k = (\eta_{1L}^k, \eta_{1R}^k; \eta_{21}^k, \eta_{2r}^k)$ and with b_1 in the perturbed strategy space $S(\eta^k)$

$$BR_2(b_1|e, v_2^k, \eta_2^k) = \begin{cases} (1 - \eta_{2R}^k, \eta_{2R}^k) & \text{if } b_{1R} > (2\varepsilon^k)b_{1L} \\ (\eta_{2L}^k, 1 - \eta_{2L}^k) & \text{if } b_{1R} < (2\varepsilon^k)b_{1L} \\ (\alpha_1,\alpha_2) & \text{if } b_{1R} = (2\varepsilon^k)b_{1L} \end{cases}$$

where

$$\eta_2^k < \alpha_1 \leq 1 - \eta_{2R}^k,\ \eta_{2R}^k \leq \alpha_2 \leq 1 - \eta_{2L}^k,\ \alpha_1 + \alpha_2 = 1.$$

In a similar manner we find (Recall $v_1^k \equiv v_1$)

$$BR_1(b|e_2, v_1^k) = \begin{cases} (1 - \eta_{1R}^k, \eta_{1R}^k) & \text{if } b_{21} < \frac{2}{3} \\ (\eta_{1L}^k, 1 - \eta_{1L}^k) & \text{if } b_{21} > \frac{2}{3} \\ (\gamma_1,\gamma_2) & \text{if } b_{21} = \frac{2}{3} \end{cases}$$

where

$$\eta^k_{1L} \le \gamma_1 \le 1 - \eta^k_{2L},\ \eta^k_{1R} \le \gamma_2 \le 1 - \eta^k_{2R},\ \gamma_1 + \gamma_2 = 1.$$

(Recall unperturbed games players play their pure best responses as much as we let them and their nonbest responses as little as we let them.)

Now to support $\hat{b} = (1,0;0,1)$ we need to choose the relative size of the η's and ε so that along the sequence:

- player 1 plays left as much as possible (given η^k_1) and
- player 2 plays right as much as possible (given η^k_2) is an equilibrium to the perturbed game.

That is we want b^k to look like $b^k = (b^k_1;b^k_2)$ where

$$b^k_1 = (b^k_{1L}, b^k_{1R}) = (1-\eta^k_{1R}, \eta^k_{1R}),$$

$$b^k_2 = (b^k_{21}, b^k_{2r}) = (\eta^k_{21}, 1-\eta^k_{21}).$$

Now if b^k_1 is to be a best response in (Γ,η^k,v^k) to b^k_2 it must be the case that $b^k_{21} < 2/3$. To see this look at $BR_2\ (\cdot 1 \ldots)$. Thus one restriction on the η^k sequence is $\eta^k_{21} < 2/3$. Next if b^k_2 is to be a best response in (Γ,η^k,v^k) to b^k_1 it must be the case that:

$$b^k_{1R} < (2\varepsilon^k) b^k_{1L}.$$

To see this look at $BR_2(\cdot|\ldots)(\cdot|e,v^k,\eta^k_2)$. Which in terms of our equation becomes $\eta^k_{1R} < (2\varepsilon^k)(1 - \eta^k_{1R})$ or $(1 + 2\varepsilon^k)\eta^k_{1R} < 2\varepsilon^k$. Thus a second restriction on the η^k and ε^k sequence is

$$\eta^k_{1R} < \frac{2\varepsilon^k}{1 + 2\varepsilon^k}.$$

Clearly if we put

$$\eta^k_{21} = \varepsilon^k, \ \eta^k_{1R} = \frac{\varepsilon^k}{1 + 2\varepsilon^k}, \ (\eta^k_{2r} = \eta^k_{1L} = \varepsilon)$$

then the two restrictions (A) and (B) are met (at least for k large enough for $\varepsilon^k < 2/3$ or $k \ln \varepsilon < \ln 2/3$ or $k < (\ln 2/3)/(\ln \varepsilon)$ (a negative number). So $\hat{b} = (1,0;0,1)$ is a weak trembling hard perfect equation.

IV. Consistent Beliefs

You might (mistakenly) believe that the sequential equation concept we can put arbitrary beliefs at nodes that are reached with probability zero under some proposed equilibrium behavioral strategy. The argument would be that we can always choose a sequence of strategies that impose arbitrary conditional distributions along the sequence (since these strategies do not have to be an equilibrium) and so also in the limit. This argument is incorrect. Basically, we need beliefs at unreached nodes to respect the information structure of the game.

To see this consider Example 5. In it nature chooses (L,R) with problem (b_{0L}, b_{0R}). Player 1 at $e_1 = \{x_1, x_2\}$ chooses (a,u) with problem (b_{1a}, b_{1v}). Player 2 at $e_2 = \{x_3, x_1\}$ chooses (L,R) with probability (b_{2L}, b_{2R}). Now suppose $b_{1a} = 1$ and $b_{1u} = 0$, then information set e_2 is reached with zero probability. The question is can we under sequential equilibrium impose arbitrary beliefs at e_2. That is, for any possible conditional distribution at e_2 say $(\mu(x_3|e_2), \mu(x_4|e_2))$, can we find a sequence of completely mixed behavior strategies say b^k such that the induced conditional probability distribution $((P^{b^k}(x_3|e_2), P^{b^k}(x_4|e_2))$ converges to $(\mu(x_3|e_2))$, $\pi(x_4|e_2)$? The answer is no (think about it before reading on).

For any completely mixed b^k we have $e_2 \in E(b^k)$ and

$$P^{b^k}(x_3|e_2) = \frac{b_{OL}b_{1u}}{b_{OL}b_{1u} + b_{OR}b_{1u}} = \frac{b_{OL}}{b_{OL} + b_{OR}} = b_{OL}$$

and

$$P^{b^k}(x_4|e_2) = \frac{b_{OR}b_{1u}}{b_{OR}b_{1u} + b_{OR}b_{1u}} = \frac{b_{OR}}{b_{OR} + b_{OL}} = b_{OR}.$$

Since nature's move is part of the game so that P_0 is a pair of given numbers. If, for example, $P_0(x_1) = b_{OL} = 1/3$, $P_0(x_2) = b_{OR} = 2/3$ then the only conditional probability distribution on e_2 that is consistent is (1/3, 2/3).

Consider next Example 6 on the next page. Imagine the equilibrium is

$$\hat{b} = (b_{1A}, b_{1L}, b_{1R}; b_{2A}, b_{2u}; b_{3L}, b_{3R})$$

$$= (1,0,0;0,1;1,0).$$

Since 1 goes A with probability 1, the rest of the tree is reached with zero probability. In particular, e_{21}, e_3, and e_{22} are unreached information sets. Now, under sequential equilibrium can we induce any conditional probability distribution on these unreached information sets that we want? The answer is no. For example, consider the beliefs: at e_{21}: $\mu(x_2|e_{21}) = 0.9$, $\mu(x_3|e_{21}) = 0.1$ and at e_{22}: $\mu(x_6|e_{22}) = \mu(x_8|e_{22}) = 0$, $\mu(x_7|e_{22}) = 0.9$, $\mu(x_9|e_{22}) = 0.1$.

I claim these "beliefs" $\mu(\cdot|e_{21})$ and $\mu(\cdot|e_{22})$ are <u>inconsistent</u> both with each other and with player 3's strategy. (Try to figure this out before reading on.)

Imagine you were player 2 sitting at e_{21}, you have the above beliefs at e_{21} namely $(\mu|x_2|e_{21}) = 0.9$, $\mu(x_3|e_{21}) = 0.1$ and you know the equilibrium strategy vector is $\hat{b} = (1,0,0;0,1;,0,1)$, what would you compute the conditional probabilities at e_{22} to be? You would most likely compute as follows: Say, for example x_7

$$P^{\hat{b},\mu}(x_7|e_{22}) = \frac{\mu(x_2|e_{21})\hat{b}_{2u}\hat{b}_{3R}}{\mu(x_2|e_{21})\hat{b}_{2u}\hat{b}_{3R} + \mu(x_2|e_{21})\hat{b}_{2u}\hat{b}_{3L}}$$

$$= \hat{b}_{3R} = 0(\neq 0.9 = \mu(x_7|e_{22}))$$

and so on. Basically, beliefs at unreached nodes have to "respect" the information structure of the game.

Consider finally Example 7. Imagine $\hat{b} = (\hat{b}_{1L},\hat{b}_{1R};\hat{b}_{2A},\hat{b}_{2a};\ldots) = (0.1,0.9;1,0;\ldots)$ is being played and imagine we claim the following beliefs: for unreached information set

$$e_3: \ \left(\mu^{\hat{b}}(x_2|e_2),\mu^{\hat{b}}(x_3|e_2)\right) = (0.1,0.9)$$

and

$$e_4: \ \left(\mu^{\hat{b}}(x_4|e_3),\mu^{\hat{b}}(x_4|e_3)\right) = (0.9,0.1)$$

are consistent. I claim they are not (why?) Recall that to be consistent the beliefs must be the limit of the conditional probability distributions induced by a sequence of completely mixed behavioral strategy that converges to the conjectured equilibrium behavioral strategy, say $\hat{b}$.

Now for any completely mixed b we know $e_3 \in E(b)$, so

$$\mu^b(x_3|e_3) = P^b(x_3|e_3) = \frac{b_{1L}b_{2u}}{b_{1L}b_{2u} + b_{1R}b_{2u}} = \frac{b_{1L}}{b_{1L} + b_{1R}} = b_{1L}$$

and similarly

$$\mu^b(x_4|e_3) = b_{1R}$$

So for $\hat{b} = (0.1,0.9;1,0)$ we know along the sequence

$$\left(\mu^{b^k}(x_3|e_3),\mu^{b^k}(x_4|e_3)\right) = (b^k_{1L},b^k_{1R}) \to (0.1,0.9)$$

for any $b^k \to \hat{b}$. This is a final implication of the KW consistency condition on beliefs (see KW for more detailed discussion).

V. <u>The Structure of the Set of Sequential Equation may be Complicated</u>

KW provide two interesting examples in which they show the set of sequential equilibrium may have isolated points. Let us compute the set of sequential equilibrium, for Example 8.

Step 1. <u>Compute the Local Best Replies for Arbitrary Beliefs</u>

For player 1, his only information set is a singleton and it starts the game so he needs no (extra) beliefs.

$$v_1(b|e_1) = b_{1A} + b_{1L}[3b_{2L} - 2b_{2R}] + b_{1R}[2b_{2L} - b_{2R}]$$

$$= b_{1A} + b_{1L}[5b_{2L} - 2] + b_{1R}[3b_{2L} - 1].$$

So A is preferred to both L and R if $3/5 > b_{2L}$ and $2/3 > b_{2L}$ or $0 \le b_{2L} < 3/5$. Likewise R is preferred to both A and B if $2/3 < b_{2L}$ and $b_{2L} < 1/2$, but this is empty. Finally, L is preferred to both A and R if $3/5 < b_{2L}$ and $b_{2L} > 1/2$ or $1 \ge b_{2L} > 3/5$. If $b_{2L} = 3/5$ then 1 mixes between A and L. Thus for any beliefs μ, player 1's best reply is

$$BR^{\mu}_1(b_2|e_1) = \begin{cases} (1,0,0) & 0 \le b_{2L} < \frac{3}{5} \\ (\alpha,1-\alpha,0) & b_{2L} = \frac{3}{5} \\ (0,1,0) & \frac{3}{5} < b_{2L} \le 1 \end{cases}$$

for any α in $[0,1]$.

For Player 2. The only information set is e_2, let us denote beliefs there as $(\mu(x_1|e_2), \mu(x_2|e_2))$ or simply $(\mu(x_1), \mu(x_2))$. With

$$V_2^{\mu}(b|e_2) = \mu(x_1)b_{2L} + \mu(x_2)b_{2R}.$$

Given μ

$$BR_1^{\mu}(b|e_2) = \begin{cases} (1,0) & \text{if } \quad \mu(x_1) > \frac{1}{2} \\ (0,1) & \qquad\quad \mu(x_1) < \frac{1}{2} \\ (\alpha, 1-\alpha),\ \alpha \in [0,1], & \mu(x_1) = \frac{1}{2} \end{cases}.$$

Now look back at the picture. There are two cases:

Case I. If 1 plays a with probability 1, then e_2 is an unreached information set and we will have a lot of flexibility in assigning consistent beliefs at e_2 (i.e., conditional probability distributions over x_1 and x_2).

Case II. If 1 does not play a with problem 1, then e_2 is reached with positive probability and the behavior of 1 pins down the beliefs at e_2.

Let us deal with Case II first. Inspection of 1's best response tells us the only way he will play (0,*,*) is if $3/5 > b_{2L} \leq 1$ then he will play (0,1,0) (i.e., left for sure), but if he does that pins down 2's beliefs to be $\mu(x_1) = 1$ and $\mu(x_2) = 0$ and with these beliefs 2 plays (1,0), so the only type of equilibrium for this case is

$$\hat{b} = (0,1,0;1,0) \text{ with } \mu^{\hat{b}} = (1,0)$$

that is 1 plays L with probability 1, 2 believes this and plays l with probability 1. It is clear

$$b^{\varepsilon} = (\varepsilon_1, 1-2\varepsilon_1, \varepsilon_1; 1-\varepsilon_2, \varepsilon_2) \rightarrow \hat{b} = (0,1,0;1,0)$$

and

$$\mu^{b^\varepsilon} = \left(\frac{1 - 2\varepsilon_1}{1 - \varepsilon_1}, \frac{\varepsilon_1}{1 - \varepsilon_1}\right) \to \mu^{\hat{b}} = (1,0).$$

For Case I

- for 1 to play (1,0,0) we need 2 to play b_2 with $0 \le b_{21} < 3/5$
- for 2 to do this he must believe $\mu(x_1) \le 1/2$
- for any such beliefs we have an equilibrium i.e. the set of equilibrium for case I is:

$\{(b,\mu) | b = (1,0,0;\alpha,1-\alpha)$ with $0 \le \alpha < 3/5$

together with $\mu^b = (\gamma,1-\gamma)$ $0 \le \gamma \le 1/2\}$.

In more detail, we have two types of equilibrium in Case I

Type I.A. $b = (1,0,0;0,1)$ with $\mu^b = (\alpha,1-\alpha)$ $\alpha < 1/2$

Type I.B. $b = (1,0,0;\alpha,1-\alpha)$ with $\mu^b = (1/2,1/2)$.

To support this consider:

$$b^\varepsilon = (1-\varepsilon,-\varepsilon_2,\varepsilon_1,\varepsilon_2;1-\varepsilon_3,\varepsilon_3)$$

which implies beliefs μ^b

$$\mu^{b^\varepsilon} = \left(\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}, \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right).$$

To generate beliefs for type I.B. put $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$. To generate beliefs for type I.A. put $\varepsilon_2 = (1/\alpha - 1)\varepsilon_1$ because then

$$\mu^{b^\varepsilon} = \left(\frac{\varepsilon_1}{\frac{1}{\alpha}\varepsilon_1}, \frac{(\frac{1}{\alpha} - 1)\varepsilon_1}{\frac{1}{\alpha}\varepsilon_1}\right) = (\alpha,1-\alpha).$$

(To see how to get this put $\varepsilon_2 = \delta\varepsilon_1$ and solve $\varepsilon_1/(1+\delta)\varepsilon_1 = 1/1+\delta = \alpha$, for δ in terms of α.)

Let us plot (project) these equilibria in $(b_{2L}, \mu^b(x_1))$ space: (see Figure 8).

Consider next the game represented in Figure 9. We compute the sequential equilibria for this as follows:

Step 1. Compute the best responses at each information set, for arbitrary beliefs.

Step 2. Break strategy space into "cases" (into strategies that lead to all nodes being reached and into those that leave some unreached nodes.

Step 3. Construct completely mixed strategies that support the proposed equilibrium.

We start with Step 1.

Player 1. only one information set e_1

$$V_1(b|e_1) = b_{1A} + b_{1L}[2b_{2L}b_{3L} - 0.1b_{2L}b_{3R} + 2b_{2R}b_{3L} - 0.1b_{2R}b_{3R}]$$

$$= b_{1A} + b_{1L}[2b_{3L} - 0.1b_{3R}] + b_{1R}[2.1b_{3L}]$$

$$= b_{1A} + b_{1L}[2.1b_{3L} - 0.1] + b_{1R}[2.1b_{3L}].$$

Since e_1 is reached with probability 1 for any strategies, 1's strategies are independent of his/her belief's.

Clearly L is dominated by R, so 1 won't play L. 1 will play A with probability 1 if $1 > (2.1)b_{3L}$ or $1/2.1 > b_{3L}$. Thus 1's best response is

$$BR_1^{\mu}(b|e_1) = \begin{cases} (1,0,0) & 0 \le b_{3L} < \frac{1}{2.1} \\ (\alpha,0,1-\alpha),\ \alpha \in [0,1] & b_{3L} = \frac{1}{2.1} \\ (0,0,1) & \frac{1}{2.1} < b_{3L} \le 1 \end{cases} .$$

Player 2. The only information set is $e_2 = \{x_1, x_2\}$ which may or may not be reached in equilibrium. So for arbitrary beliefs $(\mu(x_1|e_2), \mu(x_2|e_2))$ which we denote by $(\mu(x_1), \mu(x_2))$ we have:

$$V_2^{\mu}(b|e_2) = \mu(x_1)[b_{2L}b_{3L} + b_{2R}b_{3R}] + \mu(x_2)[b_{2L}b_{3L}]$$

$$= b_{2L}[b_{3L}] + b_{2R}[\mu(x_1)b_{3R}]$$

$$= b_{2L}[b_{3L}] + b_{2R}[\mu(x_1) - \mu(x_1)b_{3L}].$$

So $b_{2L} = 1$ if $b_{31} > \mu(x_1) - \mu(x_1)b_{3L}$ or $\mu(x_1)/(1+\mu(x_1)) < b_{3L} \le 1$

$$BR_2^{\mu}(b|e_2) = \begin{cases} (1,0) & \text{if } \mu(x_1)/(1+\mu(x_1)) < b_{3L} \le 1 \\ (\alpha,1-\alpha) & \text{if } \mu(x_1)/(1+\mu(x_1)) = b_{3L} \\ (0,1) & \text{if } \mu(x_1)/(1+\mu(x_1)) > b_{3L} \ge 0 \end{cases} .$$

Player 3. The only information set is e_3 which may or may not be reached. Denote arbitrary beliefs are $e_3 = \{x_3, x_4, x_5, x_6\}$ by

$$\{\mu(x_3), \mu(x_4), \mu(x_5), \mu(x_6)\}$$

then

$$V_3^{\mu}(b|e_3) = \mu(x_3)b_{3R} + \mu(x_4)b_{3L} + \mu(x_5)b_{3L} + \mu(x_6)b_{3L}$$

$$= b_{3L}[\mu(x_4) + \mu(x_5) + \mu(x_6)] + b_{3R}[\mu(x_3)]$$

$$= b_{3L}[1-\mu(x_3)] + b_{3R}[\mu(x_3)].$$

So

$$BR_3^{\mu}(b|e_3) = \begin{cases} (1,0) & 0 \le \mu(x_3) < \frac{1}{2} \\ (\alpha,1-\alpha),\ \alpha \in [0,1] & \mu(x_3) = \frac{1}{2} \\ (0,1) & \frac{1}{2} < \mu(x_3) \le 1 \end{cases} .$$

Step 2. Break the strategy space into cases.

Case I. Strategies that lead to all information sets being reached.

Case II. Strategies that leave some information sets unreached.

Case I. Unless 1 plays A with probability 1, both e_2 and e_3 will be reached. Now 1 plays (0,0,1) for any b_3 with $1/2.1 < b_{3L} \le 1$ and 1 mixes $(\alpha,0,1-\alpha)$ for any b_3 with $1/2.1 = b_{3L}$.

Notice under either of 1's strategies we have e_2 being reached so

$$\hat{\mu}(x_2) = \frac{b_{1L}}{b_{1L} + b_{2L}} = \frac{1}{1} \text{ (or } \frac{1-\alpha}{1-\alpha}) = 1 \text{ and } \hat{\mu}(x\ \ ) = 0.$$

- Given the beliefs on e_2 are pinned down by b_1 we have 2 plays left with probability 1 (to see this substitute $\hat{\mu}$ into 2's best response.
- For player 3, given 1 plays (0,1,0) or $(\alpha,1-\alpha,0)$ (and 2 plays (1,0)) we have the information set e_3 having a conditional probability $\mu(x_3) = 0$, so 3 plays L with probability 1 for case I.
- But given the strategies of 2 and 3; 1 plays (0,1,0).

So for case I we have a unique equilibrium

$$\hat{b} = (0,1,0;1,0;1,0)$$

with

$$\mu^{\hat{b}} = (\mu^{\hat{b}}(x_1), \mu^{\hat{b}}(x_2); \mu^{\hat{b}}(x_3), \mu^{\hat{b}}(x_4), \mu^{\hat{b}}(x_5), \mu^{\hat{b}}(x_6))$$

$$= (0,1;0,0,1,0).$$

I claim $b^{\varepsilon} = (\varepsilon_1, 1-2\varepsilon_1, \varepsilon_1; 1-\varepsilon_2, \varepsilon_2; 1-\varepsilon_3, \varepsilon_3)$ with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$ supports this. Clearly

$$b^{\varepsilon} \to \hat{b} \text{ and } \mu^{b^{\varepsilon}} = \left(\frac{\varepsilon_1}{1-\varepsilon_1}, \frac{1-2\varepsilon_1}{1-\varepsilon_1}; \frac{\varepsilon_1(1-\varepsilon_2)}{1-\varepsilon_1}, \frac{\varepsilon_1\varepsilon_2}{1-\varepsilon_1}, \frac{(1-2\varepsilon_1)(1-\varepsilon_2)}{1-\varepsilon_1}, \frac{(1-2\varepsilon_1)\varepsilon_2}{1-\varepsilon_1}\right)$$

is such that $\mu^{b^{\varepsilon}} \to \hat{\mu}$ and by above the sequential best reply holds for $(\hat{\mu}, \hat{b})$.

Case II. Information sets e_2, e_3 will be unreached if 1 plays (1,0,0)

1. 1 will play this if $b_{3L} < 1/2.1$.
2. 3 will play such a strategy if $\mu(x_3) \geq 1/2$. Notice that we can set $\mu(x_1)$ arbitrarily but that given $\mu(x_1)$, player 2's strategy pins down player 3's beliefs according to
3. $\mu(x_3) = b_{2L}\mu(x_1)$.

We'll use (1), (2), and (3) in what follows:

Fix $\mu(x_1) = \mu_1$ (a) If $b_{3L} > \mu_1/(1+\mu_1)$ then $b_{2L} = 1$ and by (3), $\mu(x_3) = \mu_1$. Now if $\mu_1 = 1/2$ then player 3 mixes any $(\alpha, 1-\alpha)$, but by (1) we consider only $b_{2L} = \alpha < 1/2.1$ and by hypothesis

$$b_{3L} < \frac{\frac{1}{2}}{1+\frac{1}{2}} = \left(\frac{\mu_1}{1+\mu_1}\right) = \frac{1}{3}$$

so $\hat{b} = (1,0,0;1,0;\alpha,1-\alpha)$ with $1/3 < \alpha < 1/2.1$ with beliefs

$$\mu^{\hat{b}}(x_1) = \mu^{\hat{b}}(x_3) = \frac{1}{2}$$

is an equilibrium. (b) If $b_{3L} = \mu_1/(1+\mu_1)$ then b_{2L} is any $\delta \in [0,1]$ and 3 plays any $\gamma \in [0,1]$ if $\mu_3 = b_{2L}\mu_1 = 1/2$. Given $b_{2L} = \delta$, this becomes 3 plays any γ if $\delta\mu_1 = 1/2$. So $\hat{b} = (1,0,0;\delta,1-\delta;\gamma,1-\gamma)$ together with $(\mu_1,\mu_3) = (1/2\delta,1/2)$ is an equilibrium, with $\gamma < 1/(2.1)$ but since $b_{3L} = \mu_1/(1+\mu_1)$ and $b_{3L} < 1/(2.1)$ implies $\mu_1/(1+\mu_1) < 1/(2.1)$ or $\mu_1 < 1/(1.1)$. So for $\mu_1 \in [1/2,1/1.1]$, $\mu_3 = 1/2$ we have

$$\hat{b} = (1,0,0;\hat{b}_{2L},\hat{b}_{3R};\hat{b}_{3L},\hat{b}_{3R})$$

with $\mu_1/(1+\mu_1) = b_{3L} < 1/(2.1)$, $b_{2L} = 1/2\mu_1$ and its easy to support this with a sequence of completely mixed strategy $b^{\varepsilon} = (1-\varepsilon_1,-\varepsilon_2,\varepsilon_1,\varepsilon_2;\alpha,1-\alpha;\gamma,1-\gamma)$ for α, $\gamma > 0$

$$\mu_1^{b^{\varepsilon}} = \left(\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}, \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right) \to (\mu_1,\mu_2)$$

if

$$\varepsilon_1 = \mu_1\varepsilon \quad \varepsilon_2 = \left(\frac{1}{\mu_1} - 1\right)\varepsilon.$$

VI. <u>Sequential Equilibria are not Invariant to Innocuous Changes in the Game Tree</u>.

(This section is from Kohlberg-Mertens "On the Strategic Stability of Equilibria," hereafter referred to as (KM).) K-M provide the following examples:

<u>Examples 10 and 11</u>. In Example 10 player 1 has one information set e_1 at which he chooses {T,M,B}. In Example 11 player 1 has two information sets e_1 and e_1'. At e_1, player 1 chooses either T or D while e_1', player 1 chooses

either M or B. Player 2's situation is the same in both. Basically, these two games are the same and all we have done is to add an "irrelevant" move for 1 (by splitting his decision into two stages). Although these games are equal (up to this innocuous addition of an extra move for 1) the sequential equilibria do not coincide.

In both games "3,3" is a sequential equilibrium, however, in the first game, "2,2" is a sequential equation, but it is not in the second.

$\Gamma(x)$: Step 1.

Player 1.

$$V_1(b|e_1) = b_{1T}[2] + b_{1m}[3b_{2L} + 1.5b_{2R}] + b_{1B}[b_{2R}]$$

$$= b_{1T}(2) + b_{1m}[1.5 + 1.5b_{2L}] + b_{1B}[1-b_{2L}].$$

B is strictly dominated by T, that is, T is preferred to M if $2 > 1.5 + 1.5b_{2L}$ which simplifies to $1/3 > b_{2L}$.

Thus for arbitrary μ (since e_1 reached with probability 1) the best response of player 1 is,

$$BR_1^{\mu}(b|e_2) = \begin{matrix} (1,0,0) & b_{2L} < \frac{1}{3} \\ (\alpha, 1-\alpha, 0) & b_{2L} = \frac{1}{3} \\ (0,1,0) & b_{2L} > \frac{1}{3} \end{matrix} .$$

Player 2.

$$V_2^{\mu}(b|e_2) = \mu(x_1)[3b_{2L}] + \mu(x_2)[b_{2R}]$$

$$= b_{2L}[3\mu(x_1)] + b_{2R}\left[1-\mu(x_1)\right].$$

L is preferred to R if $3\,\mu(x_1) > 1 - \mu(x_1)$ which simplifies to $\mu(x_1) > 1/4$. Thus the best response of 2 is,

$$BR_2^{\mu}(b|e_2) = \begin{cases} (1,0) & \mu(x_1) > \frac{1}{4} \\ (\alpha, 1-\alpha) & \mu(x_1) = \frac{1}{4} \\ (0,1) & \mu(x_1) < \frac{1}{4} \end{cases} .$$

<u>Step 2</u>. Case I: Strategies s.t. e_2 is reached. Case II: Strategies s.t. e_2 <u>not</u> reached.

<u>Case I</u>. e_2 will be reached if $b_{2L} \geq 1/3$ and 1 plays M with positive probability. This pins down beliefs $\mu(x_1) = P^b(x_1|e_2) = b_{1M}/(b_{1M} + b_{1B}) = (1-\alpha)/(1-\alpha) = 1$ and $\mu(x_2) = 0$, so 2 responds with L. The set of equilibria for this case have $\hat{b} = (0,1,0;1,0)$ with beliefs $\mu^{\hat{b}} = (\mu^{\hat{b}}(x_1), \mu^{\hat{b}}(x_2)) = (1,0)$. Clearly $b^{\varepsilon} = (\varepsilon, 1-2\varepsilon, \varepsilon; 1-\varepsilon, \varepsilon)$ with $\mu^{b^{\varepsilon}} = (1-2\varepsilon/1-\varepsilon, \varepsilon/1-\varepsilon)$ supports this.

<u>Case II</u>. e_2 will not be reached if 1 plays (1,0,0), 1 will play this if $b_{2L} < 1/3$. 2's best response includes this if $\mu(x_1) \geq 1/4$. For this case the set of equilibrium are

$$\{\hat{b} = (1,0,0;\alpha,1-\alpha),\ \mu^{\hat{b}} = (\gamma, 1-\gamma) \text{ with } \alpha < \tfrac{1}{3},\ \gamma \leq \tfrac{1}{4}\}.$$

To support this put

$$b^{\varepsilon} = (1-\varepsilon_1-\varepsilon_2, \varepsilon_1, \varepsilon_2; \alpha, 1-\alpha) \text{ (for } \alpha > 0)$$

$$b^{\varepsilon} = (1-\varepsilon_1-\varepsilon_2, \varepsilon_1, \varepsilon_2; \varepsilon_3, 1-\varepsilon_3) \text{ (for } \alpha = 0)$$

with beliefs μ

$$b^{\varepsilon} = \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}, \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right).$$

To get these beliefs to converge to $(\gamma, 1 - \gamma)$ put $\varepsilon_2 = A\varepsilon_1$, so

$$\left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}, \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) = \left(\frac{1}{1+A}, \frac{A}{1+A}\right) = (\gamma, 1-\gamma)$$

which implies

$$1 + A = \frac{1}{\gamma} \text{ or } A = \frac{1}{\gamma} - 1.$$

So put

$$b^{\varepsilon} = \left(1 - \frac{1}{\gamma}\varepsilon, \varepsilon, (\frac{1}{\gamma} - 1)\varepsilon; \alpha, 1-\alpha\right)$$

or $$b^{\varepsilon} = \left(1 - \frac{1}{\gamma}\varepsilon, \varepsilon, (\frac{1}{\gamma} - 1)\varepsilon; \varepsilon, 1-\varepsilon\right).$$

Clearly this works.

$\underline{\Gamma'(x)}$: This is basically the same game except that we have introduced an extra (strategically) irrelevant move. However, we are not forced to check the sequential best reply at both e_1 and e_1'. The key node is in e_1'

$$V_1'(b|e_1') = b_{1M}[1.5 + 1.5b_{2L}] + b_{1B}[1 - b_{2L}]$$

$$BR_1^{\mu}(b|e_1') = (0,1,0) \ \underline{\text{all}}\ b_{2L}.$$

This pins down $\mu(x_1) = 1$ and eliminates all case II equilibria of Γ. So here the <u>only</u> sequential equilibrium is $\hat{b} = (0,1,0;1,0)$, $\mu^{\hat{b}} = (1,0)$.

<u>Remark</u>: Notice (2.2) is not even subgame perfect. Kohlberg and Mertens believe this type of example points out a serious defect of sequential equation, namely: The set of sequential equation may change a great deal as we change the game tree in seemingly innocuous ways.

VII. Refinements of Sequential Equilibria for Signaling Games

[This section is from Kreps "Signaling Games and Stable Equilibria."] Kreps and others have proposed a variety of criteria that eliminate "bad" sequential equilibria. For the most part these criteria and their associated refinements have been developed for a simple class of (extensive form) games called signaling games. We will begin by defining a signaling game and then will consider several examples.

7.1 A Signaling Game (with 2 players, one-sided uncertainty and 2 stages)

Consider a simple (Bayesian) game with two players: A and B.

Stage 1. Player A learns some private information, namely that his type is some specific type t of some finite list of types T_A. Given this information, A selects an action, here, to send a "message" M to B from some finite list of messages M.

Stage 2. Player B receives the message from A, and then chooses some action, here called a "response" r from some finite list of responses R.

If A's type is t, and A plays M and B plays r. the utility of A is $U_A(t,m,r)$ and the utility of B is $U_B(t,m,r)$.

At stage 1, A knows B's type (since his type set is a singleton), so A's beliefs will be trivial. At stage 2, in order to select an action B must have some well-defined beliefs about A's type. B's beliefs will depend on A's message. Let $\mu(t_A|m)$ = the (subjective) conditional probability that A's type is t_A given message M is sent from A to B.

Now we let $b_A = (b_{Am}(t)|t \in T_A)$ denote the behavioral strategy of A where $b_{Am}(t)$ is a probability distribution over possible messages M (notice we let each type of player A have a different distribution). Let b_B =

$(b_{Br}(m) | m \in M)$ denote a behavioral strategy of B. For each message m that A sends, B will be at a different information set (which we index here by m itself). For each information set B will choose a probability distribution over possible responses $r \in R$. $b_{Br}(m)$ = a probability distribution over R, given that information set indexed by m is reached. A's utility depends on his type t_A and the behavior strategy b. Let

$$V_A(b|t_A) = \sum_{m\varepsilon M} \sum_{r\varepsilon R} b_{Am}(t_A) b_{Br}(m) U_A(t_A, m, r)$$

and

$$V_B(b|m) = \sum_{t_A \varepsilon T_A} \sum_{r\varepsilon R} \mu(t_A|m) b_{Br}(m) U_B(t_A, m, r).$$

For A: Given A's type is t_A, A chooses m with probability $b_{Am}(t_A)$. Given this choice, B chooses r with probability $b_{Br}(m)$. So given t_A, the probability of m and r is $b_{Am}(t_A)b_{Br}(m)$, then we just sum over all m and r.

For B: Conditional on receiving m, B believes A is type t_A with probability $\mu(t_A|m)$. B plays r with probability $b_{Br}(m)$, then just sum over possible types of A and possible actions of B.

7.2 The Vodka-Quiche Example

We will motivate the general case with a simple example. Confer Example 12 at end of text, notice the basic structure can be summarized as follows: A is one of two types tough or wimp, $T_A = \{T,W\}$. A knows his type by the time he moves. {"Nature" selects type T with probability $P_T = 0.9$ and W with $P_W = 0.1$ or we may simply say B's priors over T_A are $P_T = 0.9$ and $P_W = 0.1$.}

B will eventually decide to duel or not duel A

- If A is tough, B will lose by dueling.
- If A a wimp, B will win by dueling.

Before B decides he gets to see what A has for breakfast, Vodka or Quiche (that is the signal sent by A)

- If A is tough, he prefers Vodka.
- If A is a wimp, he prefers Quiche.

In both cases the cost to A of having the less preferred breakfast is smaller than the cost of having to engage in a duel (i.e., the tough guy would eat quiche if it meant B would not challenge and the wimp would drink Vodka if it meant B would not challenge).

In our notation:

$$T_A = \{T,W\},\ M = \{v,q\},\ R = \{d,n\}.$$

A's messages = drink vodka, eat quiche, B's responses: duel/not duel. A's information sets: $e_A(T)$ = sees he is tough or $e_A(w)$ = sees he is wimp. B's information sets: $e_B(v)$ = sees A drink vodka, $e_B(q)$ = sees A eat quiche. $b_A = \left(b_{Av}(T), b_{Aq}(T); b_{Av}(W), b_{Aq}(W)\right)$ (which, respectively, represent the probabilities of playing: vodka if tough, quiche if tough; vodka if wimp, quiche if wimp). $b_B = \left(b_B d(V), b_{Bn}(v); b_{Bd}(q), b_{Bn}(q)\right)$ (which respectively, the probabilities of playing duel if vodka, do not duel if vodka; duel if quiche, not duel if quiche).

Kreps claims there are two classes of Sequential Equilibria:

Class I. A has vodka regardless of type. (I.1) B duels if quiche, not duels if vodka, and this is supported with off the equation path beliefs; if A has quiche then A is more likely to be a wimp [or (I.2) B randomizes if quiche].

Class II. A has quiche regardless of type. (II.1) B duels if vodka, not duels if quiche and this is supported by out of equation beliefs: If A has vodka, then A is more likely to be a wimp then a tough [or (II.2) B randomizes if vodka]. Kreps thinks the Class II. equilibria are "unintuitive" since B interprets A's drinking vodka as a good indication A is a wimp, even though it is the tough A that prefers vodka.

In the class II equilibrium A could make the following speech:

> "I will have vodka and you should conclude I am tough, because if you concluded this then you will not duel and I am better off. You should conclude this because if I were a wimp then I would have no incentive to make this defection from equilibrium. Because were I a wimp, no matter what conclusion you made about my type from me drinking vodka, I would be worse off then if I stuck to the equilibrium."

Kreps then rules out equilibria of class II as unintuitive. Let us first show these are sequential equilibria.

Step 1. Compute the best responses for A at each information set, A has two information sets $e_A(T)$, $e_A(w)$ (denoted "T" and "W"). At information set $e_A(W)$:

$$V_A(b|W) = b_{Av}(W)[2b_{Bn}(v)] + b_{Aq}(W)[3b_{Bn}(q) + b_{Bd}(q)]$$

$$= b_{Av}(W)[2b_{Bn}(v)] + b_{Aq}(W)[1+2b_{Bn}(q)].$$

So the wimp eats <u>quiche</u> if $1 + 2b_{Bn}(q) > 2b_{Bn}(v)$

$$BR_A(b|w) = b_{Av}(w), b_{Aq}(w)) = \begin{cases} (1,0) & b_{Bn}(q) < b_{Bn}(v) - \frac{1}{2} \\ (\alpha, 1-\alpha) & = \\ (0,1) & > \end{cases} .$$

At information set $e_A(T)$:

$$V_A(b|T) = b_{Av}(T)\left[3b_{Bn}(v) + b_{Bd}(v)\right] + b_{Aq}(T)\left[2b_{Bn}(q)\right]$$

$$= b_{Av}(t)\left[1 + 2b_{Bn}(v)\right] + b_{Aq}(T)\left[2b_{Bn}(q)\right].$$

So tough guy drinks vodka if $1 + 2\, b_{Bn}(v) > 2b_{Bn}(q)$

$$BR_A(b|T) = \left(b_{Av}(T), b_{Aq}(T)\right) = \begin{cases} (1,0) & \text{if } b_{Bn}(q) < b_{Bn}(v) - \frac{1}{2} \\ (\alpha, 1-\alpha) & = \\ (0,1) & > \end{cases} .$$

Compute the best responses for B at each information set; B has two information sets $e_B(v) + e_B(q)$ (denoted "v" and "q").

Let $\left(\mu(T|v), \mu(W|v)\right)$ be the conditional probability distribution over $e_B(v)$.

$$V_B^{\mu}(b|v) = \mu(T|v)\left[b_{Bn}(v)\right] + \mu(W|v)\left[b_{Bn}(v) + 2b_{Bd}(v)\right]$$

$$= b_{Bn}(v)\left[\mu(T|v) + \mu(W|v)\right] + b_{Bd}(v)\left[2\mu(W|v)\right]$$

$$= b_{Bn}(v) + b_{Bd}(v)\left[2\mu(W|v)\right].$$

So B does <u>not</u> duels vodka drinker if $\mu(W|v) < 1/2$ (i.e., $\mu(T|v) > 1/2$) that is, if vodka "signals" A is tough guy with probability $> 1/2$).

$$BR_B^{\mu}(b|v) = \left(b_{Bn}(v), b_{Bd}(v)\right) = \begin{cases} (1,0) & \mu(W|v) < \frac{1}{2} \\ (\alpha, 1-\alpha) & \mu(W|v) = \frac{1}{2} \\ (0,1) & \mu(W|v) > \frac{1}{2} \end{cases} .$$

Let $\left(\mu(T|q), \mu(W|q)\right)$ be the conditional probability distribution on $e_B(q)$.

$$V_B^{\mu}(b|q) = b_{Bn}(q) + b_{Bd}(q)\left[2\mu(w|q)\right].$$

B does not duel quiche eater if $\mu(w|q) < 1/2$. (If eating quiche "signals" B is a tough guy.)

$$BR_B^{\mu}(b|q) = \left(b_{Bn}(q), b_{Bd}(q)\right) = \begin{cases} (1,0) & < \frac{1}{2} \\ (\alpha, 1-\alpha) & \mu(w|q) = \frac{1}{2} \\ (0,1) & > \frac{1}{2} \end{cases} .$$

Step 2. Break analysis into cases:

Case I. Both tough and wimp drink vodka with probability one.

Case II. Both tough and wimp eat quiche with probability one.

Case I. If T and W play $b_{Av}(T) = b_{Av}(w) = 1$ then B's information set $e_B(q)$ is unreached.

- To get T to play this we need $b_{Bn}(q) < b_{Bn}(v) - 1/2$.
- To get w to play this we need $b_{Bn}(q) < b_{Bn}(v) - 1/2$.

For both inequalities to hold we need $b_{Bn}(q) < b_{Bn}(v) - 1/2$ (we need B to not duel vodka drinkers much)

- B's information set $e_B(v)$ has the conditional pinned down to the prior (since both T and W mimic each other there is no new information) so $\mu(T|v) = P = 0.9$, $\mu(W|v) = P_W = 0.1$. Since $\mu(W|v) = 0.1 < 1/2$ this I implies B plays $b_{Bn}(v) = 1$ (i.e., B doesn't duel the vodka drinker since he believes the vodka drinker has a 0.9 (> 1/2) chance of being a tough guy).

- B's information set $e_B(q)$ is unreached under our proposed strategies so we can impose arbitrary beliefs here. Now in order to get T and W to play their strategies we need $b_{Bn}(q) < 1/2$ (substitute $b_{Bn}(v) = 1$ into the above inequality so we need B to duel quiche eater with probability $\geq 1/2$). Beliefs $\mu(W|q) \geq 1/2$ will support this. So

$$b_A = (b_{Av}(W), b_{Aq}(W); b_{Av}(T), b_{Aq}(T) = (1,0;1,0)$$

$$b_B = (b_{Bn}(v), b_{Bd}(v); b_{Bn}(q), b_{Bd}(q)) = (1,0;\alpha,1-\alpha)$$

with beliefs

$$\mu = (\mu(W|v), \mu(T|v); \mu(W|q), \mu(T|q))$$

$$= (0,1;\alpha,1-\alpha)$$

with $\alpha < 1/2$, $\alpha \geq 1/2$ is a set of sequential equilibrium of class I.

Case II. If T and W play $b_{Aq}(T) = b_{Aq}(W) = 1$ then B's information set $e_B(v)$ is unreached.

- To get T to play this we need $b_{Bn}(q) > b_{Bn}(v) - 1/2$.
- To get W to play this we need $b_{Bn}(q) > b_{Bn}(v) + 1/2$.

Both hold if $b_{Bn}(q) > b_{Bn}(v) + 1/2$ (we need B to not duel quiche eaters very much)

- B's information set $e_B(q)$: has the conditional pinned down to the prior (seeing quiche eater gives B no new information). So $\mu(T|q) = 0.9$, $\mu(W|q) = 0.1$. Since $\mu(W|q) = 0.9 > 1/2$ this implies B plays $b_{Bn}(q) = 1$. (B doesn't duel quiche either since he believes a quiche eater is a tough guy with probability $= 0.9 > 1/2$.)

- B's information set $e_B(v)$: is unreached under proposed strategies to get T and W to play their strategies we need $b_{Bn}(v) < 1/2$ (subsequent $b_{Bn}(q) = 1$ into above box). Beliefs $\mu(W|v) \geq 1/2$ will support this.

7.3 The Intuitive Criterion for Signaling Games

Return to the general notation of (7.1). For each message $m \in M$, B winds up at a different information set say $e_B(m)$ which we denote simply by m. Thus B solves at $e_B(m)$

$$\max_{r \in R} V_B^{\mu}(b|m).$$

Let $BR_B^{\mu}(m)$ = subset of R of best responses by player B given message m. For some set S of types of A, which is a subset of T, let $BR_B(S,m)$ = Best responses by B, for all beliefs μ assign probability 1 to the subset S of types.

$$BR_B(S,m) = U\{BR^{\mu}(m)|\mu(t|m) \text{ such that } \sum_{t \in S} \mu(t|m) = 1\}.$$

Let there be a sequential equilibrium in which the expected utility to type to player A is $\hat{u}_A(t)$, $t \in T_A$.

Intuitive Criterion:

Definition. A sequential equilibria for this signaling game fails the intuitive criterion if we can find

(i) a message m' that is unsent in equilibrium.

(ii) a proper subset S of types, (SCT).

(iii) a certain type $t' \in T/S$, subject to:

1. For all $t \in S$, and all $r \in BR(T,m')$, $\hat{u}_t > u(t,m',r)$.
2. For all $r \in BR(T/S,m')$, $u^*_{t'} < u(t',m',r)$.

In words a sequential equilibrium fails the intuitive criterion if some type t' can profitably distinguish himself from a set of types S by sending a message m'

- Where _if_ this type t' can convince the receiver of m' that t' is _not_ of type S then t' does strictly better then in original equilibrium as long as B concludes A is _not_ in type S (for all $r \in BR(T/S,m')$, $\hat{u}_{t'} < u(t',m',r)$.

- And if any player with a type in S tries to do the same he will end up worse off than he was in the original equilibrium, _no matter what_ type B concludes he is

$$(\text{all } t \in S, \text{ all } r \in BR(T,m'), \hat{u}_t > u(t',m',r).$$

Example 1

$X = \{x_0, x_1, \ldots, x_6\}$

$X_0 = \{x_0\}$

$X_1 = \{x_1, x_2\}$

$X_2 = \{x_3, \ldots, x_6\}$

$Z = \{x_7, \ldots, x_{14}\}$

$pd_1(x_7) = x_3$

$pd_2(x_7) = x_1$

$pd_3(x_7) = x_0$

$S(x_1) = \{x_3, x_4\}$

$Z(x_1) = \{x_7, x_8, x_9, x_{10}\}$

$E_1 = \{e_{11} = \{x_1\}, e_{12} = \{x_2\}\}$

$E_2 = \{e_{21} = \{x_3, x_4\}, e_{22} = \{x_5, x_6\}\}'$,

$C_{e_{11}} = \{L, R\} = \{x_3, x_4\}$

$C_{e_{21}} = \{l, r\} = \{x_7, x_9\},\ \{x_8, x_{10}\}$.

$v_1(x_9) = 0$

$v_2(x_8) = 4$

Example 2

$$b_1 = (b_{1e_{11}};b_{ie_{12}})$$

$$b_{1e_{11}} = \{b_1(L|e_{11}),b_1(R|e_{11})\},\ b_1(\cdot|e_{11}) \geq 0,$$

$$b_1(L|e_{11}) + b_1(R|e_{11}) = 1$$

$$b_{1e_{12}} = \{b_1(L|e_{12}),b_1(R|e_{12})\}$$

$$b_2 = (b_{2e_{21}},b_{2e_{22}})$$

$$b_{2e_{21}} = \{b_2(l|e_{21}),b_2(r|e_{21})\},\ b_2(\cdot|e_{22}) > 0$$

$$b_{21e_{22}}$$

$$b_{21e_{22}} = \{b_2(l|e_{21}),b_2(r|e_{21})\}$$

$$b_2(l|e_{21}) + b_2(r|e_{21}) = 1$$

Example 3

The Problem of Unreached Information sets:

(A Slight Modification of Selten's Example 2)

Example 4

Example 5 (KW p. 873)

Example 6

$$\varepsilon_1 = e_1 \{x_1\}$$

$$\varepsilon_2 = e_{21} = \{x_2, x_3\}, \; e_{22} = \{x_6, x_7, x_8, x_9\}$$

$$\varepsilon_3 = e_3 = \{x_4, x_5\}$$

Consider finally Example 7

$e_2 = \{x_{21}, x_3\}$

$e_3 = \{x_4, x_5\}$

Example 8

$e_1 = \{x_0\}$

$e_2 = \{x_1, x_2\}$

Example 9: Consider another example from KW.

$e_1 = \{x_0\}$

$e_2 = \{x_1 x_2\}$

$e_3 = \{x_3, \ldots, x_1\}$

Example 10:

Example 11:

$e_1 = \{x_0\}$

$e_2 = \{x_1, x_2\}$

$e_1\{x_0\}$

$e_1' = \{x'\}$

$e_2 = \{x_1, x_2\}$

Example 12: (The Vodka-Quiche Example)