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NONMONETARY STEADY STATES IN STATIONARY OVERLAPPING GENERATIONS MODELS WITH LONG LIVED AGENTS AND DISCOUNTING: MULTIPLICITY, OPTIMALITY, AND CONSUMPTION SMOOTHING

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Abstract

We construct a sequence of pure exchange, stationary OLG economies in which generations have longer and longer life spans and all agents maximize a discounted sum of utilities with a fixed, positive, and common discount rate. Period utility functions and endowment patterns are subject to mild restrictions and within generation heterogeneity is permitted. We show that:

(i) Every sequence of equilibrium interest rates converges to the discount rate.

(ii) Eventually every nonmonetary steady state is optimal and a monetary steady state will never exist.

(iii) For any agent consumption at any fixed age converges to permanent income evaluated using the utility discount rate.

I. Introduction

Versions of the overlapping generations model (hereafter, OLG) and the model of a fixed number of infinitely lived agents have been popular frameworks for dynamic macroeconomic analysis. The particular version of an infinitely lived agents model that we are interested in here is that where a fixed number of infinitely lived agents maximize a discounted sum of one-period utilities with a common, fixed and positive rate of time preferences in a stationary environment. The one-sector growth models of Cass [1965], and Brock and Mirman [1972] are classic examples. In this paper we undertake a comparison of nonmonetary steady state interest rates[1/] and consumption profiles between this class and a comparable class of OLG models; essentially discrete time versions of Cass and Yaari [1967]. We consider a stationary pure exchange environment with no uncertainty; however, extensions to include production and capital accumulation and uncertainty via random endowments are briefly discussed.

The aspects of the two models that this paper is concerned with are the following. It is well known that OLG models may have nonmonetary steady states that are nonoptimal. In simple cases this also implies that a fixed supply monetary steady state with valued fiat money exists (Wallace [1980]). In contrast, the infinitely lived agents model always yields equilibria that are optimal and contains no natural role for fiat money.[2/]

Secondly, in the class of infinitely lived agents models we are considering (even if agents have different one-period utility functions), it is easy to see that if: (i) aggregate consumption is constant over time (a steady state condition), and (ii) all agents are at an interior optimum, then the interest rate must be constant over time and equal to the fixed, common rate of time preference. Therefore, individual consumptions must also be constant over time and hence equal to permanent income. However, analogous statements do not hold in an OLG model with similar preferences. The reason why could be that when we look at two successive dates, the newly born in the latter and the oldest agent in the former are, in a sense, not at interior optima.[3/] Presumably, this difficulty should disappear with long lived generations; i.e., the "overlapping" structure should become less and less important.

The analysis in the present paper is motivated by the above observations. We construct a sequence of OLG economies indexed by T, the length of life of each generation. Each generation is taken to consist of H types of agents who are distinguished by their one-period utility functions and lifetime endowment patterns. All agents have a fixed, common and positive rate of time preference. Endowments are generated as follows. We take H infinite sequences of nonnegative numbers. The lifetime endowment vector of a type h agent in a T-period lived agent OLG economy (henceforth, OLG(T)) is then taken to be given by the first T elements of the corresponding infinite sequence. We then establish the following results.

If r(T) is any nonmonetary steady state interest rate for the above OLG(T) economy then, as T becomes large, any such sequence $\{r(T)\}$ converges to the rate of time preference. Further, a monetary steady state does not exist for any large T. Under a slightly stronger assumption we also show that consumption of a type h agent at any fixed age converges to his/her permanent income evaluated using the utility discount factor. These results obtain under relatively mild assumptions on the one-period utility functions of agents and the infinite sequences of endowments. The following conclusions emerge from the above results for the class of OLG economies considered:

(i) If generations are sufficiently long lived then the interest rate in any nonmonetary steady state is determined (almost) entirely by the rate of time preference. This is true even if there are multiple nonmonetary steady states.

(ii) It follows that multiplicity of nonmonetary steady states becomes less important since the entire set of steady state interest rates converges to the common rate of time preference as life spans become large.

(iii) With sufficiently long lived agents, every nonmonetary steady state is optimal (since the interest rate is positive) and monetary steady states do not exist. In Gale's [1973] terminology, we only have "classical" cases; there are no "Samuelson" cases. This implication is of interest in light of the literature on asset price bubbles. Tirole [1986] shows that the existence of such bubbly equilibria in OLG economies is intimately linked to the inefficiency of the equilibrium without rents and bubbles, and cannot arise in the efficient case. In particular, such bubbles do not arise with a finite number of infinitely lived agents (Tirole [1982]).

(iv) The consumption smoothing result is, to our knowledge, new in the OLG context. It is interesting that borrowing and lending across generations, rather than within cohorts, makes it possible for agents to smooth consumption. The interest rate and consumption profiles are both determined in equilibrium. As agents live longer the overlap among agents of different generations also increases. This enables them to smooth out fluctuating patterns of lifetime endowments.

(v) Lastly, the above results strongly suggest that empirically, infinitely lived agents models of the class considered here would be good approximations to the corresponding class of long (but, finite) lived overlapping generations models. That is, the "overlapping" structure does not make much difference.

The particular specification of preferences considered here for the OLG, as well as the infinitely lived agents model, is quite strong. It is motivated by the fact that sharp results are available for such a specification in the latter class of models which no doubt explains their popularity. In order to have a meaningful comparison, we therefore adopt a similar specification of preferences for the OLG model and consider what happens as lifetimes get large.

Another reason for our specification of preferences is the following. Suppose that each generation in the OLG model also cares about the welfare of the next generation and (additively) discounts it with the same common discount factor and is allowed to leave (optimally chosen levels of) bequests. Then, provided the discount rate is positive, it can be shown that steady states with operative bequests of such an OLG model are identical to those of a fixed number of infinitely lived agents who maximize a discounted sum of utilities with the same discount factor. Thus, the comparison we make should be viewed as reflecting on the unimportance of generationally dependent preferences and operative bequest motives (for this class of preferences) when generations are long lived.

Within the specification adopted here, the assumption of a strictly positive discount rate is critical to all of our results. That this is also the case where the transformation from the OLG model with bequest motives and operative bequests to the analogous infinitely lived agents model is well defined, is, in our opinion, a reasonable justification. It is, however, not essential that all agents in a generation have the same rate of time preference. It is not difficult to show, using the same method, that in such a case the sequence of equilibrium interest rates will converge to the smallest time preference rate; i.e., to the discount rate of the most patient type of agents.

We suspect, however, that the results established here will carry over for more general preference structures. What we have in mind are recursive (but not necessarily time separable) preferences over consumption streams with a well defined notion of a rate of time preference as in Epstein [1983] or Lucas and Stokey [1984]. It seems that for the results described earlier, the important thing is the notion of time preference and impatience and this need not be restricted to preferences of the discounted sum of utilities type with a fixed discount rate.

It should be noted that throughout our focus is on steady states and that we omit consideration of the difficult issue of convergence to a steady state from given initial conditions.[4/] We hope that this is not a very serious omission since the entire set of steady state interest rates converges, as life spans increase, to the rate of time preference. Thus, it would be adequate for our purposes if the set of steady states for each OLG economy were stable; i.e., from given initial conditions, an equilibrium path converges to some steady state. Further consideration of these issues is, however, beyond the scope of the present paper.

The rest of this paper is organized as follows. Section II describes the sequence of OLG economies we consider, the assumptions imposed on preferences and endowment patterns, and characterizes the steady states. In section III we prove the main results of the paper described before. Section IV contains a discussion of the assumptions and some extensions. Section V concludes. Some of the lengthier proofs are relegated to Appendices A and B.

II. Sequence of OLG Economies

The model described here is similar to that in Cass and Yaari [1967]. At each date t ($t = 1, 2, \ldots$), a continuum of agents whose size is normalized to unity are born, each of whom lives T periods. At a given date t, agents of different generations are indexed by their current age s, which runs from 0 through (T-1) and s = 0 describes the newly born agents. Heterogeneity is introduced by indexing agents also by their type h which runs from 1 through H. The fraction of each generation which is type h is given by γ^h and these fractions sum to one. At the initial date (t=1) there also exist agents born at dates 0, -1, -2, ..., -(T-2).

Endowments are described as follows. For fixed h, let $\{\alpha_s^h\}_{s=0}^{\infty}$ be an infinite sequence of nonnegative numbers. Let $\omega_s^h(t,T)$ be the (nonstorable) endowment of agent type h of age s at time t in an OLG model with T period lived agents. Then we put

$$\omega_s^h(t,T) = \alpha_s^h \qquad h = 1, 2, \ldots, H, \quad s = 0, 1, \ldots, T-1. \tag{2.1}$$

The numbers $\{\alpha_s^h\}$ for s = 0, 1, ..., (T-1) and h = 1, 2, ..., H describe the distribution of endowments among members of a given generation and across members of different generations. The economy is stationary since neither the total endowment nor its distribution is time dependent.

Let $c_s^h(t,T)$ be the consumption of agent type h of age s at time t. Preferences of such an agent are described by

$$\sum_{\tau=0}^{T-s-1} \beta^{\tau} U_h(c_{s+\tau}^h(t+\tau,T)), \qquad 0 < \beta < 1. \tag{2.2}$$

h = 1, 2, ..., H, s = 0, 1, ...,T-1. Note that the one-period utility function may be different for different agents in a given generation though it is the same for all type h individuals who differ only by dates of birth.

We also assume that in the aggregate there is a fixed positive quantity M of fiat money held by the initial old (s = 1, 2, ..., T-1). We will distinguish between steady state equilibria with zero value of money (non-monetary) and those with positive value of money (monetary).

The above economy is well defined for every T. This is because the endowments in (2.1) are taken to be the truncations of the H infinite sequences $\{\alpha_s^h\}_{s=0}^{\infty}$, truncated at (T-1). Similarly, the preferences defined in (2.2) can be extended naturally as T is increased.

A steady state competitive equilibrium for the above economy is described by the following. Let r_T be the interest rate and let $\ell_s^h(T)$ be claims to consumption (loans due plus real value of money holdings) held by agent (h,s) at any date. Then an agent faces the following sequence of budget constraints.

$$(2.3) \qquad \ell_s^h(T) + \alpha_s^h = c_s^h(T) + \frac{\ell_{s+1}^h(T)}{1 + r_T} \qquad \begin{array}{l} s = 0, 1, \ldots, T-1 \\ h = 1, 2, \ldots, H \end{array}$$

$$(2.4) \qquad \ell_0^h(T) = \ell_T^h(T) = 0.$$

Since preferences are strictly selfish no intergenerational bequests are allowed in the budget constraints. Per-capita assets of the population, denoted a_T, are given by

$$(2.5) \qquad a_T = \frac{1}{T} \sum_{s=0}^{T-1} \sum_{h=1}^{H} \gamma^h \ell_s^h(T)$$

The first order necessary conditions for a utility maximum subject to the budget constraints are:

$$(2.6) \qquad \frac{\beta U_h'(c_{s+1}^h(T))}{U_h'(c_s^h(T))} = \frac{1}{1 + r_T} \qquad \begin{array}{l} s = 0, 1, \ldots, T-2 \\ h = 1, 2, \ldots, H. \end{array}$$

By virtue of (2.4), the sequence of budget constraints in (2.3) can be collapsed into the following single lifetime budget constraint

$$(2.7)\qquad \sum_{s=0}^{T-1} \frac{1}{(1+r_T)^s} c_s^h(T) = \sum_{s=0}^{T-1} \frac{\alpha_s^h}{(1+r_T)^s} \qquad h = 1, 2, \ldots, H.$$

We now develop an alternative expression for per-capita assets, a_T. From (2.3) and (2.4) we have

$$\ell_{T-1}^h(T) = c_{T-1}^h(T) - \alpha_{T-1}^h$$

$$\ell_{T-2}^h(T) = c_{T-2}^h(T) - \alpha_{T-2}^h + \frac{\ell_{T-1}^h(T)}{1 + r_T}$$

$$= c_{T-2}^h(T) - \alpha_{T-2}^h + \frac{c_{T-1}^h(T) - \alpha_{T-1}^h}{1 + r_T}.$$

Proceeding backwards in this way we get

$$\ell_1^h(T) = c_1^h(T) - \alpha_1^h + \frac{c_2^h(T) - \alpha_2^h}{1 + r_T} + \ldots + \frac{c_{T-1}^h(T) + \alpha_{T-1}^h}{(1+r_T)^{T-2}}.$$

Substituting the above expressions in (2.5), rearranging terms and noting that $\ell_0^h(T)$ is zero, we obtain

$$(2.8)\qquad a_T = \frac{1}{T} \sum_{s=1}^{T-1} \sum_{j=0}^{s-1} (1+r_T)^{-j} \sum_{h=1}^{H} \gamma^h (c_s^h(T) - \alpha_s^h).$$

If $r_T \neq 0$, the above can be simplified by noting that

$$\sum_{j=0}^{s-1} (1+r_T)^{-j} = (1+r_T) \frac{\{1-(1+r_T)^{-s}\}}{r_T}.$$

Hence, we have

$$(2.9)\qquad \frac{r_T a_T}{1+r_T} = \frac{1}{T}\sum_{s=1}^{T-1}\left(1-(1+r_T)^{-s}\right)\sum_{h=1}^{H}\gamma^h(c_s^h(T)-\alpha_s^h)$$

$$= \frac{1}{T}\sum_{s=0}^{T-1}\gamma^h(c_s^h(T)-\alpha_s^h)$$

in view of the budget constraints (2.7) multiplied by γ^h and then summed over h. The aggregate resource constraint for the economy is given by

$$(2.10)\qquad \sum_{s=0}^{T-1}\sum_{h=1}^{H}\gamma^h(c_s^h(T)-\alpha_s^h) = 0$$

When $r_T = 0$, the budget constraints (2.7) automatically imply the resource constraint (2.10). Per capita assets a_T from (2.8) may be expressed as

$$(2.11)\qquad a_T(r_T=0) = \frac{1}{T}\sum_{s=0}^{T-1}\sum_{h=1}^{H}s\gamma^h(c_s^h(T)-\alpha_s^h)$$

The above considerations lead to the following:

Definition 1 (nonmonetary steady state). A nonmonetary steady state for the OLG(T) economy consists of r_T and $\{c_s^h(T)\}$ which satisfy equations (2.6), (2.7) and $a_T = 0$.

In view of (2.9) the last condition may be replaced by (2.10) provided $r_T \neq 0$. If $r_T = 0$, then the appropriate expression for a_T is given by (2.11). In such an equilibrium the real value of the aggregate money stock is zero, or equivalently, the price level is infinite.

We also have:

Definition 2 (monetary steady state). A monetary steady state for the OLG(T) economy consists of $\{c_s^h(T)\}$ that satisfy (2.6) and (2.7) with $r_T = 0$ and such that $a_T(r_T=0)$ from (2.11) is positive.

If p(t) is the price level at t, then in such an equilibrium

$$\frac{M}{Tp(t)} = a_T(r_T=0) > 0$$

and obviously,

$$r_T = \frac{p(t)}{p(t+1)} - 1 = 0.$$

We now describe the assumptions imposed on preferences and endowment sequences.

Assumption 1. $U_h(c)$ is twice continuously differentiable, strictly increasing, strictly concave, and satisfies

$$\lim_{c \to 0} U_h'(c) = \infty, \lim_{c \to \infty} U_h'(c) = 0.$$

Let p be the marginal utility associated with a given consumption level, i.e.,

$$p = U_h'(c^h).$$

Assumption 1 allows us to implicitly define consumption as a function of marginal utility, $c^h(p)$. It also follows that $c^h(p)$ is continuously differentiable, strictly decreasing, and satisfies

$$\lim_{p \to 0} c^h(p) = \infty, \lim_{p \to \infty} c^h(p) = 0.$$

The elasticity of marginal utility, denoted $\mu^h(p)$, can be expressed as

$$\mu^h(p) = \frac{-c^h(p)}{p(dc^h/dp)} > 0.$$

In a context with uncertainty, $\mu^h(p)$ would be the measure of relative risk aversion. The next assumption puts uniform upper bounds on the functions, $\mu^h(p)$.

Assumption 2. $\mu^h(p) \leq \mu_2 < \infty$ for all h and p.

We now impose the following assumptions on the sequences $\{\alpha_s^h\}$:

Assumption 3.

$$0 < a \leq \sum_{h=1}^{H} \gamma^h \alpha_s^h \leq A < \infty \quad \text{for all } s$$

In the next section we state and prove the main results.

III. Nonmonetary Steady States

To start with, we prove existence of a non-monetary steady state equilibrium for every T (Theorem 1) so that the results of this paper are not vacuous. The principal results of this paper are contained in Theorems 2, 3, 4 and 5 which are interspersed with propositions leading up to them.

In what follows, T takes values from 2 through infinity, s takes values from 0 through (T-1) and h from 1 through H. Similarly, summations over s run from 0 through (T-1) and summations over h from 1 through H. Unless qualified, statements are assumed to hold for all admissible values of T, s and h.

Theorem 1. Under assumptions 1 and 3, a non-monetary steady state equilibrium exists for every T.

Proof. The proof is along the lines of Gale [1973, p. 34, Theorem 6] and is given in Appendix A.

Next, we will show that every sequence $\{r_T\}$ where r_T is a non-monetary steady state interest rate, converges to $(1-\beta)/\beta$. The intuition behind this is quite straightforward. From utility maximization (equations (2.6)) it follows that if $(1+r_T)$ exceeds $1/\beta$, consumption profiles will be increasing with age and it can be shown that per capita consumption demand will be unbounded as T increases, whereas per capita endowment is bounded. Consequently, for large T, excess demand will result. Conversely, if $(1+r_T)$ is less than $1/\beta$, consumption profiles will be decreasing with age and it can be shown that excess supply will result. The method used to obtain the result is by contradiction; we show that if any subsequence of $(1+r_T)$ does not converge to $1/\beta$, a contradiction will arise either due to excess demand or excess supply as outlined above.

We now make some changes in notation and establish a preliminary result (proposition 1) which is used repeatedly in the main proofs. Let,

(3.1) $$\lambda_T = \frac{1}{\beta(1+r_T)}$$

(3.2) $$p_s^h(T) = U_h'(c_s^h(T))$$

From equation (2.6) and assumption 1, we can then write

(3.3) $$p_{s+1}^h(T) = \lambda_T p_s^h(T), \quad s < T-1.$$

$$(3.4) \qquad c_s^h(T) = c^h(p_s^h(T))$$

Proposition 1. Under assumptions 1 and 2,

(i) If $\lambda_T < 1$, then

$$(3.5) \qquad c_{s+1}^h(T)/c_s^h(T) \geq (\lambda_T)^{-1/\mu_2}, \; s < T-1.$$

(ii) If $\lambda_T > 1$, then

$$(3.6) \qquad c_{s+1}^h(T)/c_s^h(T) \leq (\lambda_T)^{-1/\mu_2}, \; s < T-1.$$

Proof. We will omit the indexes h and T in the proof as these are not relevant and should not cause any confusion.

Let p be fixed and let,

$$x = \ell n\, \lambda, \; f(x) = \ell n\, c(e^x p)$$

Therefore,

$$f'(x) = \frac{pe^x c'(pe^x)}{c(pe^x)} = -\frac{1}{\mu(pe^x)} \leq -\frac{1}{\mu_2}$$

Then we have,

$$\frac{f(x)-f(0)}{x} = f'(\hat{x}) \leq -\frac{1}{\mu_2}$$

where $\hat{x}$ is between 0 and x. Equivalently, we have

$$\frac{\ell n\, c(\lambda p) - \ell n\, c(p)}{\ell n\, \lambda} \leq -\frac{1}{\mu_2}$$

If $\lambda < 1$, then $\ell n\, \lambda < 0$ and we have,

$$c(\lambda p)/c(p) \geq (\lambda)^{-1/\mu_2}$$

whereas, if $\lambda > 1$, then $\ell n\, \lambda > 0$ and we obtain,

$$c(\lambda p)/c(p) \le (\lambda)^{-1/\mu_2}$$

Inequalities (3.5) and (3.6) then follow because, $c_{s+1}/c_s = c(\lambda p_s)/c(p_s)$.

In what follows assumptions 1, 2, and 3 are in force, unless otherwise noted.

Proposition 2. The sequence $\{\lambda_T\}$ is bounded.

Proof. Suppose to the contrary that it is unbounded. Then it has a subsequence that diverges to plus infinity. To save on notation suppose $\{\lambda_T\}$ diverges to plus infinity. Pick $M > 1$. Then for all T sufficiently large, $\lambda_T > M$. From (3.6) we have

$$c^h_{s+1}(T)/c^h_s(T) \le (\lambda_T)^{-1/\mu_2} \le (M)^{-1/\mu_2} = \delta < 1, \; s < T-1.$$

From the budget constraint (2.7) we have

$$\sum_s (\beta\lambda_T)^s \alpha^h_s = \sum_s (\beta\lambda_T)^s c^h_s(T)$$
$$< c^h_0(T) \sum_s (\beta\delta\lambda_T)^s.$$

Therefore,

$$(\delta)^T \sum_h \gamma^h c^h_0(T) > \frac{\sum_s \sum_h (\beta\lambda_T)^s \gamma^h \alpha^h_s}{\sum_s (\beta\lambda_T)^s} \cdot \frac{\sum_s (\beta\lambda_T)^s \delta^T}{\sum_s (\beta\delta\lambda_T)^s}.$$

which is bounded away from zero. This is a contradiction because the resource constraint (2.10) implies that:

$$(\delta)^T \sum_h \gamma^h c_0^h(T) \le (\delta)^T \sum_s \sum_h \gamma^h c_s^h(T)$$
$$\le (\delta)^T \sum_s \sum_h \gamma^h \alpha_s^h$$
$$\le (\delta)^T TA$$
$$\to 0.$$

Hence, the sequence $\{\lambda_T\}$ is bounded.

It follows that $\{\lambda_T\}$ has convergent subsequences. We next show (in Appendix A) in several parts and again by contradiction that every convergent subsequence must converge to one.

Proposition 3. Let $\{\lambda_{Tk}\}$ be a convergent subsequence of $\{\lambda_T\}$ converging to λ. Then $\lambda = 1$.

Proof: In Appendix A.

Theorem 2. Let $\{r_T\}$ be any sequence of nonmonetary steady state interest rates for the given sequence of OLG(T) economies. Then the sequence converges to $(1-\beta)/\beta$ as T becomes large.

Proof. By propositions 2 and 3, any sequence $\{\lambda_T\}$ is bounded and every convergent subsequence of it converges to one. Hence $\{\lambda_T\}$ converges to one and therefore $\{r_T\}$ converges to $(1-\beta)/\beta$.

Theorem 3. For all T sufficiently large, every nonmonetary steady state is optimal.

Proof. Suppose that there were infinitely many T's with at least one non-optimal (and hence with $r_T < 0$) non-monetary equilibrium. Then there exists a subsequence of non-monetary equilibrium interest rates which is negative and hence a convergent subsequence that cannot converge to $(1-\beta)/\beta$ which contradicts Theorem 2.

Theorem 4. For any T sufficiently large, there does not exist a monetary steady state.

Proof. Suppose that there were infinitely many T's for each of which a monetary steady state exists, i.e., $a_T(r_T=0)$ is positive. Then by the proof of Theorem 1 [case (i)], each of these economies also has at least one non-monetary equilibrium with r_T being negative. The argument for Theorem 3 applies resulting in a contradiction.

Consumption smoothing. In this subsection we briefly discuss a strengthening of assumption 2 that is needed to establish consumption smoothing. We then state the main result (Theorem 5) whose lengthy proof is relegated to Appendix B.

Assumption 2'. $0 < \mu_1 \leq \mu^h(p) \leq \mu_2 < \infty$ for all h and p, i.e., the elasticity of marginal utility is bounded and bounded away from zero.

Assumption 2' arises due to the following considerations. The first order conditions (2.6), together with proposition 3, already imply that for any fixed age s and T > s + 1,

$$p^h_{s+1}(T)/p^h_s(T) = \lambda_T = \frac{1}{\beta(1+r_T)} \rightarrow 1$$

as T becomes large. However, this need not imply that $c^h_{s+1}(T)/c^h_s(T)$ (for $T > s + 1$) also converges to one. The problem arises due to the possibility that $p^h_s(T)$ may diverge to infinity.[5/] If the elasticity $\mu^h(p)$ is not bounded away from zero, then it is quite possible for $c^h_{s+1}(T)/c^h_s(T)$ not to converge to one. For instance, if we consider the example $c(p) = e^{-p}/(1-e^{-p})$ then $\mu(p)$ is given by $(1-e^{-p})/p$ which is bounded above but goes to zero as p becomes large. It is then not difficult to construct examples where $p_{s+1}(T)/p_s(T)$ converges to one but the ratio of consumptions does not.

The main result in this sub-section is:

<u>Theorem 5</u> (permanent income/consumption smoothing). Let $\{r_T\}$ be any sequence of nonmonetary steady state equilibrium interest rates and let $\{c^h_s(T)\}$ be the corresponding sequence of consumptions at any fixed age s. Then, under assumptions 1, 2' and 3,

$$\lim_{T\to\infty} c^h_s(T) = (1-\beta) \sum_{j=0}^{\infty} (\beta)^j \alpha^h_j \qquad \text{for fixed } s,\ h.$$

<u>Proof</u>: In Appendix B

IV. Discussion

The methods of proof make heavy use of the boundedness of the elasticity of marginal utility. For the permanent income result, we also require that the elasticity be bounded away from zero. As can be seen from proposition 1 and proposition 4 (in appendix B), these assumptions serve to impose bounds on the growth rate of an agent's consumption and, thereby, on aggregate consumption demand relative to aggregate endowment. These assumptions are stronger than usual but appear to be indispensable. It is, however, possible to slightly weaken assumption 3 on endowment patterns (see, Aiyagari [1986]) to permit the aggregate endowment of some generations in each OLG(T) economy to be zero. It is also possible to let the aggregate endowment of a sequence of generations (across T) to go to zero or be unbounded.

It is also possible to extend the results obtained here to include production and capital accumulation. The results parallel those in the classic one-sector growth model with a representative infinitely lived agent. A more difficult extension, attempted in Aiyagari [1986], is to environments with uncertainty. For the special case of a logarithmic one-period utility function and random endowments, we were able to show that the one period ahead contingent claims prices converge (as T gets large) to the prices that would prevail in the analogous infinitely lived agent model. This was not possible for other utility functions because it turns out that so long as agents live three or more periods, consumption allocations and contingent claims prices depend on the entire infinite history of endowment realizations. This is essentially the same problem that arises in Spear [1985] even though the intertemporal preferences here are time-separable.

V. Conclusion

This paper has investigated the behavior of steady state interest rates and consumption allocations in a class of OLG models in which agents have longer and longer lifetimes. The environment is one of pure exchange, endowment patterns are fairly general but intertemporal preferences are restricted to be of the discounted sum of utilities type with a common, fixed, and positive discount rate. It has been shown that every sequence of equilibrium steady state interest rates converges to the common rate of time preference. The associated sequence of consumptions at any fixed age converge to permanent income. These results can also be extended to include production and capital accumulation.

The chief implications are as follows. In this class of OLG economies with long lived agents, nonmonetary steady state interest rates are almost completely determined by time preference and nearly independent of one-period utility functions or the distribution of endowments either within or across generations. Further, multiplicity of nonmonetary steady states almost disappears when agents are long lived and every such steady state will be optimal and a monetary steady state will not exist. This result is interesting in light of the OLG literature on money since the existence of monetary equilibria in simpler versions of such models is closely linked to the nonoptimality of nonmonetary equilibria (Wallace [1980]).

The consumption smoothing/permanent income result is, to our knowledge, new in the OLG context. It is interesting that borrowing and lending across generations rather than within cohorts makes it possible for each generation to approximately smooth out its consumption pattern, whatever be the pattern of lifetime endowments. Further, both the interest rate and consumption allocations are determined simultaneously in equilibrium. As agents live longer, the overlap across members of different generations becomes greater and this makes it possible for them to smooth out consumption.

The above results and their implications suggest a strong similarity between this class of OLG models with long lived agents and the corresponding class of models of a fixed number of infinitely lived agents. Even in the absence of any intergenerational bequest motives linking members of successive generations, the latter would appear to be reasonably good approximations to the former. However, these results are for steady states only and their implications for nonstationary equilibrium paths are unclear. Even though we show that the multiplicity of steady states almost disappears, it does not seem likely that the continuum of nonstationary equilibrium paths that generally exist in OLG models will also disappear. Their importance may be diminished if the set of steady states is stable.

The case of uncertainty has proved to be difficult (except for log utility) and is currently being studied. Another issue that is clearly important is the speed of convergence of the set of nonmonetary steady state interest rates to the rate of time preference. This will clearly depend on the discount factor itself, the nature of the utility function, and the extent of variability in lifetime endowment patterns. This is also being studied.

Footnotes

1/In the OLG model these are variously referred to as "balanced" by Gale [1973], "barter" by Cass, Okuno, and Zilcha [1980], and "real" by Kehoe and Levine [1985]. The term "nonmonetary" is from Wallace [1980]. For us, a nonmonetary equilibrium is one where the aggregate assets of the population are zero or equivalently, money has zero value. A monetary equilibrium is one with a fixed positive quantity of valued fiat money. Note that in our terminology, in contrast to Kehoe and Levine [1985], golden rule steady states with "negative" money are not monetary equilibria.

2/See, for example, Bewley [1972] or Wilson [1981, section 5].

3/I am grateful to Neil Wallace for the exposition in this paragraph.

4/Some discussion of these is contained in Kehoe and Levine [1985].

5/$p_s^h(T)$ is bounded away from zero by virtue of the fact that $(1+r_T)^{-1} \to \beta < 1$. The budget constraint, together with assumption 3, then guarantees that $c_s^h(T)$ is bounded above for any fixed s. However, $c_s^h(T)$ may not be bounded away from zero.

Appendix A

Proof of Theorem 1.

For purposes of this proof T can be taken to be fixed and is hence suppressed. Differentiate the budget constraint (2.7) with respect to r to get,

$$\sum_s (-s)(1+r)^{-s-1}(c_s^h-\alpha_s^h) + \sum_s (1+r)^{-s} \frac{\partial}{\partial r} (c_s^h-\alpha_s^h) = 0$$

Evaluating the above at r = 0, multiplying by γ^h and summing over h and using (2.11) we have,

$$\text{(A1)} \qquad \frac{\partial}{\partial r}\Big[\sum_s \sum_h \gamma^h(c_s^h-\alpha_s^h)\Big]_{r=0} = \Big[\sum_s \sum_h s\gamma^h(c_s^h-\alpha_s^h)\Big]_{r=0} = Ta(r=0)$$

If a(r=0) is zero, then by definition 1, r = 0 is a non-monetary steady state. So, suppose that a(r=0) is non-zero. We have two cases,

i) a(r=0) is positive: Note that when r equals zero, the budget constraints (2.7) multiplied by γ^h and summed over h imply the resource constraint,

$$\sum_s \sum_h \gamma^h(c_s^h-\alpha_s^h) = 0 \text{ when } r = 0$$

Therefore, by (A1), there exists an $\varepsilon > 0$ such that,

$$\text{(A2)} \qquad \sum_s \sum_h \gamma^h(c_s^h-\alpha_s^h) < 0 \text{ for } r \in (-\varepsilon,0)$$

Multiply the budget constraint (2.7) by $\gamma^h(1+r)^{T-1}$ and sum over h to get,

$$\text{(A3)} \qquad \sum_s \sum_h (1+r)^{T-1-s}\gamma^h(c_s^h-\alpha_s^h) = 0$$

and let $(1+r) \to 0$ from above. It must be that c_s^h for some h and some s = 0, 1, ..., T-2 must be unbounded above because, if each of them is bounded above then we have from (2.6),

$$U'(c_{T-1}^h) = [\beta(1+r)]^{s+1-T}U'(c_s^h), \; s = 0, 1, \ldots, T-2$$

so that $c_{T-1}^h \to 0$ as $(1+r) \to 0$. But then the aggregate budget constraint (A3) will be violated because by assumption $\sum_h \gamma^h \alpha_{T-1}^h > 0$. Hence c_s^h must be unbounded for some h and s and therefore eventually,

$$\sum_s \sum_h \gamma^h(c_s^h - \alpha_s^h) \to \infty \text{ as } (1+r) \to 0$$

Together with (A2), this implies the existence of a strictly negative r which satisfies the resource constraint (2.10) and hence from (2.9) the condition that a = 0.

ii) <u>a(r=0) is negative</u>: By the same reasoning as in (i), using (A1) we have that there is an $\varepsilon > 0$ such that

(A4) $$\sum_s \sum_h \gamma^h(c_s^h - \alpha_s^h) < 0 \text{ for } r \in (0,\varepsilon)$$

Now let $r \to \infty$. It must be that c_s^h for some h and some s = 1, 2, ..., T - 1 is unbounded above because if each of them is bounded above, then we have from (2.6),

$$U'(c_0^h) = [\beta(1+r)]^s U'(c_s^h), \; s = 1, 2, \ldots, T-1$$

so that $c_0^h \to 0$ as $r \to \infty$. But then the aggregate budget constraint corresponding to (2.7) (obtained by multiplying (2.7) by γ^h and summing over h) will be violated since by assumption, $\sum_h \gamma^h \alpha_0^h > 0$. Therefore, c_s^h must be unbounded for some h and s and therefore eventually,

$$\sum_s \sum_h \gamma^h (c_s^h - \alpha_s^h) \to \infty \text{ as } r \to \infty$$

Together with (A4), this proves the existence of a strictly positive r which satisfies the resource constraint (2.10) and hence from (2.9) the condition that a = 0.

Proof of proposition 3. To avoid clutter in notation, assume that $\{\lambda_T\}$ converges to λ. The proof is in several parts.

(i) $\lambda \geq 1$. Suppose to the contrary that $\lambda < 1$. Pick $\varepsilon > 0$ such that $1 > \lambda + \varepsilon$. Then for all T sufficiently large, $\lambda_T < \lambda + \varepsilon$. It then follows from (3.5) that

$$c_{s+1}^h(T)/c_s^h(T) \geq (\lambda_T)^{-1/\mu_2} \geq (\lambda+\varepsilon)^{-1/\mu_2} = \delta > 1, \; s < T-1$$

From the budget constraint (2.7) we have

$$\sum_s (\beta\lambda_T)^s \alpha_s^h = \sum_s (\beta\lambda_T)^s c_s^h(T)$$

$$\leq \frac{c_{T-1}^h(T)}{(\delta)^{T-1}} \sum_s \left(\beta\delta(\lambda+\varepsilon)\right)^s$$

We can always pick ε so that $\lambda + \varepsilon$ is close to (but not equal to) one and hence $\beta(\lambda+\varepsilon)\delta$ is less than one. Therefore,

$$\sum_h \gamma^h c_{T-1}^h(T)/(\delta)^{T-1} \geq \sum_s \sum_h (\beta\lambda_T)^s \gamma^h \alpha_s^h / \sum_s \left(\beta\delta(\lambda+\varepsilon)\right)^s$$

is bounded away from zero by assumption 3. This results in a contradiction because of the resource constraint (2.10) which implies

$$\sum_h \gamma^h c^h_{T-1}(T)/(\delta)^{T-1} \le \sum_s \sum_h \gamma^h c^h_s(T)/(\delta)^{T-1}$$

$$= \sum_s \sum_h \gamma^h \alpha^h_s/(\delta)^{T-1}$$

$$\to 0 \text{ as } T \to \infty$$

(ii) $\underline{\lambda \notin (1,1/\beta)}$. Suppose to the contrary that $\lambda \in (1,1/\beta)$. Choose ε so that $1 < \lambda - \varepsilon < \lambda + \varepsilon < 1/\beta$. Then for all T sufficiently large, $|\lambda_T - \lambda| < \varepsilon$. Using (3.6) we have

$$c^h_{s+1}(T)/c^h_s(T) \le (\lambda_T)^{-1/\mu_2} \le (\lambda-\varepsilon)^{-1/\mu_2} = \delta, \quad s < T-1$$

where δ is between zero and one. Hence, we have

$$\sum_h \gamma^h c^h_0(T) \le \sum_s \sum_h (\beta\lambda_T)^s \gamma^h c^h_s(T)$$

$$\le \sum_s \sum_h (\beta\lambda_T)^s \gamma^h \alpha^h_s$$

$$\le \sum_s \sum_h \left[\beta(\lambda+\varepsilon)\right]^s \gamma^h \alpha^h_s$$

which is bounded above by assumption 3 since $\beta(\lambda+\varepsilon)$ is less than one.

However, this implies that

$$\sum_s \sum_h \gamma^h \alpha^h_s = \sum_s \sum_h \gamma^h c^h_s(T)$$

$$\le \sum_h \gamma^h c^h_0(T) \sum_s (\delta)^s$$

which is bounded above which contradicts assumption 3.

(iii) $\underline{\lambda \le 1/\beta}$. Suppose to the contrary that $\lambda > 1/\beta$ and pick ε so that $\varepsilon < \lambda - 1/\beta$. Then, for all T sufficiently large, $|\lambda_T - \lambda| < \varepsilon$. Just as in part (ii), using (3.6) we have

$$c^h_{s+1}(T)/c^h_s(T) \le (\lambda_T)^{-1/\mu_2} \le (\lambda-\varepsilon)^{-1/\mu_2}$$

$$\le (\beta)^{1/\mu_2} \equiv \frac{(\lambda-\varepsilon)\delta(\varepsilon)}{\lambda+\varepsilon}, \qquad s < T-1$$

where $\delta(\varepsilon)$ is defined in the obvious way. It is obvious that $\delta(0)$ is less than one; hence, for sufficiently small ε (which we assume), $\delta(\varepsilon)$ is less than one. We now have from the budget constraint

$$\sum_s (\beta\lambda_T)^s \alpha^h_s = \sum_s (\beta\lambda_T)^s c^h_s(T)$$

$$\le \sum_s \left(\beta(\lambda-\varepsilon)\delta(\varepsilon)\right)^s c^h_0(T)$$

It follows from the resource constraint (2.10) that

$$\sum_s \sum_h \gamma^h \alpha^h_s = \sum_s \sum_h \gamma^h c^h_s(T)$$

$$\ge \sum_h \gamma^h c^h_0(T)$$

$$\ge \frac{\sum_s \sum_h \left(\beta(\lambda-\varepsilon)\right)^s \gamma^h \alpha^h_s}{\sum_s \left(\beta(\lambda-\varepsilon)\delta(\varepsilon)\right)^s}$$

If $\beta(\lambda-\varepsilon)\delta(\varepsilon)$ is less than one, the above inequality, together with assumption 3, implies that

$$\sum_s \sum_h \gamma^h \alpha^h_s / \sum_s \left(\beta(\lambda-\varepsilon)\right)^s$$

is bounded away from zero which is a contradiction since $\beta(\lambda-\varepsilon) > 1$. Similarly, if $\beta(\lambda-\varepsilon)\delta(\varepsilon)$ exceeds one, then the above inequality implies that

$$\left(\delta(\varepsilon)\right)^T \sum_s \sum_h \gamma^h \alpha^h_s$$

is bounded away from zero which is again a contradiction. In the borderline case where $\beta(\lambda-\varepsilon)\delta(\varepsilon)$ is exactly one, we have

$$\frac{T \sum_s \sum_h \gamma^h \alpha_s^h}{\sum_s \left(\beta(\lambda-\varepsilon)\right)^s} \geq \frac{\sum_s \sum_h \left(\beta(\lambda-\varepsilon)\right)^s \gamma^h \alpha_s^h}{\sum_s \left(\beta(\lambda-\varepsilon)\right)^s}$$

Since $\beta(\lambda-\varepsilon)$ exceeds one, the left hand side above goes to zero, whereas the right hand side is bounded away from zero; resulting again in a contradiction. Hence the result.

(iv) $\underline{\lambda \neq 1/\beta}$. Suppose to the contrary that $\lambda = 1/\beta$. Using the budget constraint, equation (3.6), and the resource constraint we have

$$\sum_s \sum_h (\beta\lambda_T)^s \gamma^h \alpha_s^h = \sum_s \sum_h (\beta\lambda_T)^s \gamma^h c_s^h(T)$$

$$\leq \sum_h \gamma^h c_0^h(T) \sum_s (\beta\lambda_T)^s (\lambda_T)^{-s/\mu_2}$$

$$\leq \sum_s \sum_h \gamma^h \alpha_s^h \sum_s (z_T)^s$$

where $z_T = \beta(\lambda_T)^{1-\frac{1}{\mu_2}}$.

For T sufficiently large, z_T will be less than one. Using assumption 3, we can conclude that $\sum_s (\beta\lambda_T)^s/T$ is bounded above. In a like fashion, we have

$$\sum_s \sum_h \gamma^h \alpha_s^h = \sum_s \sum_h \gamma^h c_s^h(T)$$

$$\leq \sum_h \gamma^h c_0^h(T) \sum_s (\lambda_T)^{-s/\mu_2}$$

$$\leq \sum_j \sum_h (\beta\lambda_T)^j \gamma^h \alpha_j^h \sum_s (\lambda_T)^{-s/\mu_2}$$

It then follows that

$$\sum_s (\beta\lambda_T)^s/T$$

is bounded away from zero. Now let $\delta_T = \beta\lambda_T$ and

$$f(\delta_T,T) = \sum_s (\delta_T)^s/T = \frac{(\delta_T)^T-1}{T(\delta_T-1)}$$

If $(\delta_T)^T \to +\infty$, then $T(\delta_T-1) \to +\infty$. However, since $f(\delta,T)$ is a convex function of δ and $f(1,T) = 1$, we have

$$\begin{aligned} f(\delta_T,T) &\geq 1 + (\delta_T-1)f_1(1,T) \\ &= 1 + \frac{(\delta_T-1)(T-1)}{2} \end{aligned}$$

which implies that $f(\delta_T,T) \to +\infty$, which is a contradiction. In a similar fashion, if $(\delta_T)^T \to 0$, then it must be that $T(\delta_T-1) \to$ minus infinity. If we have to the contrary that $T(\delta_T-1) \geq a$, then $\delta_T \geq 1 + a/T$, and $(\delta_T)^T \geq (1+a/T)^T \to e^a > 0$ which is a contradiction. However, if $(\delta_T)^T \to 0$ and $T(\delta_T-1) \to$ minus infinity, then $f(\delta_T,T) \to 0$, which is again a contradiction.

It follows that $(\delta_T)^T$ is bounded and bounded away from zero. Next, we will show that

$$\sum_{j=0}^{s-1} (\delta_T)^j/s$$

is bounded and bounded away from zero for all $s \leq T - 1$. Let $0 < a \leq (\delta_T)^T \leq A < \infty$. If $\delta_T \geq 1$, then $1 \leq (\delta_T)^j \leq (\delta_T)^T \leq A$. Hence,

$$1 \leq \sum_{j=0}^{s-1} (\delta_T)^j/s \leq A$$

If $\delta_T \leq 1$, then $a \leq (\delta_T)^T \leq (\delta_T)^j \leq 1$ and hence

$$a \le \sum_{j=0}^{s-1} (\delta_T)^j/s \le 1$$

Hence the result.

Next, it is easy to see that

$$\sum_s \sum_h s\gamma^h \alpha_s^h / \sum_s \sum_h \gamma^h \alpha_s^h \ge \frac{a}{A} \sum_s s \to \text{plus infinity}$$

We are now ready to show that $\lambda = 1/\beta$ leads to a contradiction. From (2.5), (2.8), and using (3.1), we have

$$a \sum_s \sum_h s\gamma^h \alpha_s^h \le \sum_s \sum_{j=0}^{s-1} (\beta\lambda_T)^j \sum_h \gamma^h \alpha_s^h$$

$$= \sum_s \sum_{j=0}^{s-1} (\beta\lambda_T)^j \sum_h \gamma^h c_s^h(T)$$

$$\le A \sum_s \sum_h s\gamma^h c_s^h(T)$$

$$\le A \sum_h \gamma^h c_0^h(T) \sum_s s(\lambda_T)^{-s/\mu_2}$$

$$\le A \sum_j \sum_h \gamma^h \alpha_j^h \sum_s s(\lambda_T)^{-s/\mu_2},$$

which is bounded above leading to a contradiction.

It follows that every convergent subsequence of $\{\lambda_T\}$ converges to one.

Appendix B

The proof of Theorem 5 is established by means of a series of propositions.

Proposition 4.

(i) If $\lambda_T < 1$, then

(B1) $$c^h_{s+1}(T)/c^h_s(T) \le (\lambda_T)^{-1/\mu_1}, \quad s < T-1.$$

(ii) If $\lambda_T > 1$, then

(B2) $$c^h_{s+1}(T)/c^h_s(T) \ge (\lambda_T)^{-1/\mu_1}, \quad s < T-1.$$

Proof. Similar to that for proposition 1.

We start by considering the solution to an individual agent's optimization problem which is characterized by equations (2.6) and (2.7). The (unique) solution may be characterized by:

(B3) $$p^h_0(T) = f^h(\lambda_T,T)$$

(B4) $$p^h_s(T) = (\lambda_T)^s p^h_0(T)$$

(B5) $$\sum_s (\beta\lambda_T)^s(c^h_s(T)-\alpha^h_s) = 0$$

where $f^h(+,+)$ is a continuously differentiable function of λ_T for each T. Next we show that there is a neighborhood of one in which the sequence (in T) of functions $f^h(+,T)$ is (uniformly) bounded and bounded away from zero. Further, the sequence of derivatives $f^h_1(\cdot,T)$ is also uniformly bounded. Since $\lambda_T \to 1$, it will then follow that

$$\lim_{T\to\infty} f^h(\lambda_T,T) = \lim_{T\to\infty} f^h(1,T) = U'_h(y^h)$$

where

$$y^h \equiv (1-\beta)\sum_{j=0}^{\infty}(\beta)^j\alpha_j^h$$

Proposition 5. There exists an interval $[\lambda_1,\lambda_2]$ where $0 < \lambda_1 < 1 < \lambda_2 < \infty$ and numbers b, B such that $0 < b \le f^h(\lambda,T) \le B < \infty$ for all h, T and $\lambda \in [\lambda_1,\lambda_2]$.

Proof. Let $\lambda \le 1$.

$$p_{s+1}^h(T) = \lambda p_s^h(T) \le p_s^h(T), \quad s < T-1$$

and hence

$$c_{s+1}^h(T) \ge c_s^h(T), \quad s < T-1$$

It follows that

$$c_0^h(T) \le \sum_s (\lambda\beta)^s c_s^h(T) = \sum_s (\lambda\beta)^s \alpha_s^h$$

$$\le \sum_s (\beta)^s \alpha_s^h$$

which is bounded above.

define, $z(\lambda) = \beta(\lambda)^{1-\frac{1}{\mu_1}}$

Now, using (B2), we have

$$\sum_s (\lambda\beta)^s \alpha_s^h = \sum_s (\lambda\beta)^s c_s^h(T)$$

$$\leq c_0^h(T) \sum_s (\lambda\beta)^s (\lambda)^{-s/\mu_1}$$

$$= c_0^h(T) \sum_s (z(\lambda))^s.$$

Since $z(1) = \beta < 1$, we can find $\lambda_1 < 1$ such that

$$z = \sup \{z(\lambda),\ \lambda_1 \leq \lambda \leq 1\} \leq 1.$$

It then follows that

$$c_0^h(T) \geq (1-z) \sum_s (\lambda_1\beta)^s \alpha_s^h$$

which is bounded away from zero.

Next fix λ_2 so that $1 < \lambda_2 < 1/\beta$ and let λ be such that $1 \leq \lambda \leq \lambda_2$. It follows that

$$p_{s+1}^h(T) = \lambda p_s^h(T) \geq p_s^h(T),\ s < T-1$$

and hence

$$c_{s+1}^h(T) \leq c_s^h(T),\ s < T-1$$

Hence,

$$\sum_s (\lambda\beta)^s \alpha_s^h = \sum_s (\lambda\beta)^s c_s^h(T)$$

$$\leq c_0^h(T) \sum_s (\lambda\beta)^s.$$

It follows that

$$c_0^h(T) \geq (1-\lambda_2\beta) \sum_s (\beta)^s \alpha_s^h$$

which is bounded away from zero.

We also have

$$c_0^h(T) \leq \sum_s (\lambda\beta)^s c_s^h(T) = \sum_s (\lambda\beta)^s \alpha_s^h$$

$$\leq \sum_s (\lambda_2\beta)^s \alpha_s^h$$

which is bounded above.

Since $c_0^h(T) = c^h(p_0^h(T))$ and $p_0^h(T) = f^h(\lambda,T)$, the proposition is proved for $\lambda \in [\lambda_1,\lambda_2]$.

Proposition 6. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that whenever λ', $\lambda'' \in [\lambda_1,\lambda_2]$ and $|\lambda'-\lambda''| < \delta$,

$$|f^h(\lambda',T)-f^h(\lambda'',T)| < \varepsilon.$$

Further, δ can be chosen independently of h and T.

Proof. The result follows by differentiating the budget constraint

$$\sum_s (\lambda\beta)^s \alpha_s^h = \sum_s (\lambda\beta)^s c_s^h(T)$$

with respect to λ and noting that

$$c_s^h(T) = c^h(p_s^h(T))$$

$$p_s^h(T) = (\lambda)^s p_0^h(T) = (\lambda)^s f^h(\lambda,T).$$

We then obtain

$$\left|\frac{\lambda}{f^h} f_1^h\right| \le \frac{\sum_s s(\lambda\beta)^s \alpha_s^h + (1+\frac{1}{\mu_1}) \sum_s s(\lambda\beta)^s c_s^h(T)}{\frac{1}{\mu_2} \sum_s (\lambda\beta)^s c_s^h(T)}.$$

(a) For $\lambda \varepsilon [\lambda_1,\lambda_2]$ we have that:

(i) $\sum_s s(\lambda\beta)^s \alpha_s^h$ is bounded above since $\lambda\beta < 1$, and

(ii) $\sum_s (\lambda\beta)^s c_s^h(T) \ge c_0^h(T)$ which is bounded away from zero.

(b) If $\lambda \varepsilon [1,\lambda_2]$, then $c_s^h(T) \le c_0^h(T)$ and hence,

$$\sum_s s(\beta\lambda)^s c_s^h(T) \le c_0^h(T) \sum_s s(\lambda_2\beta)^s$$

which is bounded above.

(c) If $\lambda \varepsilon [\lambda_1,1]$, then

$$\sum_s s(\beta\lambda)^s c_s^h(T) \le c_0^h(T) \sum_s s(\lambda\beta)^s(\lambda)^{-s/\mu_1}$$

$$\le c_0^h(T) \sum_s s(z)^s$$

which is bounded above (z is as defined in the proof of proposition 5).

From (a), (b), and (c) and because $f^h(\lambda,T)$ is uniformly bounded and bounded away from zero, it follows that for $\lambda \varepsilon [\lambda_1,\lambda_2]$, $f_1^h(\lambda,T)$ is also uniformly bounded. Therefore, let M $> \left|f_1^h(\lambda,T)\right|$ and given $\varepsilon > 0$, choose $\delta = \varepsilon/M$. We then have

$$\left|f^h(\lambda',T)-f^h(\lambda'',T)\right| \le \left|\lambda'-\lambda''\right| \left|f_1^h(\hat{\lambda},T)\right|$$

$$< \delta M = \varepsilon$$

where $\hat{\lambda}$ is between λ' and λ''.

Proof of Theorem 5.

Let $\tilde{p}_0^h(T) = f^h(1,T)$. Then from (B3)-(B5), we have

$$c^h(\tilde{p}_0^h(T)) = (\sum_s (\beta)^s)^{-1} \sum_s (\beta)^s \alpha_s^h.$$

Therefore,

$$\{\tilde{p}_0^h(T)\} \to p_0^h \text{ where } c^h(p_0^h) = y^h \equiv (1-\beta) \sum_{j=0}^{\infty} (\beta)^j \alpha_j^h$$

Given $\varepsilon > 0$, there exists T_1 such that

(B6) $\left|\tilde{p}_0^h(T)-p_0^h\right| < \varepsilon/2$ for all $T > T_1$.

By the previous proposition, there exists $\delta > 0$ such that, whenever $|\lambda-1| < \delta$,

(B7) $\left|f^h(\lambda,T)-f^h(1,T)\right| < \varepsilon/2$ for all T.

Since $\{\lambda_T\} \to 1$, there exists T_2 such that

(B8) $\left|\lambda_T-1\right| < \delta$ for all $T > T_2$.

Combining (B6)-(B8), we have that for all $T > \max\,[T_1,T_2]$,

$$\begin{aligned}\left|p_0^h(T)-p_0^h\right| &= \left|f^h(\lambda_T,T)-p_0^h\right| \\ &\le \left|f^h(\lambda_T,T)-\tilde{p}_0^h(T)\right| + \left|\tilde{p}_0^h(T)-p_0^h\right| \\ &< \left|f^h(\lambda_T,T)-f^h(1,T)\right| + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Hence, $\{p_0^h(T)\} \to p_0^h$. Further, it follows from (B4) and Theorem 2 that for fixed h and s,

$$\{p_s^h(T)\} = \{p_0^h(T)(\lambda_T)^s\} \to p_0^h.$$

Hence,

$$c_s^h(T) = c^h(p_s^h(T)) \xrightarrow[T\to\infty]{} c^h(p_0^h) = y^h.$$

That is, consumption at age zero (or at any fixed age) for any agent h converges to permanent income.

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