

Federal Reserve Bank of Minneapolis
Research Department Working Paper

STATIONARY DETERMINISTIC CYCLES IN A CLASS
OF OVERLAPPING GENERATIONS MODELS
WITH LONG LIVED AGENTS

S. Rao Aiyagari

Working Paper 319

Revised April 1987

NOT FOR DISTRIBUTION
WITHOUT AUTHOR'S APPROVAL

I am grateful to the Sloan Foundation and the NSF for financial support. Comments from seminar participants (especially, Neil Wallace) at the Federal Reserve Bank of Minneapolis were very helpful. Any errors of omission, commission or attribution are my sole responsibility.

The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. The material contained is of a preliminary nature, is circulated to stimulate discussion, and is not to be quoted without permission of the author.

Abstract

We consider the existence of deterministically cycling steady state equilibria in a class of stationary overlapping generations models with sufficiently long (but, finite) lived agents. Preferences are of the discounted sum of utilities type with a fixed discount rate. Utility functions with large coefficients of relative risk aversion which generate strong income effects (relative to substitution effects) and backward bending offer curves are permitted. Lifetime endowment patterns are quite arbitrary. We show that if agents have a positive discount rate, then as agents' lifespans get large, short period non-monetary cycles will disappear. Further, constant monetary steady states do not exist and therefore, neither do stationary monetary cycles of any period. We then consider the case where agents have a negative discount rate and show that there are robust examples in which constant monetary steady states as well as stationary monetary cycles (with undiminished amplitude) can occur no matter how long agents live.

I. Introduction

In this paper we investigate the occurrence of short period deterministic cycles in stationary overlapping generations (hereafter, OLG) models with long lived agents. Grandmont's [1985] discussion shows how monetary cycles can arise due to strong income effects (relative to substitution effects) which lead to backward bending offer curves.^{1/} He argues that such endogenous cycles can be consistent with some observed business cycle relationships and that monetary policy can be effective in eliminating cycles. However, all of Grandmont's discussion is in the context of a two period lived agent OLG model and hence all of the cycles in this model have periods greater than the agents' lifespans. This has prompted the comment (Sims [1986]) that observed business cycles have periods much shorter than agents' lifespans and that short period cycles would either be unlikely to exist or be quantitatively insignificant in amplitude in OLG models with long lived agents. The argument for this is presumably based on the incentive (due to concave utility functions) and the opportunity (as agents live many more periods relative to a cycle and hence will overlap with many other generations) to avoid fluctuating lifetime consumptions.

The above argument, however, does not seem entirely convincing. It is true that agents who face a constant interest rate and fluctuating incomes would wish to have smooth consumptions. But, when interest rates themselves are fluctuating, they would not choose to smooth consumption, even when incomes are constant. It is, therefore, not obvious that such short period cycles cannot exist; whatever their magnitude.

This paper considers this issue, for monetary as well as non-monetary cycles in a class of stationary pure exchange OLG economies with long lived agents. The method used is similar to that in Aiyagari [1986a]. We construct a sequence of such OLG economies with longer and longer lived agents. Preferences are of the discounted sum of utilities type with a fixed discount rate and lifetime endowment patterns are quite arbitrary. We fix attention on cycles of a given period and consider what happens as lifetimes become large. The discussion is restricted to cycles of period two for simplicity but as will be seen the method carries over for cycles of any period. In addition, and again for simplicity, we initially take the utility function to be of the constant relative risk aversion type. We believe that this brings out most clearly the reason why such equilibria may or may not exist. It will be seen, however, that this simplification, too, can be dispensed with.

We first consider the case when agents have a positive discount rate. A result from an earlier paper (Aiyagari [1986a]), reproduced here, shows that constant monetary steady states do not exist for any large length of life (denoted T). It is an immediate implication that monetary cycles of any period cannot exist for any T large. Therefore, for this case, we focus on short-period non-monetary cycles and show that these, too, must disappear (i.e., do not exist) as soon as T becomes large. Note that this is stronger than asserting only that the amplitude of cycles goes to zero as agents live longer.

Since much of the discussion of cycles has taken place in the context of monetary equilibria, we need to allow for the existence of (at least) constant monetary steady states when agents have long life-times. This leads us to consider the case where agents have a negative discount rate. In such a case, we show that it is possible to construct robust examples such that both constant monetary steady states as well as cyclical monetary equilibria (with undiminishing amplitude) exist no matter how long agents live. Thus, the intuition referred to earlier seems valid for the case of a positive discount rate but not so when the discount rate is negative.

The above result may also have implications for the existence of stationary "sunspots" equilibria. Spear [1984] shows that these too arise due to strong income effects. Azariadis and Guesnerie [1986] show that such (two-state) sunspots equilibria arise if and only if there are (two-period) deterministic cycles. This connection, however, is not pursued here due to the difficulties inherent in analyzing stochastic steady states in OLG models with more than two period lived agents (Aiyagari [1986a]).^{2/} Another implication would be for endogenous fluctuations in asset prices. The analysis suggests that in positive discount rate OLG economies, sunspots equilibria and endogenous cyclical fluctuations in asset prices unrelated to dividend fluctuations would not occur.

In section II, we lay down the model and exhibit the result for the case of a positive discount rate. A discussion of monetary cycles when the discount rate is negative is contained in

Section III. Section IV concludes. Appendix A shows that constant monetary steady states (and therefore, monetary cycles) do not exist when the discount rate is positive, but can exist when it is negative.

II. Deterministically Cycling Steady States

The model used is a simplified version of Aiyagari [1986a] without any intragenerational heterogeneity. Consider a stationary OLG economy with one representative agent per generation who lives for T periods. At any given date there are T agents of different generations indexed by their current age s which runs from 1 (for the newly born) to T (for the about to die). If we let $c(s)$ be the consumption of an agent at age s , then a newly born agent has preferences given by,

$$\sum_{s=1}^T \beta^{s-1} U(c(s))$$

where $0 < \beta < 1$ and $U(c) = \frac{c^{1-\alpha} - 1}{1-\alpha}$, $\alpha > 0$. Lifetime endowments are given by $\{w_s, s=1,2,..,T\}$. These endowments are viewed as truncations (at T) from a given infinite sequence $\{w_s\}_{s=1}^{\infty}$ which is taken to be bounded and bounded away from zero. As we increase T , we get a sequence of OLG economies with longer and longer lived agents.^{3/}

We let $r_1, r_2, r_1, r_2, \dots$ be a two-period cycle in interest rates with $r_1 > r_2$. Correspondingly $\gamma_1, \gamma_2, \gamma_1, \gamma_2, \dots$ is the two-period cycle in discount factors where

$$\gamma_1 = \frac{1}{1+r_1}$$

and obviously $\gamma_1 < \gamma_2$. Due to stationarity and the focus on steady states we only need to consider two types of agents. Let Agent 1 be the one who faces the sequence r_1, r_2, \dots over his life and let $c^1(s)$, $s = 1, 2, \dots, T$ be his lifetime consumptions. Agent 2 faces the sequence r_2, r_1, \dots and let $c^2(s)$, $s = 1, 2, \dots, T$ be his lifetime consumptions.

The agents solve the following optimization problems.

Agent 1:

$$(1) \quad \max \sum_{s=1}^T \beta^{s-1} U(c^1(s))$$

s.t.

$$\begin{aligned} c^1(1) + \gamma_1 c^1(2) + \gamma_1 \gamma_2 c^1(3) + \gamma_1^2 \gamma_2 c^1(4) + \dots \\ = w_1 + \gamma_1 w_2 + \gamma_1 \gamma_2 w_3 + \gamma_1^2 \gamma_2 w_4 + \dots \end{aligned}$$

Agent 2:

$$(2) \quad \max \sum_{s=1}^T \beta^{s-1} U(c^2(s))$$

s.t.

$$\begin{aligned} c^2(1) + \gamma_2 c^2(2) + \gamma_2 \gamma_1 c^2(3) + \gamma_2^2 \gamma_1 c^2(4) + \dots \\ = w_1 + \gamma_2 w_2 + \gamma_2 \gamma_1 w_3 + \gamma_2^2 \gamma_1 w_4 + \dots \end{aligned}$$

Market clearing: Again, due to stationarity it is enough to look at market clearing at two consecutive dates as shown.

$$\begin{array}{cccccccc}
 r_1 & r_2 & r_1 & r_2 & r_1 & r_2 & r_1 & r_2 & \dots \\
 & & c^1(1) & c^1(2) & c^1(3) & c^1(4) & c^1(5) & \dots & \\
 & & c^2(1) & c^2(2) & c^2(3) & c^2(4) & c^2(5) & c^2(6) & \dots \\
 c^1(1) & c^1(2) & c^1(3) & c^1(4) & c^1(5) & \dots & & &
 \end{array}$$

These yield:^{4/}

$$(3) \quad c^1(1) + c^1(3) + c^1(5) + \dots + c^2(2) + c^2(4) + c^2(6) + \dots = \sum w_s$$

$$(4) \quad c^2(1) + c^2(3) + c^2(5) + \dots + c^1(2) + c^1(4) + c^1(6) + \dots = \sum w_s.$$

Utility maximization now implies that for any agent i ,

$$(5) \quad \frac{c^i(s+1)}{c^i(s)} = \begin{cases} x_1 & \text{if } r_t = r_1 \\ x_2 & \text{if } r_t = r_2 \end{cases}$$

and that $x_1 > x_2$. Note that r_t is the interest rate from t to $(t+1)$. This follows because the FONC for max yields

$$\frac{U'(c_s^i)}{\beta U'(c_{s+1}^i)} = 1 + r_t$$

which implies,

$$\frac{c_{s+1}^i}{c_s^i} = \left(\frac{\beta}{\gamma_t}\right)^{1/\alpha}.$$

Therefore,

$$x_1 = \left(\frac{\beta}{\gamma_1}\right)^{1/\alpha} > x_2 = \left(\frac{\beta}{\gamma_2}\right)^{1/\alpha}.$$

At this point we assume that T is an odd number. The case when T is even is considered after this. Using (5) in the market clearing conditions (3) and (4) we have

$$(6) \quad c^1(1)[1+x_1x_2+(x_1x_2)^2+\dots+(x_1x_2)^{(T+1)/2}] \\ + x_2c^2(1)[1+x_1x_2+(x_1x_2)^2+(x_1x_2)^{(T-1)/2}] = \sum w_s$$

$$(7) \quad c^2(1)[1+x_1x_2+(x_1x_2)^2+\dots] + x_1c^1(1)[1+x_1x_2+(x_1x_2)^2+\dots] = \sum w_s.$$

Let,

$$A = 1 + x_1x_2 + \dots + (x_1x_2)^{(T+1)/2}$$

$$B = 1 + x_1x_2 + \dots + (x_1x_2)^{(T-1)/2}$$

and,

$$\Delta = A^2 - x_1x_2B^2 = A^2 - B(A-1) = A(A-B) + B > 0.$$

We then have,

$$(9) \quad c^1(1)/\sum w_s = (A-x_2B)/\Delta$$

$$(10) \quad c^2(1)/\sum w_s = (A-x_1B)/\Delta$$

$$(11) \quad \frac{c^1(1) - c^2(1)}{\sum w_s} = \frac{B(x_1-x_2)}{\Delta} > 0.$$

$$(12) \quad \frac{c^1(2) - c^2(2)}{\sum w_s} = \frac{x_1c^1(1) - x_2c^2(1)}{\sum w_s} = \frac{(x_1-x_2)A}{\Delta} > 0.$$

It then follows that

$$(13) \quad \frac{c^1(s) - c^2(s)}{\sum w_s} = (x_1x_2)^{(s-1)/2} \frac{B(x_1-x_2)}{\Delta} > 0$$

if s is odd and

$$(14) \quad \frac{c^1(s) - c^2(s)}{\sum w_s} = (x_1 x_2)^{(s-2)/2} \frac{A(x_1 - x_2)}{\Delta} > 0$$

if s is even. In general, we conclude

$$(15) \quad c^1(s) > c^2(s) \quad \forall s$$

i.e., Agent 1, who is born when the interest rate is high, must have a uniformly higher lifetime consumption profile as compared to Agent 2.

When can this happen? We now illustrate the role of a high risk aversion coefficient (high α) in generating large income effects (relative to substitution effects) and backward bending offer curves which can lead to the desired effect on the consumption profiles of the two agents. For this purpose, it is convenient to rewrite the optimization problems of the two agents in the following manner which takes advantage of separability.

For Agent 1:

Let

$$(16) \quad v_1(e^1(1)) = \max \sum_{s \text{ odd}} \beta^{s-1} U(c^1(s))$$

s.t.

$$\sum_{s \text{ odd}} (\gamma_1 \gamma_2)^{(s-1)/2} c^1(s) = e^1(1)$$

$$(17) \quad v_2(e^1(2)) = \max \sum_{s \text{ even}} \beta^{s-2} U(c^1(s))$$

s.t.

$$\sum_{s \text{ even}} (\gamma_1 \gamma_2)^{(s-2)/2} c^1(s) = e^1(2)$$

$$(18) \quad \max V_1(e^1(1)) + \beta V_2(e^1(2))$$

s.t.

$$e^1(1) + \gamma_1 e^1(2) = \tilde{w}_1 + \gamma_1 \tilde{w}_2.$$

For Agent 2:

Same except replace $c^1(s)$, $e^1(1)$, $e^1(2)$ by $c^2(s)$, $e^2(1)$, $e^2(2)$ and in the last step

$$(19) \quad \max V_1(e^2(1)) + \beta V_2(e^2(2))$$

s.t.

$$e^2(1) + \gamma_2 e^2(2) = \tilde{w}_1 + \gamma_2 \tilde{w}_2$$

$$(20) \quad \tilde{w}_1 = \sum_{s \text{ odd}} (\gamma_1 \gamma_2)^{(s-1)/2} w_s$$

$$(21) \quad \tilde{w}_2 = \sum_{s \text{ even}} (\gamma_1 \gamma_2)^{(s-2)/2} w_s.$$

Obviously, we need only consider the two-period optimization problem

$$(22) \quad \max V_1(e_1) + \beta V_2(e_2)$$

s.t.

$$e_1 + \gamma e_2 = \tilde{w}_1 + \gamma \tilde{w}_2$$

because this yields

$$(23) \quad \begin{cases} (e^1(1), e^1(2)) = (e_1, e_2) | \gamma = \gamma_1 \\ (e^2(1), e^2(2)) = (e_1, e_2) | \gamma = \gamma_2. \end{cases}$$

The requirement on consumption profiles derived earlier in (15) then implies that we must have

$$(24) \quad \begin{cases} e^1(1) > e^2(1) \\ e^1(2) > e^2(2) \end{cases}$$

from (16) and (17) and the corresponding problems for the second agent.

Looking at (22) and (23), and keeping in mind that $\gamma_1 < \gamma_2$, we see that this can only happen if the offer curve is positively sloped and e_1 is a gross complement for e_2 ; i.e., e_1 falls as γ rises in the relevant neighborhood. This in turn requires that excess demand for good 2 be positive ($e_2 - \tilde{w}_2 > 0$) and that the risk aversion coefficient for $V_2(\cdot)$ be greater than one, so that the situation is as shown in Figure 1 (all figures are at the end).

Formally, it is easy to verify that

$$(25) \quad \frac{de_1}{d\gamma} = \frac{V_2' [1 - \alpha_2 (e_2 - \tilde{w}_2) / e_2]}{\Delta'}$$

$$(26) \quad \frac{de_2}{d\gamma} = -V_1' [1 + \alpha_1 (\tilde{w}_1 - e_1) / e_1] / \beta \Delta'$$

$$(27) \quad \frac{de_2}{de_1} = \frac{V_1' [1 + \alpha_1 (\tilde{w}_1 - e_1) / e_1]}{\beta V_2' [\alpha_2 (e_2 - \tilde{w}_2) / e_2 - 1]}$$

where

$$\Delta' = V_1'' \gamma^2 / \beta - V_2'', \quad \alpha_1 = - \frac{e_1 V_1''}{V_1'} > 0, \quad \alpha_2 = - \frac{e_2 V_2''}{V_2'} > 0.$$

The parameters α_1 and α_2 will be inherited by $V_1(\cdot)$ and $V_2(\cdot)$ respectively from $U(\cdot)$ via (16) and (17). In fact, for the case of constant relative risk aversion it is easily seen that $\alpha_1 = \alpha_2 = \alpha$. As is also obvious from the picture, to get $e_2 - \tilde{w}_2 > 0$, we need \tilde{w}_1 to be significantly larger than \tilde{w}_2 . This, together with an α sufficiently larger than one, may generate cycles.

It is possible to get a rough idea of magnitudes as follows. We have from (22) and (23) that

$$(28) \quad \frac{V_1'(e^1(1))}{\beta V_2'(e^1(2))} = \frac{1}{\gamma_1} > \frac{1}{\gamma_2} = \frac{V_1'(e^2(1))}{\beta V_2'(e^2(2))}.$$

First, it is not difficult to show (along the lines of Aiyagari [1986a]) that as T gets large both γ_1 and γ_2 converge to β .^{5/} Further, the functions $V_1(\cdot)$ and $V_2(\cdot)$ are nearly identical for large T . The only reason for any difference between them is that we took T to be an odd number so that the definition of $V_1(\cdot)$ contains one additional term as compared to $V_2(\cdot)$. But this difference will tend to zero for large T .^{6/} It then follows from (28) that as T gets large,

$$(29) \quad \begin{cases} e^1(1) \approx e^1(2) \\ e^2(1) \approx e^2(2). \end{cases}$$

Therefore, we get from the budget constraint (22) that

$$(30) \quad e_2 \approx \frac{\tilde{w}_1 + \beta \tilde{w}_2}{1 + \beta}.$$

Plugging this in (25) we get our condition as:

$$1 - \alpha(e_2 - \tilde{w}_2)/e_2 < 0$$

$$(31) \quad \tilde{w}_2 < \left(\frac{\alpha-1}{\alpha+\beta}\right)\tilde{w}_1$$

and approximately we have

$$\tilde{w}_1 = \sum_{s \text{ odd}} \beta^{s-1} w_s$$

$$\tilde{w}_2 = \sum_{s \text{ even}} \beta^{s-2} w_s.$$

This requires (in addition to $\alpha > 1$) that endowment streams be larger in odd periods of life as compared to even periods in the above (present value) sense. As an example, if $\alpha = 2$ and $\beta \approx 1$, we need $\tilde{w}_2 < 1/3 \tilde{w}_1$. Such a requirement may not seem odd in the context of a two-period lived agent OLG model. It does seem a little strange in the context of a many period lived agent OLG model. This consideration in itself may be deemed sufficient to make cycles seem unlikely. We will show, however, that cycles can be ruled out independently of the pattern of lifetime endowments as well as the risk aversion parameter.

Looking at (11) and (12), and in view of footnote 5, we see that,

$$\frac{c^1(1) - c^2(1)}{c^1(2) - c^2(2)} = B/A \rightarrow 1 \text{ as } T \rightarrow \infty.$$

It must then follow from (16) and (17) that

$$\frac{e^1(1) - e^2(1)}{e^1(2) - e^2(2)} \rightarrow 1 \text{ as } T \rightarrow \infty.$$

Looking at Figure 1 and noting that $\gamma_1, \gamma_2 \rightarrow \beta$ we conclude that the slope of the offer curve at $\gamma = \beta$ is

$$\frac{\partial e_2}{\partial e_1} \Big|_{\gamma=\beta} \rightarrow 1 \text{ as } T \rightarrow \infty.$$

This can be seen to be impossible because at $\gamma = \beta$, $V_1' \cong V_2'$ and

$$e_1 \cong e_2 = \frac{\tilde{w}_1 + \beta \tilde{w}_2}{1 + \beta}.$$

Hence we get from (27) that,

$$\frac{\partial e_2}{\partial e_1} \Big|_{\gamma=\beta} = \frac{1 + \frac{\alpha\beta(\tilde{w}_1 - \tilde{w}_2)}{\tilde{w}_1 + \beta\tilde{w}_2}}{\beta \left\{ \frac{\alpha(\tilde{w}_1 - \tilde{w}_2)}{\tilde{w}_1 + \beta\tilde{w}_2} - 1 \right\}} = 1 + \frac{1 + \beta}{\beta \left\{ \frac{\alpha(\tilde{w}_1 - \tilde{w}_2)}{\tilde{w}_1 + \beta\tilde{w}_2} - 1 \right\}}.$$

Therefore

$$\frac{\partial e_2}{\partial e_1} \Big|_{\gamma=\beta}$$

is strictly greater than and bounded away from one.

We now briefly look at the case when T is even. Equations (6) and (7) become

$$[c^1(1) + x_2 c^2(1)] [1 + x_1 x_2 + \dots + (x_1 x_2)^{T/2}] = \sum w_s$$

$$[c^2(1) + x_1 c^1(1)] [1 + x_1 x_2 + \dots + (x_1 x_2)^{T/2}] = \sum w_s.$$

The above two equations imply that, $c^1(1) + x_2 c^2(1) = c^2(1) + x_1 c^1(1)$ and hence, $c^1(1)(1-x_1) = c^2(1)(1-x_2)$. Therefore, either $x_1, x_2 < 1$ or $x_1, x_2 > 1$. Further, we have

$$\frac{c^1(1)}{\sum w_s} = \frac{1 - x_2}{1 - (x_1 x_2)^{(T/2)+1}}$$

$$\frac{c^2(1)}{\sum w_s} = \frac{1 - x_1}{1 - (x_1 x_2)^{(T/2)+1}}$$

Consider what happens if $x_1, x_2 > 1$. Then,

$$(32) \quad \frac{c^1(1) - c^2(1)}{\sum w_s} = \frac{x_1 - x_2}{1 - (x_1 x_2)^{(T/2)+1}} < 0$$

$$(33) \quad \frac{c^1(2) - c^2(2)}{\sum w_s} = \frac{x_1 c^1(1) - x_2 c^2(1)}{\sum w_s} = \frac{x_1 - x_2}{1 - (x_1 x_2)^{(T/2)+1}} < 0$$

and, in general, $c^1(s) < c^2(s) \forall s$. From (16) and (17) we then see that this requires $e^1(1) < e^2(1)$, $e^1(2) < e^2(2)$ and note that $\gamma_1 < \gamma_2 < \beta$ and $V_1(\cdot) \equiv V_2(\cdot)$. In terms of the offer curve picture the situation must look like that in Figure 2 (dashed budget lines).

This will again require that the offer curve be positively sloped in a neighborhood of $\gamma = \beta$ as shown. This will take a high α and a low \tilde{w}_1 relative to \tilde{w}_2 ; i.e., endowments should be relatively larger in even periods of life compared to odd periods. However, the same consideration that was used previously can be appealed to again to eliminate these cycles for all sufficiently large T . From (32) and (33):

$$\frac{c^2(1) - c^1(1)}{c^2(2) - c^1(2)} = 1.$$

This will require

$$\frac{e^2(1) - e^1(1)}{e^2(2) - e^1(2)} \rightarrow 1 \text{ as } T \rightarrow \infty$$

which cannot happen since the offer curve (this time) will have a slope that is strictly less than and bounded away from one as T gets large.

Lastly, consider the case $x_1, x_2 < 1$. Then we have $c^1(1) - c^2(1) = c^1(2) - c^2(2) > 0$ and this time, $c^1(s) > c^2(s) \forall s$. From (16) and (17) this will take $e^1(1) > e^2(1), e^1(2) > e^2(2)$ and this time, $\gamma_2 > \gamma_1 > \beta$ and $V_1(\cdot) \equiv V_2(\cdot)$.

The offer curve picture must look as shown in Figure 1 (solid budget lines).

Again, the same argument as used before (namely the slope of the offer curve at $\gamma=\beta$) will lead to the elimination of these cycles.

Thus, in all cases, cycles cannot survive large T .

Extension to k-period Cycles:

The above analysis extends to cycles of any fixed period k . One can demonstrate that in the limit (as $T \rightarrow \infty$) there is an inconsistency between the requirements of market clearing and properties of the demand functions. It follows that for each k , there is a T_k finite such that for all T exceeding T_k , there cannot be cyclic equilibria with period k . An important, but unanswered question is: Is there a \hat{T} such that for all T exceeding \hat{T} there are no cycles with periods less than some fraction (possibly one) of T ?

Extension to Other Utility Functions

For simplicity we restrict attention to two-period cycles. Following Aiyagari [1986a] we assume that the risk aversion coefficient is bounded and bounded away from zero.^{7/} We let T be even so that there are an equal number of odd and even periods in an agent's life. This makes the functions $V_1(\cdot)$ and $V_2(\cdot)$ in (16) and (17) identical. Utility maximization now implies that,

$$\frac{U'(c^1(s+1))}{U'(c^1(s))} \Big|_{s \text{ odd}} = \frac{U'(c^2(s+1))}{U'(c^2(s))} \Big|_{s \text{ even}} = \frac{\gamma_1}{\beta}$$

$$\frac{U'(c^1(s+1))}{U'(c^1(s))} \Big|_{s \text{ even}} = \frac{U'(c^2(s+1))}{U'(c^2(s))} \Big|_{s \text{ odd}} = \frac{\gamma_2}{\beta}.$$

It is easy to see that either γ_1, γ_2 both exceed β or that they are both less than β . For, suppose to the contrary that $\gamma_1 < \beta < \gamma_2$. Then, it follows from above that,

$$c^1(1) < c^1(2), c^1(3) < c^1(4), c^1(5) < c^1(6) \dots \text{etc, and}$$

$$c^2(1) > c^2(2), c^2(3) > c^2(4), c^2(5) > c^2(6) \dots \text{etc.}$$

The above are inconsistent with the market clearing conditions (3) and (4).

Next, it is easy to see that either $c^1(s) > c^2(s)$ for all s or that $c^1(s) < c^2(s)$ for all s . This follows because,

$$\frac{U'(c^1(s+2))}{U'(c^1(s))} = \frac{U'(c^2(s+2))}{U'(c^2(s))} = \frac{\gamma_1 \gamma_2}{\beta^2} \text{ for all } s.$$

If $c^1(1) > c^2(1)$ then $c^1(3) > c^2(3)$, $c^1(5) > c^2(5)$ and so on. We cannot have $c^1(2) < c^2(2)$ because this will imply, $c^1(4) < c^2(4)$, $c^1(6) < c^2(6)$ and so on, which is inconsistent with market clearing. Therefore, we must have, $c^1(s) > c^2(s)$ for all s . Similarly, if $c^1(1) < c^2(1)$ then, $c^1(s) < c^2(s)$ for all s .

It follows from (16) and (17) and the analogous problems for agent 2 that we must have, either

$$e^1(1) > e^2(1) \text{ and } e^1(2) > e^2(2)$$

or

$$e^1(1) < e^2(1) \text{ and } e^1(2) < e^2(2).$$

From (22) and (23) this leads to the conclusion that the offer curve must be positively sloped. This leads to four possible situations as shown in Figures 1 and 2 (solid or dashed budget lines).

As in the case of constant relative risk aversion, here too it is not difficult to show that $\gamma_1 \gamma_2 \rightarrow \beta^2$ and hence that $\gamma_1, \gamma_2 \rightarrow \beta$ (the latter follows because either $\gamma_1 < \gamma_2 < \beta$ or $\beta < \gamma_1 < \gamma_2$). Therefore, each of the $e^i(s) \rightarrow e^* = (\tilde{w}_1 + \beta \tilde{w}_2) / (1 + \beta)$ where,

$$\tilde{w}_1 = \sum_{s \text{ odd}}^{\infty} \beta^{s-1} w_s$$

$$\tilde{w}_2 = \sum_{s \text{ even}}^{\infty} \beta^{s-2} w_s.$$

The offer curves in the diagrams are therefore drawn in a small neighborhood of e^* . As before, the important fact about the offer curves is the following.

In Figure 1, $\left. \frac{de_2}{de_1} \right|_{e^*} > 1$ where as,

in Figure 2, $\left. \frac{de_2}{de_1} \right|_{e^*} < 1$.

These follow from (27) because at e^* , $V_1' = V_2'$ and $\alpha_1 = \alpha_2$. Consider, first, the situation in Figure 1. We have that for all T sufficiently large,

$$e^1(2) - e^2(2) \geq (1+\epsilon) (e^1(1) - e^2(1))$$

for some ϵ positive. In view of (16) and (17) and the analagous Problems for agent 2 we have that,

$$c^1(2) - c^2(2) > c^1(1) - c^2(1)$$

$$c^1(4) - c^2(4) > c^1(3) - c^2(3)$$

and so on. These inequalities are inconsistent with market clearing because when we subtract (4) from (3) and rearrange terms, we get

$$c^1(1) - c^2(1) + c^1(3) - c^2(3) + c^1(5) - c^2(5) - - =$$

$$c^1(2) - c^2(2) + c^1(4) - c^2(4) + c^1(6) - c^2(6) + - - .$$

The situation in Figure 2 is exactly the opposite because it implies that for all large T ,

$$e^2(2) - e^1(2) \leq (e^2(1) - e^1(1)) / (1+\delta)$$

for some δ positive. This will imply that,

$$c^2(2) - c^1(2) < c^2(1) - c^1(1)$$

$$c^2(4) - c^1(4) < c^2(3) - c^1(3)$$

and so on, which again contradicts market clearing.

Thus, such two-period cycles cannot persist as T gets large and must disappear.

Discussion. There are (at least) two possible objections to viewing the results of this section as reflecting negatively on Grandmont [1985]. One is that Grandmont [1985] is concerned with monetary cycles whereas the set up in this section rules out a constant monetary steady state for all large T which is a prerequisite for obtaining monetary cycles (see footnote 5, (c)). Thus, the cycles analyzed in this section are non-monetary cycles. Of course, to the extent that one regards the assumption of a positive discount rate in this section as reasonable, one could conclude that it does not matter that monetary cycles (of any period) do not exist because a constant monetary steady state does not exist. However, there is an asymmetry with non-monetary cycles because a constant non-monetary steady state always exists for every T whereas (as we have just shown) non-monetary cycles of a fixed period do not exist for any large T .^{8/} We will, in the next section, discuss the existence (and robustness) of monetary steady states, both constant and cyclical. Be that as it may, we think that it is of interest that short period cycles (of either variety) do not exist in this class of OLG models with long lived agents who discount the future. This also suggests that the neglect of cyclical steady states in Aiyagari [1986a] is not of much consequence.

A second objection relates to the specification of preferences here because the period utility function $U(\cdot)$ is the same in every period. Grandmont's [1985] discussion is in terms of the risk aversion coefficient being relatively high for old agents. Quite aside from how to separate the young from the old when people live many (as opposed to only two) periods, our specification with identical $U(\cdot)$ across periods and constant relative risk aversion may not capture this. Even if the risk aversion coefficient were varying instead of being constant, this may not help because consumption at every age converges to the same value i.e., permanent income (see footnote 5). Thus, in the limit, risk aversion coefficients would be equalized across any two (fixed) periods.^{9/}

One possible way of handling this is the following. Still focusing on two-period cycles, suppose we alter preferences as follows. For $i = 1, 2$, agent i maximizes

$$\sum_{s \text{ odd}} \beta^{s-1} \{ (c^i(s))^{1-\alpha_1} - 1 \} / (1-\alpha_1) \\ + \sum_{s \text{ even}} \beta^{s-1} \{ (c^i(s))^{1-\alpha_2} - 1 \} / (1-\alpha_2).$$

Thus, agents have different risk aversion coefficients in odd as opposed to even periods of life. It then follows from (16) and (17) that,

$$(35) \quad V_1(e) = K_1(T)e^{1-\alpha_1} / (1-\alpha_1), \quad V_2(e) = K_2(T)e^{1-\alpha_2} / (1-\alpha_2).$$

Equations (22) and (23) now indicate the sense in which this is comparable to a two-period lived agent problem with different risk aversion coefficients for the young and the old. The young in one generation face γ_1 while those in the next face γ_2 and so on. In fact the analogy can be made a lot closer. The market clearing conditions (3) and (4) can be written,

$$(36) \quad e^1(1)A_1B_1 + e^2(2)A_2B_2 = \sum w_s$$

$$(37) \quad e^2(1)A_1B_1 + e^1(2)A_2B_2 = \sum w_s$$

where,

$$A_1 = \left\{ \sum_{s \text{ odd}} (\gamma_1 \gamma_2)^{\frac{s-1}{2}} (\beta^2 / \gamma_1 \gamma_2)^{\frac{(s-1)}{2\alpha_1}} \right\}^{-1}$$

$$A_2 = \left\{ \sum_{s \text{ even}} (\gamma_1 \gamma_2)^{\frac{s-2}{2}} (\beta^2 / \gamma_1 \gamma_2)^{\frac{(s-2)}{2\alpha_2}} \right\}^{-1}$$

$$B_1 = \sum_{s \text{ odd}} (\beta^2 / \gamma_1 \gamma_2)^{\frac{s-1}{2\alpha_1}}, \quad B_2 = \sum_{s \text{ even}} (\beta^2 / \gamma_1 \gamma_2)^{\frac{s-2}{2\alpha_2}}.$$

These follow because from utility maximization we have,

$$\begin{aligned} \frac{c^i(s+2)}{c^i(s)} &= (\beta^2 / \gamma_1 \gamma_2)^{\frac{1}{\alpha_1}} \text{ for } s \text{ odd} \\ &= (\beta^2 / \gamma_1 \gamma_2)^{\frac{1}{\alpha_2}} \text{ for } s \text{ even.} \end{aligned}$$

Further, we can directly solve problems (16) and (17) to see that,

$$c^i(1) = A_1 e^i(1) \text{ and } c^i(2) = A_2 e^i(2).$$

Now, consider the case, $\gamma_1 \gamma_2 = \beta^2$ and $w(s) = w_1$ for s odd and $w(s) = w_2$ for s even. Then the market clearing conditions reduce to:

$$(38) \quad e^1(1) + e^2(2) = \tilde{w}_1 + \tilde{w}_2$$

$$(39) \quad e^2(1) + e^1(2) = \tilde{w}_1 + \tilde{w}_2.$$

Together with (22) and (23) this is exactly analagous to a two-period lived agent model. However, such a two-period cycle cannot exist because it requires $\gamma_1 \gamma_2 = 1$ to support it, which is not the case.

However, it may be possible to get non-monetary cycles with $\gamma_1 \gamma_2 \neq \beta^2$ and $\alpha_1 \neq \alpha_2$. In this case, equations (36) and (37) reduce to:

$$e^1(1) (\lim_{T \rightarrow \infty} 2B_1/T) + e^2(2) (\lim_{T \rightarrow \infty} 2B_2/T) = \tilde{w}_1 + \tilde{w}_2$$

$$e^2(1) (\lim_{T \rightarrow \infty} 2B_1/T) + e^1(2) (\lim_{T \rightarrow \infty} 2B_2/T) = \tilde{w}_1 + \tilde{w}_2$$

because,

$$A_1, A_2 \neq (1-\beta^2).$$

However, since $\alpha_1 \neq \alpha_2$, it need not be the case that,

$$\lim(2B_1/T) = \lim(2B_2/T)$$

and hence,

$$\frac{e^1(2) - e^2(2)}{e^1(1) - e^2(1)}$$

need not converge to one.

Thus, in this case having $\alpha_1 \neq \alpha_2$ may permit such cycles to persist (but, with amplitude going to zero) even as T tends to infinity.

III. Monetary Cycles: Appendix A shows that neither constant nor cyclical monetary steady states can exist for any large T if the discount rate is positive (i.e., $\beta < 1$). It also shows that constant monetary steady states can exist for all large T if the discount rate is negative. Here, we will exhibit robust examples of monetary cycles when $\beta > 1$. In a two-period monetary cycle, $\gamma_1 \gamma_2 = 1$ and therefore (36) and (37) reduce to (38) and (39) where

$$\tilde{w}_1 = \sum_{s \text{ odd}} w(s) \text{ and } \tilde{w}_2 = \sum_{s \text{ even}} w(s).$$

This happens because $A_1 B_1 = A_2 B_2 = 1$. Note that we do not require $w(s)$ to be constant over s odd (or s even). If $\alpha_1 = \alpha_2 = \alpha$ and T is even, then the functions $V_1(\cdot)$ and $V_2(\cdot)$ are identical (see footnote 6) and utility maximization implies,

$$e^1(2)/e^1(1) = (\beta/\gamma_1)^{\frac{1}{\alpha}}, e^2(2)/e^2(1) = (\beta/\gamma_2)^{\frac{1}{\alpha}} = (\beta\gamma_1)^{\frac{1}{\alpha}}.$$

Hence, equating (38) and (39) we have,

$$e^1(1) \left(1 - (\beta/\gamma_1)^{\frac{1}{\alpha}} \right) = e^2(1) \left(1 - (\beta\gamma_1)^{\frac{1}{\alpha}} \right).$$

Since, $\beta > 1 > \gamma_1$, it follows that

$$\beta > \gamma_1^{-1} > 1 > \gamma_1.$$

Further, the budget constraints are,

$$e^1(1) + \gamma_1 e^1(2) = \tilde{w}_1 + \gamma_1 \tilde{w}_2$$

$$e^2(1) + \gamma_1^{-1} e^2(2) = \tilde{w}_1 + \gamma_1^{-1} \tilde{w}_2.$$

Multiplying the second by γ_1 and adding the two we have,

$$(e^1(1) + e^2(2)) + \gamma_1 (e^1(2) + e^2(1)) = (1 + \gamma_1)(\tilde{w}_1 + \tilde{w}_2).$$

It follows that one of (38) and (39) is redundant. It is straightforward to compute the demand functions for $e^1(1)$ and $e^2(2)$ and use (38) to obtain,

$$\frac{\tilde{w}_1 + \gamma_1 \tilde{w}_2}{\left(1 + \beta^{\frac{1}{\alpha}} \gamma_1^{1 - \frac{1}{\alpha}}\right)} + \frac{(\beta \gamma_1)^{\frac{1}{\alpha}} (\tilde{w}_1 + \gamma_1^{-1} \tilde{w}_2)}{\left(1 + \beta^{\frac{1}{\alpha}} \gamma_1^{-\left(1 - \frac{1}{\alpha}\right)}\right)} = \tilde{w}_1 + \tilde{w}_2.$$

Positive solutions for \tilde{w}_1 and \tilde{w}_2 will exist provided,

$$1 - \gamma_1 < \beta^{\frac{1}{\alpha}} \left[\gamma_1^{\frac{1}{\alpha}} - \gamma_1^{1 - \frac{1}{\alpha}} \right] < \beta^{\frac{2}{\alpha}} (1 - \gamma_1).$$

This requires an α of at least 2. In fact, it requires

$$1 < \beta^{\frac{1}{\alpha}} \frac{\left[\gamma_1^{\frac{1}{\alpha}} - \gamma_1^{1 - \frac{1}{\alpha}} \right]}{(1 - \gamma_1)} < \beta^{\frac{1}{\alpha}} \left(1 - \frac{2}{\alpha}\right)$$

and therefore, $\beta > (1-2/\alpha)^{-\alpha}$. This implies incredibly large values of either α or β or both. For example, if $\alpha = 3$, $\beta > 27$ or $\beta > 7.39$ even if $\alpha = \infty$. However, robust examples of cycles (for all large T , even) do exist. For instance, pick $\alpha = 10$, $\beta = 20$, $\gamma_1 = .99$, $\gamma_1^{-1} = 1.01$, $w(s) = .097$ for s odd, $w(s) = .903$ for s even. This is a stationary monetary cycle that persists with constant amplitude for all large T (even). Graphically, the situation is as shown in Figure 3 with $(\tilde{w}_1, \tilde{w}_2)$ increasing along a ray through the origin as T increases.

One reason why such large values of α and β are required may be that we imposed $\alpha_1 = \alpha_2 = \alpha$. If we allow α_1 and α_2 to differ, then there is an extra degree of freedom which may expand the set of robust examples.^{10/} It should be noted that in these monetary cycles with β exceeding one, the offer curve is positively sloped but consumption in even periods ($e(2)$) is a gross complement for consumption in odd periods ($e(1)$). This requires (From (25)-(27)) a sufficiently small \tilde{w}_1 relative to \tilde{w}_2 and a sufficiently large α_1 . This happens because we took T to be even.

If T is odd, then the functions $V_1(\cdot)$ and $V_2(\cdot)$ are not identical because, as noted earlier, the definition of $V_1(\cdot)$ contains an additional term in the budget constraint than $V_2(\cdot)$. Since, $\gamma_1\gamma_2 = 1$ and $\beta > 1$, we see from footnote (6) that,

$$\frac{V_2(e)}{V_1(e)} \equiv k(T) \rightarrow \frac{1}{\beta^2}.$$

As the discussion on page 10 shows, we require the offer curve to be positively sloped and $e(1)$ to be a gross complement to $e(2)$. From (25)-(27) this requires a large \tilde{w}_1 relative to \tilde{w}_2 and a large α_2 . From utility maximization we have,

$$\begin{aligned} V_1'(e_1)/\beta V_2'(e_2) &= \gamma_1^{-1} \text{ for agent 1} \\ &= \gamma_1 \text{ for agent 2.} \end{aligned}$$

Hence, we have,

$$e^1(2)/e^1(1) = (\beta k(T)/\gamma_1)^{\frac{1}{\alpha}}, \quad e^2(2)/e^2(1) = (\beta k(T)\gamma_1)^{\frac{1}{\alpha}}.$$

This together with market clearing then requires that $e^2(2) < e^2(1)$ and $e^1(2) < e^1(1)$ so that we want,

$$\beta k(T) < \gamma_1 < 1 < \gamma_1^{-1}.$$

Since $k(T)$ is converging to $1/\beta^2$ and β exceeds one the situation is as graphed in Figure 4, which is similar to the case for a two-period lived agent model. It is clear that robust examples of cycles corresponding to figure 4 can easily be constructed. It should be noted that the offer curve in this case crosses the 45° line at a gross interest rate equal to β (and not β^{-1}) because $V_1'(e)/V_2'(e)$ is converging to β^2 .

What is interesting is that the type of endowment patterns that generate cycles (of period two) for T even do not generate cycles for T odd and vice versa. The former requires the total endowment in odd periods to be much smaller than that in even periods while the converse is required for the latter.

The most important aspect of the examples is clearly that monetary cycles can persist with undiminished amplitude. There is no tendency towards damping as there is for the case of a positive discount rate.

If the specification of a negative utility discount rate for agents seems odd, one alternative would be to adopt the scenario in Aiyagari [1986b]. There, the discount rate was taken to be positive but population growth was allowed for. It was shown that a constant monetary steady state exists if and only if the discount rate is less than the growth rate, provided the risk aversion coefficient is small. The proviso, clearly works against the possibility of getting monetary cycles.

IV. Conclusions: The main results are,

- (i) In stationary no growth OLG models where agents have a positive discount rate ($\beta < 1$), short-period cyclical non-monetary steady states cannot exist. Constant monetary steady states do not exist for any T sufficiently large and consequently monetary cycles of any period cannot exist. Non-monetary cycles may exist and persist if agents exhibit systematically oscillating (say, over odd and even periods of life) patterns of risk aversion coefficients and endowments.
- (ii) If agents exhibit a negative discount rate ($\beta > 1$), then constant monetary steady states exist for all T sufficiently large and cyclical monetary steady states also can exist and be undamped given suitable preferences and life-time patterns of endowments. An example in which a two-period cycle can

arise would be one in which agents exhibit a systematically oscillating pattern of endowments (and, possibly, but not necessarily, of preferences) as in (i) above.

Thus, the comment of Sims [1986], referred to in the introduction, seems reasonable when agents have a positive discount rate but not so, otherwise. We conclude that the case for deterministic cycles (and possibly also for stationary "sunspots" equilibria) is weak in a class of OLG models with sufficiently long lived agents, who discount the future positively.

Footnotes

^{1/}As he notes, this observation goes back to Gale [1973].

^{2/}These are similar in spirit to Spear [1985].

^{3/}That cycles can arise in this framework is easily shown by examples. A two-period cycle with two-period lived agents occurs when, $\alpha = 3$, $\beta < 1/27$, $W_1 = 1$, $W_2 = 0$. This cycle is characterized by interest rates r_1 and r_2 where, $(1+r_1)^{-1} = \beta/x_1^3$, $(1+r_2)^{-1} = \beta/x_2^3$, and, $x_1, x_2 = \{[1-\beta^{1/3}] \pm [(1-\beta^{1/3})^2 - 4\beta^{2/3}]^{1/2}\}/2$. Note that this is a monetary cycle; non-monetary cycles do not exist in this two-period lived agent set up. However, if we have four period lived agents, then a two-period non-monetary cycle occurs when, $\alpha = 10$, $\beta = .95$, $W_1 = .041$, $W_2 = .04$, $W_3 = .259$, $W_4 = .66$. This is characterized by, $(1+r_1)^{-1} = .055$ and $(1+r_2)^{-1} = .059$. A two-period monetary cycle can occur when, $\alpha = 28.35$, $\beta = .004225$, $W_1 = .9544$, $W_2 = W_3 = 0$, $W_4 = .0456$. This is characterized by, $(1+r_1)^{-1} = .42345$, $(1+r_2)^{-1} = 2.3616$. Any resemblance of these examples to reality is purely coincidental!

^{4/}Even though the market clearing conditions are stated only in terms of the commodity markets we also require that in addition asset markets satisfy the condition that aggregate desired assets be nonnegative. It can be shown that (following Gale [1973]) if $\gamma_1\gamma_2 \neq 1$ then commodity market clearing implies asset market clearing i.e., aggregate desired assets will be zero. If $\gamma_1\gamma_2 = 1$ then aggregate desired assets may be positive or negative. If it turns out to be positive it can be supported as a monetary equilibrium with a fixed positive quantity of valued fiat

money. However, if aggregate desired assets are negative when $\gamma_1\gamma_2 = 1$, it cannot be supported as an equilibrium (in our definition) and the only equilibria are those with $\gamma_1\gamma_2 \neq 1$ (i.e., non-monetary equilibria, following Wallace [1980]). Aiyagari [1986a] shows that constant monetary steady states do not exist for any T sufficiently large because aggregate desired assets when $\gamma_1 = \gamma_2 = 1$ diverge to minus infinity.

^{5/}In Aiyagari [1986a] attention was restricted to non-cyclical steady states ($\gamma_1 = \gamma_2 = \gamma$) but within generation heterogeneity was allowed. It was shown that,

- a) every sequence of equilibrium γ 's converges to β as T gets large,
- b) consumption at any fixed age s , converges to permanent income evaluated using β ,
- c) monetary steady states do not exist for any T sufficiently large.

In the present context, suppose that $c^1(1)$ and $c^2(1)$ remain bounded and bounded away from zero as T gets large. From (6) and (7) this implies that A/T and B/T are bounded and bounded away from zero. This immediately implies that $x_1x_2 \rightarrow 1$ and further that $(x_1x_2)^T$ is bounded. Therefore, A/B converges to 1. It must then follow from (9) and (10) that $x_1, x_2 \rightarrow 1$. Otherwise, either $c^1(1)$ or $c^2(1)$ will become negative for some finite T . Therefore, γ_1 and γ_2 converge to β .

Note that this argument only shows that the amplitude of cycles must go to zero as T gets large. It does not bear on whether such equilibria can exist.

6/ Direct computation from (16) and (17) shows that, $V_1(e) = k_1(T) e^{1-\alpha}/(1-\alpha)$ and $V_2(e) = k_2(T) e^{(1-\alpha)}/(1-\alpha)$ where,

$$k_1(T) = \left\{ \sum_{s \text{ odd}} (\gamma_1 \gamma_2)^{\frac{(s-1)}{2}} (\beta^2 / \gamma_1 \gamma_2)^{\frac{(s-1)}{2\alpha}} \right\}^\alpha$$

$$k_2(T) = \left\{ \sum_{s \text{ even}} (\gamma_1 \gamma_2)^{\frac{(s-2)}{2}} (\beta^2 / \gamma_1 \gamma_2)^{\frac{(s-2)}{2\alpha}} \right\}^\alpha.$$

Therefore, if $\gamma_1 \gamma_2 + \beta^2 < 1$, then, $k_1(T), k_2(T) \rightarrow (1-\beta^2)^{-\alpha}$.

7/ Boundedness above is sufficient for interest rates to converge to $(1-\beta)/\beta$. Boundedness away from zero also guarantees that consumptions converge to permanent income.

8/ It would have been nice if there was some way we could appeal to Sarkovskii's theorem (see Grandmont [1985], p.1019, Theorem 4.3) to rule out cycles of periodicities other than two. This does not seem possible here.

9/ The convergence of consumptions is not uniform. While it is true that $c(s)$ converges to y (say) for each fixed s (as T gets large), it is not true that $c(T)$ or $c(T-1)$ converges to y .

10/ At the expense of a rather strange specification of preferences alternating over odd and even periods of life, in addition to similarly alternating endowments.

Appendix A

Here we show that constant monetary steady states ($\gamma_1 = \gamma_2 = 1$) cannot occur for any T sufficiently large, if β is less than one.

Define consumption (c) as a function of marginal utility (p) by

$$p = U'(c(p))$$

and assume that the risk aversion coefficient is bounded above, i.e.,

$$0 \leq \alpha(p) = \frac{-cU''}{U'} = \frac{-c(p)}{pc'(p)} \leq \alpha < \infty.$$

It then follows that,

$$\begin{aligned} \ln c(\lambda p) &= \ln c(p) + \frac{d \ln c(\lambda p)}{d \ln \lambda} \Big|_{\hat{\lambda}} \ln \lambda \\ &= \ln c(p) - \ln \lambda / \alpha(\hat{\lambda} p) \end{aligned}$$

where $\hat{\lambda}$ is between λ and one.

Therefore, we have,

- i) if $\lambda > 1$ then $c(\lambda p)/c(p) \leq \lambda^{-\frac{1}{\alpha}}$
- ii) if $\lambda < 1$ then $c(\lambda p)/c(p) \geq \lambda^{-\frac{1}{\alpha}}$.

With $\gamma_1 = \gamma_2 = 1$, the budget constraint and the market clearing condition are identical and give,

$$\sum_s c(s) = \sum_s w(s).$$

Utility maximization implies, $p_{s+1}/p_s = \beta^{-1}$ where $p_s = U'(c(s))$ or equivalently $c(s) = c(p_s)$. An expression for the per capita desired assets of the population is given by [Aiyagari, 1986a],

$$a_T = \frac{1}{T} \sum_{s=1}^T s(c(s)-w(s)).$$

Now, suppose as in the paper that $\beta < 1$. Then, using (i) we have,

$$c(s+1)/c(s) \leq \beta^{\frac{1}{\alpha}}$$

and hence,

$$\begin{aligned} a_T &\leq \frac{1}{T} \sum_{s=1}^T s c(1) \beta^{\frac{(s-1)}{\alpha}} - \frac{1}{T} \sum_{s=1}^T s w(s) \\ &\leq \left(\frac{1}{T} \sum_{s=1}^T w(s) \right) \sum_{s=1}^T s \beta^{\frac{(s-1)}{\alpha}} - \frac{1}{T} \sum_{s=1}^T s w(s). \end{aligned}$$

Since the sequence $w(s)$ is bounded and bounded away from zero, the first term above remains bounded whereas the second diverges to minus infinity. Hence $a_T \rightarrow -\infty$ and a constant monetary steady cannot exist for any large T .

The proof for non-existence of cycles proceeds in a similar way. For example, for two-period cycles, the expressions for per capita desired assets in high interest rate ($a_1(T)$) and low interest rate ($a_2(T)$) periods, respectively are given by:

$$a_1(T) = \frac{1}{T} \left\{ \sum_{s \text{ odd}} \frac{(s+1)}{2} (c^1(s)-w(s)) + \gamma_1 \sum_{s \text{ even}} \frac{s}{2} (c^1(s)-w(s)) \right\}$$

$$\begin{aligned}
 & + \gamma_1 \sum_{s \text{ odd}} \frac{(s-1)}{2} (c^2(s)-w(s)) + \gamma_1^{-1} \sum_{s \text{ even}} \frac{s}{2} (c^2(s)-w(s)) \Big\} \\
 a_2(T) = & \frac{1}{T} \left\{ \gamma_1^{-1} \sum_{s \text{ odd}} \frac{(s-1)}{2} (c^1(s)-w(s)) + \sum_{s \text{ even}} \frac{s}{2} (c^1(s)-w(s)) \right. \\
 & \left. + \sum_{s \text{ odd}} \frac{(s+1)}{2} (c^2(s)-w(s)) + \gamma_1^{-1} \sum_{s \text{ even}} \frac{s}{2} (c^2(s)-w(s)) \right\}
 \end{aligned}$$

and these will tend to $-\infty$ for exactly the same reason.

On the other hand, if $\beta > 1$, then it is easy to construct robust examples where a monetary steady state exists for all large T . Suppose that the coefficient of risk aversion is constant and equal to α . Then,

$$U'(c_s)/\beta U'(c_{s+1}) = 1 \text{ implies } c_{s+1} = c_s \beta^{\frac{1}{\alpha}}.$$

From the budget constraint we have,

$$\sum w(s) = \sum c(s) = c(1) \sum \beta^{\frac{(s-1)}{\alpha}}.$$

Therefore, the expression for a_T becomes,

$$\begin{aligned}
 a_T &= \frac{1}{T} \sum s c(1) \beta^{\frac{(s-1)}{\alpha}} - \frac{1}{T} \sum s w(s) \\
 &= \frac{1}{T} \frac{(\sum s \beta^{\frac{(s-1)}{\alpha}}) \sum w(s)}{\sum \beta^{\frac{(s-1)}{\alpha}}} - \frac{1}{T} \sum s w(s) \\
 &= \sum w(s) \left\{ \frac{\sum s \beta^{\frac{(s-1)}{\alpha}}}{T \sum \beta^{\frac{(s-1)}{\alpha}}} - \frac{\sum s w(s)}{T \sum w(s)} \right\}.
 \end{aligned}$$

This expression will be positive for all large T provided that $\sum_s w(s)/T \sum w(s)$ is bounded away from one (it is always less than one). This is because the first term in parenthesis is converging to one. If, for instance, $w(s)$ is constant, then the second term converges to $1/2$. This remains true if $w(s)$ is constant separately over odd and even s . It follows that a constant monetary steady state will exist for all large T for a wide pattern of lifetime endowments.

It should be noted that each of the above results carries over even if the risk aversion coefficient fluctuates over odd and even periods. We simply have to consider separately sums over odd and even s .

References

- Aiyagari, S. Rao (1986a). "Non-Monetary Steady States in Stationary Overlapping Generations Models With Long Lived Agents and Discounting: Multiplicity, Optimality and Consumption Smoothing," Department of Economics, University of Wisconsin.
- Aiyagari, S. Rao (1986b). "Optimality and Monetary Equilibria in Stationary Overlapping Generations Models With Long Lived Agents: Growth versus Discounting," Federal Reserve Bank of Minneapolis and The University of Wisconsin.
- Azariadis, C., and R. Guesnerie (1986). "Sunspots and Cycles," The Review of Economic Studies, Vol. LIII (5), No. 176 (October), pp. 725-737.
- Gale, D. (1973). "Pure Exchange Equilibrium of Dynamic Economic Models," Journal of Economic Theory, 6, pp. 12-36.
- Grandmont, J. M. (1985). "Competitive Business Cycles," Econometrica, 53(5), pp. 995-1045.
- Sims, Christopher (1986). "Comment," in Models of Economic Dynamics, H. F. Sonnenschein (ed.), Springer-Verlag, New York, 1986.
- Spear, S. E. (1985). "Rational Expectations in the Overlapping Generations Model," Journal of Economic Theory, 35(2), pp. 251-275.
- Spear, S. E. (1984). "Sufficient Conditions for the Existence of Sunspot Equilibria," Journal of Economic Theory, 34, pp. 360-370.
- Wallace, Neil (1980). "The Overlapping Generations Model of Fiat Money," in John Kareken and Neil Wallace (eds.), Models of

Monetary Economies, Federal Reserve Bank of Minneapolis, pp.
49-82.