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OVERLAPPING GENERATIONS AND INFINITELY LIVED AGENTS

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Abstract

One purpose of this article is to exposit the relationship between overlapping generations models (with constructive immortality) and infinitely lived agents models. We use this to point out errors in "calibration," especially with regard to the use of interest rate data, in the class of representative agent models when growth in population and per capita variables is taken into account. We also point out some common misconceptions regarding the "volume of trade" in representative agent models and show how to reconcile the savings profile of the representative agent with the life cycle savings profile in a generational model.

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I. Introduction

The number of empirical (as well as theoretical) applications of the infinitely lived agents model with the familiar discounted sum of utilities preferences are, by now, too numerous to list; we will therefore not attempt to catalogue them. The most common specification consists of a single "representative" infinitely lived agent together with a technology that may (or may not) permit permanent growth in consumption, output and capital accumulation. For empirical applications the above variables are matched to the corresponding per capita magnitudes in the data. Similarly, data on interest rates is taken to correspond to the net marginal product of capital which also equals the marginal rate of substitution. This is the specification adopted in the widely known "real business cycle" model of Kydland and Prescott [1982].

However, actual economies consist of finitely lived overlapping generations (hereafter, OLG) with some net population growth as well as growth in per capita variables due to technological change. Therefore, it would be nice to have a <u>decentralized</u> justification for the matching procedure described above rather than identify the "representative" agent with a fictitious social planner. Such a decentralized justification is likely to imply some restrictions that may otherwise be missed and is also likely to throw some light on the proper matching of variables in the data (especially interest rates) to variables in the model.

As illustrations we provide three examples:

i) "Calibration" of parameters in representative agent models when growth in population and per-capita magnitudes is taken into account,

ii) "volume of trade" in representative agent models,

iii) reconciling savings profiles in representative agent models with life cycle savings in a generational model.

As is not surprising, the constructive immortality of Barro [1974] via bequest motives and operative bequests in an OLG model with a discounted sum of utilities preferences can be used to make the transition to a model with a fixed number of infinitely lived agents. This then provides the decentralized justification we are seeking. We show this in section II. Section III contains a discussion of the implied restrictions and the above mentioned illustrations. Section IV concludes.

II. OLG and Constructive Immortality

We start with an OLG model in which generations live T periods and are indexed by their age, s = 1, 2, ..., T where s = 1 denotes the newly born and s = T denotes the about to die. At date t (= 1, 2, 3, ...), the size of the newly born generation is given by $(1+n)^t$ and therefore (assuming that this has been going on for some time before t = 1) the population growth rate is n. Each generation consists of a finite number of types of agents indexed by h = 1, 2, ..., H. γ^h denotes the fraction of each generation that is type h and these fractions sum to one. Types are distinguished by their preferences, labor endowments and initial asset holdings.

Preferences are generated as follows. Let $c_s^h(t)$ be the consumption and $\ell_s^h(t)$ the labor supply of an s year old, type h agent at date t and $V_s^h(t)$ be the welfare of the same agent. Then, welfare is generationally interdependent in the following way.

$$\begin{aligned} V_s^h(t) = E_t\{&U_s^h(c_s^h(t),\ell_s^h(t),\varepsilon_s^h(t)) \\ &+\alpha U_{s+1}^h(c_{s+1}^h(t+1),\ell_{s+1}^h(t+1),\varepsilon_{s+1}^h(t+1)) \\ &+ \ldots + \alpha^{T-s}U_T^h(c_T^h(t+T-s),\ell_T^h(t+T-s),\varepsilon_T^h(t+T-s)) \\ &+ (1+n)V_{s-1}^h(t)\}, \quad s = 1, 2, 3, \ldots, T \end{aligned} \tag{2.1}$$

$$V_0^h(t) \equiv \alpha E_t\{V_1^h(t+1)\} \tag{2.2}$$

In the above, $\{U_s^h(\cdot)\}$ are period utility functions for type h and α is a fixed, common discount factor. The $\varepsilon_s^h(t)$'s can be thought of as random preference shocks. $E_t\{\cdot\}$ stands for the expectation conditional on information available at date t of the random variables in $\{\cdot\}$. Thus, a s year old agent's welfare is given by an expected discounted sum of current and future (own) utilities plus the sum of the welfare of each member of the next generation (the (s-1) year olds). A newly born (s=1) agent's welfare takes into account the welfare of his descendants (the one year olds at the next date) discounted by α, because they appear one period later.

We can use the recursive definition of welfare in the above to obtain a direct expression for the welfare of the oldest agent as follows.

$$V_1^h(t) = U_1^h\left(c_1^h(t), \ell_1^h(t), \varepsilon_1^h(t)\right) + \alpha E_t\left[V_2^h(t+1) - (1+n)V_1^h(t+1)\right]$$
$$+ (1+n)\alpha E_t\{V_1^h(t+1)\}$$

Therefore,

$$V_1^h(t) - \alpha E_t\{V_2^h(t+1)\} = U_1^h\left(c_1^h(t), \ell_1^h(t), \varepsilon_1^h(t)\right)$$

$$V_2^h(t) - \alpha E_t\{V_3^h(t+1)\} = U_2^h\left(c_2^h(t), \ell_2^h(t), \varepsilon_2^h(t)\right)$$
$$+ (1+n)\left[V_1^h(t) - \alpha E_t\{V_2^h(t+1)\}\right]$$

$$V_{T-1}^h(t) - \alpha E_t\{V_T^h(t+1)\} = U_{T-1}^h\left(c_{T-1}^h(t), \ell_{T-1}^h(t), \varepsilon_{T-1}^h(t)\right)$$
$$+ (1+n)\left[V_{T-2}^h(t) - \alpha E_t\{V_{T-1}^h(t+1)\}\right]$$

$$V_T^h(t) = U_T^h\left(c_T^h(t), \ell_T^h(t), \varepsilon_T^h(t)\right)$$
$$+ (1+n)V_{T-1}^h(t).$$

Combining the above we have,

$$(2.3) \qquad (1+n)^{-(T-1)}V_T^h(t) = \sum_{s=1}^{T}(1+n)^{-(s-1)}U_s^h\left(c_s^h(t), \ell_s^h(t), \varepsilon_s^h(t)\right)$$
$$+ \alpha(1+n)^{-(T-2)}E_t\{V_T^h(t+1)\}$$

Now let,

$$(2.4) \qquad \beta = (1+n)\alpha \text{ and assume that } \beta < 1.$$

Then we can write,

$$(2.5) \qquad (1+n)^{-(T-1)}V_T^h(1) = E_1\left\{\sum_{t=1}^{\infty}\beta^{t-1}\sum_{s=1}^{T}(1+n)^{-(s-1)}U_s^h\left(c_s^h(t), \ell_s^h(t), \varepsilon_s^h(t)\right)\right\}$$

Thus the oldest agent of type h at the initial date can be viewed as maximizing the above expression which involves the utilities of his own consumptions and labor supplies, his descendants' consumptions and labor supplies, his descendants' descendants' consumptions and labor supplies and so on.

We now develop the budget constraints subject to which agents maximize their welfare. The uncertainty in the model is represented by having J states of the world at each date and where necessary we will use j to denote the state at the current date t and j' to denote the state at the next period date (t+1). We will assume that there are, at each date, spot markets in labor and the single consumption (as well as output and capital) good and one period ahead Arrow-Debreu contingent claims markets in the consumption good. We let $q_t(j')$ be the price at t (in units of the date t consumption good) of a claim to one unit of consumption at (t+1) in the state j'. Let, w(t) be the spot real wage rate and $a_s^h(t)$ be the assets (not including bequests) held by a type h, s year old at date t. We assume that bequests are left at each date by a T year old agent to each of his (T-1) year old direct descendants and use $b_{T-1}^h(t)$ to denote the bequest received by a (T-1) year old. If we let r_t be the risk free real interest rate from t to (t+1), then it follows that,

$$(2.6) \qquad 1 + r_t = \left(\sum_{j'} q_t(j') \right)^{-1}$$

We then have the following sequence of budget constraints.

$$(2.7)\quad \begin{cases} w(t)\ell_s^h(t) + a_s^h(t) = c_s^h(t) + \sum_{j'} q_t(j')a_{s+1}^h(t+1,j'), \\ s = 1, 2, \ldots, T-2 \\ w(t)\ell_{T-1}^h(t) + a_{T-1}^h(t) + b_{T-1}^h(t) = c_{T-1}^h(t) + \sum_{j'} q_t(j')a_T^h(t+1,j') \\ w(t)\ell_T^h(t) + a_T^h(t) - (1+n)b_{T-1}^h(t) = c_T^h(t) \end{cases}$$

where we impose, $a_1^h(t) = 0$. That is, agents when they are born have no assets. Individual optimization subject to the above constraints obviously yields,

$$(2.8)\qquad \frac{-U_{s,2}^h}{U_{s,1}^h}\left(c_s^h(t),\ell_s^h(t),\varepsilon_s^h(t)\right) = w(t),\ s = 1, 2, \ldots, T$$

$$(2.9)\qquad \alpha\pi_t(j')\frac{U_{s+1,1}^h\left(c_{s+1}^h(t+1),\ell_{s+1}^h(t+1),\varepsilon_{s+1}^h(t+1)\right)}{U_{s,1}^h\left(c_s^h(t),\ell_s^h(t),\varepsilon_s^h(t)\right)} = q_t(j'),$$

$$s = 1, 2, \ldots, T-1$$

where $U_{s,i}^h$ is the partial derivative of U_s^h with respect to its ith argument and $\pi_t(j')$ is the probability of occurrence of state j' at (t+1) conditional on t. In addition, the optimal choice of bequest $b_{T-1}^h(t)$ by the T year olds at t give us

$$\frac{\partial V_T^h(t)}{\partial b_{T-1}^h(t)} = -(1+n)U_{T,1}^h\left(c_T^h(t),\ell_T^h(t),\varepsilon_T^h(t)\right) + (1+n)\frac{\partial V_{T-1}^h(t)}{\partial b_{T-1}^h(t)} \le 0$$

with equality if $b_{T-1}^h(t) > 0$. This follows from the definition of $V_T^h(t)$ and the budget constraints (2.7). Again, from the definition of $V_{T-1}^h(t)$ and the budget constraints (2.7) we have,

$$\frac{\partial V^h_{T-1}(t)}{\partial b^h_{T-1}(t)} = U^h_{T-1,1}\left(c^h_{T-1}(t), \ell^h_{T-1}(t), \varepsilon^h_{T-1}(t)\right)$$

Assuming a positive solution for bequests i.e., operative bequests the above equations then imply that,

$$(2.10)\quad \begin{cases} U^h_{s,1}\left(c^h_s(t), \ell^h_s(t), \varepsilon^h_s(t)\right) = U^h_{s+1,1}\left(c^h_{s+1}(t), \ell^h_{s+1}(t), \varepsilon^h_{s+1}(t)\right), \\ U^h_{s,2}\left(c^h_s(t), \ell^h_s(t), \varepsilon^h_s(t)\right) = U^h_{s+1,2}\left(c^h_{s+1}(t), \ell^h_{s+1}(t), \varepsilon^h_{s+1}(t)\right), \\ s = 1, 2, \ldots, T-1 \end{cases}$$

Now, let

$$(2.11)\quad \begin{cases} a^h(t) = \sum_{s=1}^{T} (1+n)^{-(s-1)} a^h_s(t) / \sum_{s=1}^{T} (1+n)^{-(s-1)} \\ \ell^h(t) = \sum_{s=1}^{T} (1+n)^{-(s-1)} \ell^h_s(t) / \sum_{s=1}^{T} (1+n)^{-(s-1)} \\ c^h(t) = \sum_{s=1}^{T} (1+n)^{-(s-1)} c^h_s(t) / \sum_{s=1}^{T} (1+n)^{-(s-1)} \end{cases}$$

and,

$$(2.12)\qquad \hat{q}_t(j') = (1+n)q_t(j')$$

Then, the equations (2.7) can be collapsed into the following,

$$(2.13)\qquad w(t)\ell^h(t) + a^h(t) = c^h(t) + \sum_{j'} \hat{q}_t(j')a^h(t+1,j'),$$

$$t = 1, 2, 3, \ldots$$

where $\left(a^h(t), \ell^h(t), c^h(t)\right)$ are the per-capita amounts of assets, labor supply and consumption of all the type h agents at date t, viewed as members of a single family. When there is no uncertainty the above sequence of budget constraints can be collapsed into a familiar single budget constraint provided that,

$$(2.14)\qquad \lim_{t\to\infty} a^h(t) \prod_{j=1}^{t-1} (1+\hat{r}_j)^{-1} = \lim_{t\to\infty} a^h(t) \prod_{j=1}^{t-1} \frac{(1+n)}{(1+r_j)} = 0$$

where,

$$(2.15)\qquad 1 + \hat{r}_t = \left(\sum_{j'} \hat{q}_t(j')\right)^{-1} = \left((1+n) \sum_{j'} q_t(j')\right)^{-1} = \frac{1 + r_t}{1 + n}$$

In such a case, (2.13) can equivalently be written as,

$$(2.16)\qquad a^h(1) + w(1)\ell^h(1) - c^h(1) + \sum_{t=2}^{\infty} \left[w(t)\ell^h(t) - c^h(t)\right]\left[\prod_{j=1}^{t-1} (1+\hat{r}_j)^{-1}\right] = 0$$

Condition (2.14) is the familiar condition that the present value of aggregate assets go to zero or that aggregate assets not grow at a faster rate than the interest rate. Of course, a single budget constraint analogous to (2.16) can be developed for the case of uncertainty also in terms of the contingent claims prices.

It can be shown that the behavior of each member of the type h family is correctly described by simply considering the behavior of the oldest member of that family since he/she takes into account the welfare of all the other members and their resources. The oldest agent at date 1 then maximizes (2.5) subject to (2.11) and (2.13) taking $a^h(1)$, $w(t)$ and the $\hat{q}_t(j')$ as given. This can be done in two steps. First, define

$$(2.17)\qquad \hat{U}^h\left[c^h(t),\ell^h(t),\varepsilon^h(t)\right] = \frac{\max \sum_{s=1}^{T} (1+n)^{-(s-1)} U_s^h\left(c_s^h(t),\ell_s^h(t),\varepsilon_s^h(t)\right)}{\sum_{s=1}^{T} (1+n)^{-(s-1)}}$$

subject to (2.11) taking $c^h(t)$, $\ell^h(t)$ as given. Note that in (2.17), $\varepsilon^h(t)$ stands for the vector of preference shocks $[\varepsilon_1^h(t), \varepsilon_2^h(t), \ldots, \varepsilon_T^h(t)]$.

We can now view the oldest agent at date 1 as maximizing,

$$(2.18) \qquad \hat{V}_T^h(1) = (1+n)^{-(T-1)} V_T^h(1) / \sum_{s=1}^{T} (1+n)^{-(s-1)}$$

$$= E_1 \{ \sum_{t=1}^{\infty} \beta^{t-1} \hat{U}^h[c^h(t), \ell^h(t), \varepsilon^h(t)] \}$$

subject to the sequence of budget constraints (2.13). In view of (2.11) and (2.17), the first order conditions for this problem will reproduce (2.8)-(2.10).

Now, let

$$(2.19) \qquad \begin{cases} \ell(t) = \sum_h \gamma^h \ell^h(t) \\ c(t) = \sum_h \gamma^h c^h(t) \end{cases}$$

so that $\ell(t)$ is the per capita labor supply at time t and $c(t)$ is per capita consumption. Similarly, let $k(t)$ be the per capita amount of capital in the economy at time t, $f(k(t), \ell(t), \theta(t))$ be the constant returns to scale production function and δ the depreciation rate of capital. The $\theta(t)$'s represent stochastic technology shocks. Then, we obviously have,

$$(2.20) \qquad w(t) = f_2[k(t), \ell(t), \theta(t)]$$

The net marginal product of capital, denoted ξ_t is given by,

$$(2.21) \qquad \xi_t = f_1[k(t), \ell(t), \theta(t)] - \delta$$

The asset market equilibrium condition is

$$(2.22) \qquad \sum_h \gamma^h a^h(t) = k(t)\left[f_1\left(k(t),\ell(t),\theta(t)\right)+1-\delta\right]$$

while the goods market equilibrium condition is,

$$(2.23) \qquad c(t) + (1+n)k(t+1) = f\left(k(t),\ell(t),\theta(t)\right)+(1-\delta)k(t)$$

The optimization problems of the oldest agents lead to the following restrictions in addition to the market clearing conditions.

$$(2.24) \qquad \frac{\beta E_t\left\{(1+\xi_{t+1})\hat{U}_1^h\left(c^h(t+1),\ell^h(t+1),\varepsilon^h(t+1)\right)\right\}}{(1+n)\hat{U}_1^h\left(c^h(t),\ell^h(t),\varepsilon^h(t)\right)} = 1$$

$$(2.25) \qquad \frac{\hat{U}_1^h\left(c^h(t),\ell^h(t),\varepsilon^h(t)\right)}{\beta E_t\left\{\hat{U}_1^h\left(c^h(t+1),\ell^h(t+1),\varepsilon^h(t+1)\right)\right\}} = \frac{1+r_t}{1+n}$$

$$(2.26) \qquad \frac{-\hat{U}_2^h}{\hat{U}_1^h}\left(c^h(t),\ell^h(t),\varepsilon^h(t)\right) = w(t) = f_2\left(k(t),\ell(t),\theta(t)\right)$$

The model can now be viewed as consisting of a fixed number of types H (where γ^h is the fraction which is type h) of infinitely lived agents each of whom maximizes (2.18) subject to (2.13) taking $a^h(1)$, $\varepsilon^h(t)$, $w(t)$ and the $\hat{q}_t(j')$ as given. In equilibrium, (2.19) through (2.26) must hold. In this fashion the OLG structure has been transformed into one of infinitely lived families. Asset holdings, consumptions and labor supplies across members of different ages of a given family have been aggregated into per family magnitudes.

We can proceed from here to a representative agent formulation in two ways. A trivial way is to assume that all the families (indexed by h) are alike. Then, in effect, there is only one family and per family magnitudes are also per capita magnitudes and the index h may be dropped. However, it is possible to allow for some diversity across families which still permits aggregation to a single representative infinitely lived agent formulation. In general, the conditions under which such aggregation obtains are quite stringent (Eichenbaum, Hansen and Richards [1984]). Assuming that these conditions are fulfilled the behavior of the per capita variables is correctly described by a representative agent who maximizes a weighted average of the welfare of the H types of agents where the weights correspond to the fraction of the population that is type h.

This solution can be obtained in two steps. First, define

$$(2.27)\qquad \hat{U}\big(c(t),\ell(t),\varepsilon(t)\big) = \max \sum_{h=1}^{H} \gamma^h \hat{U}^h\big(c^h(t),\ell^h(t),\varepsilon^h(t)\big)$$

subject to (2.19) taking $c(t)$ and $\ell(t)$ as given.

Note that in (2.27), $\varepsilon(t)$ now stands for the matrix of preference shocks $\{\varepsilon_s^h(t)\}$, h = 1, 2, ..., H, s = 1, 2, ..., T. Then maximize,

$$(2.28)\qquad \hat{V}_T(1) = \sum_{h=1}^{H} \gamma^h \hat{V}_T^h(1) = E_1\Big\{\sum_{t=1}^{\infty} \beta^{t-1}\hat{U}\big(c(t),\ell(t),\varepsilon(t)\big)\Big\}$$

subject to the economy's resource constraint (2.23). The first order conditions for this problem are the same as (2.24)-(2.26) with the h indexes being omitted everywhere. We thus obtain a representative infinitely lived agent model. In the process we have provided a decentralized justification for such a model in terms of an OLG model with bequest motives and operative bequests linking all the members of different generations of each family.

So far, we have allowed for population growth in the model. Now we will consider how to permit growth in the per capita variables $c(t)$, $k(t)$ and $y(t)$ $(=f(k(t),\ell(t),\theta(t))$ as well as the real wage $w(t)$. However we will require $\ell(t)$ and $r(t)$ to be stationary (i.e., to not exhibit any geometric growth or decline). We will consider two alternative specifications for growth in per capita magnitudes, trend stationary and difference (in logs) stationary. The former corresponds to deterministic trend growth and the latter to stochastic random walk growth. First we will consider trend stationarity. Suppose that,

$$\theta(t) = (1+\hat{n})^t\hat{\theta}(t) \tag{2.29}$$

where $\hat{\theta}(t)$ is stationary so that the productivity shock to technology is growing at the rate $\hat{n}$, and suppose that the production function is homogeneous of degree one in $(k(t),\theta(t))$ so that we can write,

$$y(t) = f(k(t),\ell(t),\theta(t)) = \theta(t)f\{k(t)/\theta(t),\ell(t)\} \tag{2.30}$$

Let,

$$\begin{cases} \hat{c}(t) = c(t)/(1+\hat{n})^t \\ \hat{k}(t) = k(t)/(1+\hat{n})^t \\ \hat{y}(t) = y(t)/(1+\hat{n})^t = \hat{\theta}(t)f\{\hat{k}(t)/\hat{\theta}(t),\ell(t)\} \end{cases} \tag{2.31}$$

Then, the economy's resource constraint (2.23) can be written as,

$$(2.32)\qquad \hat{c}(t) + (1+n)(1+\hat{n})\hat{k}(t+1) = \hat{\theta}(t)f\left(\frac{\hat{k}(t)}{\hat{\theta}(t)},\ell(t)\right) + (1-\delta)\hat{k}(t)$$

The real wage and the net marginal product of capital are obviously given as,

$$(2.33)\qquad \begin{cases} w(t) = (1+\hat{n})^t\hat{\theta}(t)f_2\left(\hat{k}(t)/\hat{\theta}(t),\ell(t)\right) \\ \xi(t) = f_1\left(\hat{k}(t)/\hat{\theta}(t),\ell(t)\right) - \delta \end{cases}$$

In terms of specifying preferences we may proceed as follows. Assume that $\hat{U}\left(c(t),\ell(t),\varepsilon(t)\right)$ is homogeneous of degree λ in $c(t)$ and $\varepsilon(t)$ and that

$$(2.34)\qquad \varepsilon(t) = (1+\hat{n})^t\hat{\varepsilon}(t)$$

where again the $\hat{\varepsilon}(t)$ are stationary. Note that the homogeneity of degree λ of $\hat{U}$ will follow if each of the primitive utility functions U_s^h is homogeneous of degree λ in $\left(c_s^h,\varepsilon_s^h\right)$. This specification is meant to capture the notion that the $\varepsilon(t)$ represent some reference standard of living which keeps rising over time and to which the consumer compares his actual standard of living to measure utility. Given the above assumption the marginal conditions (2.24)-(2.26) become

$$(2.35)\qquad E_t\left\{\frac{\hat{\beta}\left(1+\xi(t+1)\right)\hat{U}_1\left(\hat{c}(t+1),\ell(t+1),\hat{\varepsilon}(t+1)\right)}{(1+n)(1+\hat{n})\hat{U}_1\left(\hat{c}(t),\ell(t),\hat{\varepsilon}(t)\right)}\right\} = 1$$

$$(2.36)\qquad \frac{1+r_t}{(1+n)(1+\hat{n})} = \frac{\hat{U}_1\left(\hat{c}(t),\ell(t),\hat{\varepsilon}(t)\right)}{E_t\left\{\hat{\beta}\hat{U}_1\left(\hat{c}(t+1),\ell(t+1),\hat{\varepsilon}(t+1)\right)\right\}}$$

$$(2.37)\qquad \frac{-\hat{U}_2\big(\hat{c}(t),\ell(t),\hat{\varepsilon}(t)\big)}{\hat{U}_1\big(\hat{c}(t),\ell(t),\hat{\varepsilon}(t)\big)} = \frac{w(t)}{(1+\hat{n})^t} = \hat{w}(t)$$

$$= \hat{\theta}(t) f_2\Big(\frac{\hat{k}(t)}{\hat{\theta}(t)},\ell(t)\Big)$$

where,

$$(2.38)\qquad \hat{\beta} = \beta(1+\hat{n})^{\lambda}$$

and is assumed to be less than one. The above relationships follow because $\hat{U}_1$ is homogeneous of degree $(\lambda-1)$ in (c_t,ε_t) whereas $\hat{U}_2$ is homogeneous of degree λ in the same variables. The condition that $\hat{\beta}$ be less than one arises because we would like the discounted sum of utilities in (2.28) to be well defined.

Given the stationarity of $\hat{\theta}(t)$ and $\hat{\varepsilon}(t)$, it follows that $\hat{c}(t)$, $\hat{y}(t)$, $\hat{k}(t)$, $\ell(t)$ and $r(t)$ will be stationary and hence that the variables $c(t)$, $y(t)$, $k(t)$, and $w(t)$ will be growing at the rate $\hat{n}$. Thus, we can generate growth in the real wage and per capita consumption, output and capital while per capita labor supply and the interest rate will be stationary. The variables $\hat{c}(t)$, $\hat{y}(t)$ and $\hat{k}(t)$ are the variables that correspond to the residuals when the (logarithms) of aggregate variables are linearly detrended (a standard procedure) and are the focus of business cycle studies. It should be noted that gross investment $I(t)$ is equal to $(K(t+1)-(1-\delta)K(t))$ where $K(t)$ is the aggregate capital stock, and hence the per capita value of gross investment $i(t)$ corresponds to $(1+n)k(t+1) - (1-\delta)k(t)$. Hence,

$$(2.39)\qquad \hat{i}(t) = i(t)/(1+\hat{n})^t = (1+n)(1+\hat{n})\hat{k}(t+1) - (1-\delta)\hat{k}(t)$$

corresponds to the residual when $\log I(t)$ is linearly detrended.

It is perhaps worth pointing out that the usual specification with no taste shocks and a constant relative risk aversion coefficient of μ in consumption is subsumed in the above specification. In such a case we simply put, $\lambda = 1 - \mu$. The easiest way to see this is to ignore labor supply (assume that it is fixed) and write,

$$\hat{U}(c_t, \bar{\ell}, \varepsilon_t) = \varepsilon_t^{\lambda}[(c_t/\varepsilon_t)^{1-\mu} - 1]/(1-\mu)$$

However, under such an interpretation λ is restricted to be less than one whereas when taste shocks are allowed λ may well assume values greater than one. The parameter λ thus controls the growth rate of marginal utility and permits marginal utility to grow at a rate independent of the growth rate in consumption. Consequently, the parameter λ will influence the level of interest rates.

The above specifications correspond to modeling macroeconomic time series as consisting of a log-linear time trend with deviations around it, i.e., as trend stationary processes. Lately however, some researchers (King, Plosser, Stock and Watson [1987]) have questioned this method of decomposing a macro time series into a trend component and a cyclical component. There is some evidence suggesting that aggregate macro time series might be better modeled as difference stationary processes, i.e., that the logarithm of the variable consists of the sum of a stationary cyclical component and a non-stationary trend component which follows a random walk with drift (i.e., the growth rate of the trend component is itself random instead of being deterministic). This can be modeled in the following way.

Assume that,

$$\theta(t) = \theta_1(t)\theta_2(t) \tag{2.40}$$

where $\theta_2(t)$ is stationary and

$$\theta_1(t) = \theta_1(t-1)(1+\hat{n}_{t-1}) \tag{2.41}$$

where $\hat{n}_t$ is an independently and identically distributed random variable. It should be noted that if $\hat{n}_t$ were to be constant, then this specification will become identical to the previous one with $\theta_2(t)$ equalling $\theta_1(0)\hat{\theta}(t)$. Now put,

$$\begin{cases} \hat{k}(t) = k(t)/\theta_1(t) \\ \hat{y}(t) = y(t)/\theta_1(t) \\ \hat{c}(t) = c(t)/\theta_1(t) \end{cases} \tag{2.42}$$

Then the resource constraint (2.23) can be rewritten as,

$$\hat{c}(t) + (1+n)(1+\hat{n}_t)\hat{k}(t+1) = \theta_2(t)f\left(\hat{k}(t)/\theta_2(t),\ell(t)\right) + (1-\delta)\hat{k}(t) \tag{2.43}$$

Again, we assume that $\hat{U}(\quad)$ is homogeneous of degree λ in $c(t)$ and $\varepsilon(t)$ and that

$$\hat{\varepsilon}(t) = \varepsilon(t)/\theta_1(t) \text{ is stationary.} \tag{2.44}$$

The first order conditions (2.35)-(2.37) get modified to,

$$E_t\left\{\frac{\hat{\beta}_t(1+\xi_{t+1})\hat{U}_1\left(\hat{c}(t+1),\ell(t+1),\hat{\varepsilon}(t+1)\right)}{(1+n)(1+\hat{n}_t)\hat{U}_1\left(\hat{c}(t),\ell(t),\hat{\varepsilon}(t)\right)}\right\} = 1 \tag{2.45}$$

$$(2.46)\qquad \frac{1+r_t}{(1+n)(1+\hat{n}_t)} = \frac{\hat{U}_1(c(t),\ell(t),\hat{\varepsilon}(t))}{E_t\{\hat{\beta}_t\hat{U}_1(\hat{c}(t+1),\ell(t+1),\hat{\varepsilon}(t+1))\}}$$

$$(2.47)\qquad \frac{-\hat{U}_2}{\hat{U}_1}\left(\hat{c}(t),\ell(t),\hat{\varepsilon}(t)\right) = \frac{w(t)}{\theta_1(t)} = \hat{w}(t) = \theta_2(t)f_2(\hat{k}(t)/\theta_2(t),\ell(t))$$

where,

$$(2.48)\qquad \hat{\beta}_t = \beta(1+\hat{n}_t)^{\lambda}$$

is the (stochastic) discount factor and is assumed less than one. It follows that $\hat{k}(t)$, $\hat{y}(t)$ and $\hat{c}(t)$ will follow stationary processes and hence that the logarithms of the aggregate magnitudes $K(t)$, $Y(t)$ and $C(t)$ as well as the per capita magnitudes $k(t)$, $y(t)$ and $c(t)$ can be represented as a sum of two components--one of which follows a random walk with drift and the other is stationary.

III. Some Illustrations

A) Business Cycles and Calibration

The most common procedure is to detrend the macro data and regard the comovements of the deviations as the objects to be explained by theory. The theoretical model is then some version of a stationary optimal growth model in which the variables are thought of as corresponding to the deviations from trend of actual data. As the discussion in the previous section makes clear, the appropriate model is one where the representative agent maximizes,

$$(3.1)\qquad E_1\{\sum_{t=1}^{\infty}\hat{\beta}^{t-1}\hat{U}(\hat{c}(t),\ell(t),\hat{\varepsilon}(t))\}$$

subject to

$$(3.2) \qquad \hat{c}(t) + (1+n)(1+\hat{n})\hat{k}(t+1) = \hat{\theta}(t)f[\hat{k}(t)/\hat{\theta}(t),\ell(t)] + (1-\delta)\hat{k}(t)$$

and the restrictions that $[\hat{\varepsilon}(t),\hat{\theta}(t)]$ be stationary and that $\hat{\beta}$ be less than one. Note that we are considering a trend stationary specification here. The percentage deviations from the (constant) trend for the model's variables $\hat{c}(t)$, $\hat{k}(t)$, $\hat{y}(t)$, $\hat{i}(t)$ $[=(1+n)(1+\hat{n})\hat{k}(t+1)-(1-\delta)\hat{k}(t)]$ and $\hat{w}(t)$ correspond to the residuals when the logs of the corresponding aggregate variables are linearly detrended. Further, the risk free interest rate is given by (2.36). In fact one can rewrite the above problem as follows,

$$(3.3) \qquad \text{Maximize } E_1\{\sum_{t=1}^{\infty} \hat{\beta}^{t-1}\tilde{U}[c^*(t),\ell(t),\hat{\varepsilon}(t)]\}$$

subject to,

$$c^*(t) + \hat{k}(t+1) = \hat{\theta}(t)f^*[\hat{k}(t)/\hat{\theta}(t),\ell(t)] + (1-\delta^*)\hat{k}(t)$$

where

$$(3.4) \quad \begin{cases} c^*(t) = \hat{c}(t)/(1+n)(1+\hat{n}) \\ 1-\delta^* = (1-\delta)/(1+n)(1+\hat{n}) \\ i^*(t) = \hat{i}(t)/(1+n)(1+\hat{n}) = \hat{k}(t+1) - (1-\delta^*)\hat{k}(t) \\ y^*(t) = \hat{\theta}(t)f^*(\cdot,\cdot) = \hat{\theta}(t)f(\cdot,\cdot)/(1+n)(1+\hat{n}) \\ \qquad = \hat{y}(t)/(1+n)(1+\hat{n}) \end{cases}$$

The optimality equations corresponding to (2.35)-(2.37) can be rewritten as follows.

$$(3.5) \qquad E_t\left\{\frac{\hat{\beta}(1+\xi^*_{t+1})\tilde{U}_1\left(c^*(t+1),\ell(t+1),\hat{\varepsilon}(t+1)\right)}{\tilde{U}_1\left(c^*(t),\ell(t),\hat{\varepsilon}(t)\right)}\right\} = 1$$

$$(3.6) \qquad \frac{1 + r_t}{(1+n)(1+\hat{n})} = 1 + r^*_t = \frac{\tilde{U}_1\left(c^*(t),\ell(t),\hat{\varepsilon}(t)\right)}{E_t\left\{\hat{\beta}\tilde{U}_1\left(c^*(t+1),\ell(t+1),\hat{\varepsilon}(t+1)\right)\right\}}$$

$$(3.7) \qquad 1 + \xi^*_t = \frac{1 + \xi_t}{(1+n)(1+\hat{n})} = \frac{f_1\left(\hat{k}(t)/\hat{\theta}(t),\ell(t)\right) + 1 - \delta}{(1+n)(1+\hat{n})}$$

$$= f^*_1\left(\hat{k}(t)/\hat{\theta}(t),\ell(t)\right) + 1 - \delta^*$$

$$(3.8) \qquad \frac{-\tilde{U}_2\left(c^*(t),\ell(t),\hat{\varepsilon}(t)\right)}{\tilde{U}_1\left(c^*(t),\ell(t),\hat{\varepsilon}(t)\right)} = w^*(t) = \hat{\theta}(t)f^*_2(\cdot,\cdot)$$

$$= \frac{\hat{\theta}(t)f_2(\cdot,\cdot)}{(1+n)(1+\hat{n})} = \frac{\hat{w}(t)}{(1+n)(1+\hat{n})}$$

If one were to estimate such a model, one would transform the macro data on Y(t), K(t), C(t), I(t) and W(t) by detrending and also transform $(1+r_t)$ into $(1+r^*_t)$ as indicated by (3.6). The estimated parameters $\hat{\beta}$, δ^*, n and $\hat{n}$ can be transformed into actual δ as shown in (3.4).

On the other hand if one were "calibrating" the model then one has to keep in mind that the expected marginal rate of substitution is <u>not</u> the risk free real rate but the real rate corrected for the growth rate in total output. Similarly, δ^* is not the actual depreciation rate of capital but one corrected for growth. As an example, Kydland and Prescott [1982] have a stationary model similar to the one above but they fail to make the indicated corrections for growth in calibrating their model. As examples, we mention the following.

i) They take the subjective discount rate $(=\hat{\beta}^{-1}-1)$ to be 4 percent, the share of capital in output to be 36 percent and the depreciation rate to be 10 percent and deduce that the capital to annual output ratio is 2.4. Presumably, they are using the deterministic steady-state versions of (3.5) and (3.7),

$$(\hat{\beta})^{-1} - 1 = f_1 - \delta = \left(\frac{Kf_1}{Y}\right) \frac{Y}{K} - \delta$$

which yields K/Y of 2.6. However, a glance at (3.4) and (3.7) shows that this is incorrect and that the correct procedure is,

$$(\hat{\beta})^{-1} - 1 \doteq f_1 - \delta - n - \hat{n}$$

which yields a value of K/Y equal to 2.0 assuming that $(n+\hat{n})$ is roughly 3.5 percent. This implies that out of the share of capital in output (36 percent), roughly 20 percent is depreciation and the remaining 16 percent is return on capital in contrast to their values of 24 percent and 12 percent respectively.

ii) They use the first order condition for utility maximization in the form,

$$1 + r = \frac{U_1(c(t),\ldots)}{\beta U_1(c(t+1),\ldots)}$$

together with an assumption of constant relative risk aversion $[U(\cdot)$ is homogeneous of degree $1 - \mu$ in $c]$ to derive

$$1 + r = \beta^{-1} \left(\frac{c_{t+1}}{c_t}\right)^{\mu}$$

or approximately,

$$r = \beta^{-1} - 1 + \mu(c_{t+1}-c_t)/c_t$$

and then use the assumed value of four percent for $(\beta^{-1}-1)$, together with data on r and the growth rate of per capita consumption to come up with a range for the risk aversion coefficient μ. This is simply incorrect in the context of their stationary model which does not permit any steady growth in consumption. As one can see from (3.6), the correct steady-state relation is, approximately,

$$r = \hat{\beta}^{-1} - 1 + n + \hat{n}$$

which throws no light whatsoever on the risk aversion coefficient μ. This can only be used to calibrate the discount factor $\hat{\beta}$ and can lead to problems. Presumably, what we would like to use is a measure of the risk free real interest rate and not the expected real return to capital which is what Kydland and Prescott [1982] used. From (3.5) and (3.6) it is easy to see that these two will not be the same so long as the return to capital is correlated with the future marginal utility of consumption. If one attempts to use the average real return on government T-bills as a rough measure of the risk free real rate, then the above restriction can lead to nonsensical values of $\hat{\beta}$ because this average return has mostly been less than the growth rate of total output over much of the post war period.

B) Volume of Trade

In his introduction to the Prescott-Summers debate, Rodolfo Manuelli (Quarterly Review, Fall 1986) writes, "As an example, consider the models Prescott surveys (in this issue). Most of them are representative agent models. Formally, the models assume a large number of consumers, but they are specialized by assuming also that the consumers are identical. One of the consequences of this specialization is a very sharp prediction about the volume of trade: it is zero. If explaining observations on the volume of trade is considered essential to an analysis, this prediction is enough to dismiss such models."

It is not difficult to illustrate within the present context why such a conclusion may be mistaken. First of all, a representative agent formulation can be consistent with a limited amount of diversity across the H types of infinitely lived families and hence with some trade taking place across families. However, even if we assume that all the families are alike and hence that there is effectively a single family it is still possible to generate some trade (borrowing and lending in credit markets) within members of different generations. The following simple example demonstrates this.

Assume that there are three generations alive at each date. We suppress considerations relating to growth, production, capital accumulation and labor supply in order to focus on intergenerational trade. Suppose that individuals receive income only in their second period equal to y_2. They are assumed to leave bequests in their last period to their middle aged progeny. We assume that the utility functions U_s are independent of s and we also suppress the preference shocks $\{\varepsilon_s^h(t)\}$. As should be obvious from the set up individuals will borrow when they are young, pay back loans, lend and receive bequests when middle aged and finally, receive payment on loans and leave bequests when old. A quick calculation reveals the following solution in the steady state.

$$c_1 = c_2 = c_3 = y_2/3$$

$$1 + r = \beta^{-1}$$

$$b = y_2(1-\beta)/3\beta$$

$$\ell_1 = -y_2/3$$

$$\ell_2 = y_2/3$$

where $\{c_s\}$ is consumption in the s^{th} period of life and ℓ_1 and ℓ_2 are loans made (borrowed, if negative) in the first and second period. The solution is obtained as follows. Equations (2.10) imply the equality of marginal utilities and hence, in the present instance, equality of consumptions. It follows immediately from (2.9) and (2.12) that (1+r) equals β^{-1}. The budget constraints (2.7) can be collapsed into a single lifetime budget constraint as follows,

$$c_1 + \frac{c_2}{(1+r)} + \frac{(c_3+b)}{(1+r)^2} = \frac{y_2+ b}{(1+r)}$$

which yields the solution for bequests which are positive given that β is less than one. It is then easy to find the values of ℓ_1 and ℓ_2 that support these allocations. As one can see this model which is equivalent to a single representative infinitely lived agent formulation (since the bequest motive is operative) can nevertheless be consistent with a non-zero volume of trade in credit markets across members of different generations.

One can, of course, object to the above scenario on the grounds that in this model, the volume of trade in credit markets depends on the assumed pattern of bequests in which the entire bequest is passed on in a single period from the old to the middle-aged. One can imagine alternative scenarios in which the bequest is passed on over two periods, say, and this will imply a different volume of trade. Thus, the model's predictions regarding the volume of trade are completely dependent upon assumptions regarding the timing of bequests. However, one can imagine modifications to the basic model involving, say, the tax structure that may help pin down the timing of bequests and hence the volume of trade. There is, in any case, no presumption that the volume of trade will necessarily be zero.

C) Pattern of Savings

Casual observation suggests that the life cycle savings pattern of a typical generation is strongly "hump" shaped; with a large hump in the middle years surrounded by dis-savings in youth and old age. The representative infinitely lived agent formulation on the other hand yields a stationary time path for savings with no significant humps. The representative family calculates permanent income by considering total family income in each period, then chooses total family consumption to equal permanent income and then allocates family consumption across members of different generations. Roughly speaking, savings will be positive when actual total family income exceeds permanent income and negative in the contrary case.

The previous example (B) shows how these two observations can be quite consistent with each other. The time path of income is concentrated in the middle years and so are bequests received (by assumption). The desire on the part of individuals to have smooth consumption then leads them to borrow and lend via credit markets. This not only generates trade in these markets but also gives rise to hump shaped savings. As seen, individuals dissave in youth and old age and save in the middle years. This pattern is quite consistent with a constant (and zero) savings rate for the representative agent. The zero savings rate is obviously because of the absence of investment in the model.

IV. Conclusion

Part of the purpose of this note has been to exposit the relationship between OLG models with bequest motives and operative bequests and the standard model of a representative infinitely lived agent who maximizes a discounted sum of expected utilities over an infinite horizon subject to the economy's resource constraint. We have shown how this can be done and how we can take account of both population growth as well as growth in per capita variables occurring due to exogenous factors. We have therefore provided a decentralized justification for using the representative agent model. We have also shown that stationary models of this type which abstract from growth can nevertheless be used to study business fluctuations represented as deviations from a deterministic or a random walk trend.

However, while it is logically consistent to build representative agent models abstracting from growth and use them to confront the comovements of the cyclical (detrended) component of aggregate variables care must be exercised in interpreting the estimated parameters or in calibrating the model. Such estimation or calibration is not invariant to the growth path of aggregate variables. Furthermore, such models do not necessarily imply that the volume of trade must be zero and they can also be consistent with observed patterns of life cycle savings.

References

Barro, R. J., "Are Government Bonds Net Wealth?" Journal of Political Economy, 82 [1974], pp. 1095-1117.

Eichenbaum, M., Lars Hansen and Scott Richards, "The Dynamic Equilibrium Pricing of Durable Consumption Goods," GSIA, Carnegie-Mellon University, 1984.

King, Robert, Charles Plosser, James Stock and Mark Watson, "Stochastic Trends and Economic Fluctuations," NBER, Stanford, January 1987.

Kydland, F. and Edward C. Prescott, "Time to Build and Aggregate Fluctuations," Econometrica, 50 [1982], pp. 1345-1370.

Manuelli, Rodolfo E., "Modern Business Cycle Analysis: A Guide to the Prescott-Summers Debate," Quarterly Review, Federal Reserve Bank of Minneapolis, Fall 1986.