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LINKS BETWEEN STRUCTURAL AND  
REDUCED FORM SEASONALITY

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Many economic time series seem to exhibit somewhat predictable seasonal fluctuations. In modeling variables whose data series appear to be seasonal, economists must either include a source of seasonality in the model or transform the model and data to remove the seasonal fluctuations. Common examples of these two options are including seasonal intercept shifts, on the one hand, or specifying the model in terms of data transformed by the Census X-11 algorithm, on the other.

Economists have debated the relative merits of including seasonal elements in the model versus transforming the model and data to a deseasonalized form. (See Zellner [1978].) This debate, and much of the analysis of the modeling or removal of seasonality in general, has been shaped in part by assumptions about the nature of seasonality in economic time series. This paper attempts to shed new light on the debate, as well as sketch some new approaches handling seasonality, by deriving seasonal reduced forms for an important class of economic models.

#### A GENERIC LINEAR-QUADRATIC ECONOMY WITH SEASONALITY

Drawn by the tractability of linear first-order conditions, economists have for many years shown a great deal of ingenuity in representing economic phenomena by means of models that can be written in linear-quadratic form. In the last ten years, work on estimating rational expectations models has extended the linear-quadratic framework to encompass much of economic dynamics and time series. In this framework, the equilibrium laws of motion for the economy solve a constrained maximization problem with a quadratic objective and linear equality constraints. The

linearity of the constraints means they can be substituted into the objective function, so we can represent the generic linear-quadratic framework by focusing on the maximization, with respect to the series of choice vectors  $\{N_t\}_{t=0}^{\infty}$ , of the objective

$$(1) \quad R_0 = E_0 \sum_{t=0}^{\infty} b^t \{A_t(L)N_t - [B_t(L)N_t]^2 - [W_t(L)u_t][D_t(L)N_t]\},$$

where  $u_t$  is a nonchoice (exogenous or endowment) time series that obeys  $u_t = V_t(L)e_t$ ,  $e_t$  is a white noise process,  $b$  is a discount factor,  $E_0$  denotes mathematical expectation conditional on the values of variables dated 0 or earlier,  $L$  is the time series lag operator,  $L^j X_t = X_{t-j}$ , and  $A_t(L)$ ,  $B_t(L)$ ,  $D_t(L)$ , and  $[W_t(L)V_t(L)]$  are polynomials in  $L$  all assumed (for convenience and without loss of generality) to be of order  $q$ . I define the lag operator  $L$  so that it does not shift the sequences of coefficient matrices  $A_t$ ,  $B_t$ ,  $W_t$ ,  $V_t$ , or  $D_t$ . (That is,  $L^j A_t(L) = A_t(L)$ , etc.) Let  $C_t(L)$  denote  $W_t(L)V_t(L)$ .  $N_{t-1}$ ,  $N_{t-2}$ , ... are taken as given.

This generic linear-quadratic objective is generic in the sense that appropriate specifications of  $N_t$ ,  $u_t$ ,  $A_t$ ,  $B_t$ ,  $C_t$ , and  $D_t$  would reproduce a wide variety of the models economists have analyzed. The objective  $R_t$  has an additional feature--time-varying coefficients in the lag operator polynomials  $A_t$ ,  $B_t$ ,  $C_t$ , and  $D_t$ --that makes it more general than the commonly studied time-invariant cases. This feature will allow seasonality to enter the model, and show up in reduced forms, in a variety of ways.

To simplify matters, I'll assume that second-order conditions are always satisfied and that dynamic optimality requires that the stable roots of polynomials be solved backwards

(in time) and unstable roots forward. I'll also exploit certainty equivalence, first dropping and then reinserting the expectations operator. These simplifications allow me to focus on the generic first-order condition that corresponds to (1), or (dropping the expectations operator)

$$(2) \quad [2B_{t,0}B_t(L)N_t + 2bB_{t+1,1}B_{t+1}(L)N_{t+1} + \dots + 2b^q B_{t+q,q}B_{t+q}(L)N_{t+q}] \\ = [A_{t,0} + bA_{t+1,1} + \dots + b^q A_{t+q,q}] - [D_{t,0}C_t(L)e_t + bD_{t+1,1}C_{t+1}(L)e_{t+1} \\ + \dots + b^q D_{t+q,q}C_{t+q}(L)e_{t+q}].$$

I will now analyze four special cases of this first-order condition and their implications for the nature of seasonality in economic time series.

Case A: All Polynomials Time Invariant

This is a standard linear-quadratic framework of applied economics. Dropping subscripts on A, B, C, and D collapses (2) to

$$(2a) \quad [2B(bL^{-1})B(L)]N_t = A(b) - [D(bL^{-1})C(L)]e_t.$$

Noting that the roots of the polynomial on the left side of (2a) come in  $(z, bz^{-1})$  pairs, we can rewrite it as (Sargent, p. I-5),

$$(3a) \quad P(bL^{-1})P(L)N_t = A(b) - [D(bL^{-1})C(L)]e_t,$$

or

$$(4a) \quad P(L)N_t = P(bL^{-1})^{-1}A(b) - P(bL^{-1})^{-1}[D(bL^{-1})C(L)]e_t,$$

where  $P(L)$  is of order  $q$ . Reintroducing the expectations operator gives

$$(5a) \quad P(L)N_t = P(bL^{-1})A(b) - [P(bL^{-1})^{-1}D(bL^{-1})C(L)]_+ e_t,$$

where the plussing operator  $[\ ]_+$  sets to zero all polynomial terms in negative powers of  $L$ . Thus we can write the solution to the linear-quadratic model in this case in the form

$$(6a) \quad P(L)N_t = K + Q(L)e_t,$$

where  $K$  is a constant and  $Q(L)$  is a  $q$ th-order time-invariant moving average polynomial in nonnegative powers of  $L$ . This is a standard stationary ARMA  $(q,q)$  model for  $N_t$ .

The seasonal aspects of the reduced form (3), if any, are embedded in the polynomials  $P(L)$  and  $Q(L)$ . If we rewrite (6a) as

$$(7a) \quad N_t = [Q(L)/P(L)]e_t + K',$$

where  $K' = [K/P(L)]$ , it is apparent that any seasonality in  $N_t$  results from seasonal power in the spectrum of the rational linear filter  $[Q(L)/P(L)]$ .

Seasonality in  $[Q(L)/P(L)]$  could arise either from within the structure of the economy--from agents' tastes, technology, or endowments--or from without, in the exogenous processes. I will term the former endogenous seasonality and the latter exogenous seasonality.

Note that the denominator polynomial  $P(L)$  is determined by the polynomial  $B(L)$ . This polynomial describes how the choice

vector  $N_t$  affects the model. It would thus typically involve the tastes of and technologies available to the economic agents in the model, and for that reason I term any seasonal power in  $P(L)$  endogenous. In a quarterly model, an example would be using  $B(L) = (1-L^4)$  to capture costs of adjusting a capital stock relative to its year-earlier value.

The numerator polynomial is determined by both endogenous and exogenous factors. It depends on  $P(L)$  (and hence  $B(L)$ ) as well as  $C(L)$ , and  $C(L)$  in turn equals  $W(L)V(L)$ . If the seasonal power in  $Q(L)$  stems from  $B(L)$ , I would term this endogenous seasonality in  $Q(L)$ , for the reasons discussed in the previous paragraph. Similarly,  $W(L)$  describes how the indeterministic non-choice process  $u_t$  affects the objective  $R_t$ , and this also typically involves agents' tastes and technologies. I would thus term endogenous any seasonality in  $C(L)$  or  $Q(L)$  stemming from  $W(L)$ .

By contrast,  $V(L)$  describes how the nonchoice series  $u_t$  evolves over time. This evolution is unrelated (at least within the model) to agents' tastes and technologies, and I would usually term exogenous any seasonality in  $C(L)$  or  $Q(L)$  arising from the spectrum of  $V(L)$ . For example, in a small-country model with world prices assumed to be exogenous,  $V(L)$  would have to capture the exogenous seasonality of world prices. The one exception to this rule is that  $u_t$  can also include elements that represent agents' endowments of commodities. I regard as endogenous seasonality in  $V(L)$  derived from seasonality in endowments.

Case B:  $A_t(L)$  Periodic But Other Polynomials Time Invariant

This is also a common framework for applied economics. It adds deterministic seasonal components, of both endogenous and exogenous origin, to the purely indeterministic seasonality of Case A. With the polynomial  $A_t(L)$  periodic, in the sense that if  $s$  equals the number of time periods (months, quarters, etc.) per year then  $A_{t+s}(L) = A_t(L)$ , the first-order condition becomes

$$(2b) \quad [2B(bL^{-1})B(L)]N_t = \tilde{A}_t(b) - [D(bL^{-1})C(L)]e_t,$$

where  $\tilde{A}_t(L) = A_{t,0} + A_{t+1,1}L + \dots + A_{t+q,q}L^q$ . Note that  $\tilde{A}_t(L)$  inherits the periodicity (of period  $s$ ) of  $A_t(L)$ .

Following the same solution procedure as in Case A gives

$$(3b) \quad P(bL^{-1})P(L)N_t = \tilde{A}_t(b) - [D(bL^{-1})C(L)]e_t,$$

or

$$(4b) \quad P(L)N_t = P(bL^{-1})^{-1}\tilde{A}_t(b) - P(bL^{-1})^{-1}[D(bL^{-1})C(L)]e_t,$$

and, reintroducing the expectations operator,

$$(5b) \quad P(L)N_t = P(bL^{-1})^{-1}\tilde{A}_t(b) - [P(bL^{-1})^{-1}D(bL^{-1})C(L)]_+e_t,$$

or

$$(6b) \quad P(L)N_t = K_t + Q(L)e_t,$$

where  $K_t = P(bL^{-1})^{-1}\tilde{A}_t(b)$ . Because  $\tilde{A}_t(b)$  is periodic of period  $s$ , so is the sequence  $K_t$ . (6b) thus differs from (6a) in that instead of a constant mean it has a deterministic but periodically varying mean. (6b) can be reduced to a stationary time series

model by either seasonal differencing or by removal of an estimated seasonal intercept. In the former case, (6b) is represented as ARMA (q,s,q), and its differenced form would be the ARMA (q,q) model

$$P(L)(N_t - N_{t-s}) = Q(L)(e_t - e_{t-s}).$$

In the latter case, removal of the intercept term in one stage and subsequent analysis of the residuals introduces no bias. To see this, note that (6b) can be rewritten as

$$(7b) \quad N_t = [Q(L)/P(L)]e_t + K'_t,$$

where  $K'_t = [K_t/P(L)]$  is also a deterministic sequence with period  $s$ . Since the deterministic  $K'_t$  process is obviously uncorrelated with the white noise process  $e_t$ , unbiased estimates of its  $s$  distinct values can be obtained by regressing  $N_t$  on a set of seasonal dummies.

The sources of seasonality in this model include all of those discussed in Case A plus those that give rise to periodicity in the polynomial  $A_t(L)$ . To make the generic model (1) as general as possible, it is necessary to allow the periodicity of  $A_t(L)$  to be attributed to exogenous as well as endogenous sources. One reason for this is that  $A_t(L)$  incorporates any deterministic components related to the nonchoice variables  $u_t$ , which may be either exogenous or endogenous. Suppose  $u_t$  is the indeterministic component of a stochastic process

$$v_t = k_t + u_t,$$



where  $k_t$  is periodic of period  $s$ . Suppose also that it is  $v_t$ , not  $u_t$ , that directly enters the planning problem. Then, recalling that  $B_t$ ,  $W_t$ , and  $D_t$  are time-invariant here, we can rewrite (1) as

$$\begin{aligned}
 (1b) \quad R_t &= E_0 \sum_{j=0}^{\infty} b^j \{ A_t(L)N_t - [B(L)N_t]^2 - [W(L)v_t] [D(L)N_t] \} \\
 &= E_0 \sum_{j=0}^{\infty} b^j \{ A_t(L)N_t - [B(L)N_t]^2 - [W(L)u_t + W(L)k_t] [D(L)N_t] \} \\
 &= E_0 \sum_{j=0}^{\infty} b^j \{ A_t(L)N_t - W_0^* k_t D(L)N_t - W_1^* k_{t-1} D(L)N_t - \dots \\
 &\quad - W_{s-1}^* k_{t-s+1} D(L)N_t - [B(L)N_t]^2 - [W(L)u_t] [D(L)N_t] \} \\
 &= E_0 \sum_{j=0}^{\infty} b^j \{ A_t^*(L)N_t - [B(L)N_t]^2 - [W(L)u_t] [D(L)N_t] \},
 \end{aligned}$$

where

$$W_0^* = W_0 + W_s + \dots + W_{q_0 s}; \quad q-s < q_0 s \leq q,$$

$$W_1^* = W_1 + W_{1+s} + \dots + W_{1+q_1 s}; \quad q-s-1 < q_1 s \leq q-1,$$

⋮

$$W_{s-1}^* = W_{s-1} + W_{s-1+s} + \dots + W_{s-1+q_s s}; \quad q-2s+1 < q_s s \leq q-s+1,$$

and

$$A_t^*(L) = A_t(L) + W_0^* k_t D(L) + W_1^* k_{t-1} D(L) + \dots + W_{s-1}^* k_{t-s+1} D(L)$$

has the same form as  $A_t(L)$ . In the typical case the periodic deterministic components in  $u_t$  correspond to exogenous variables and would be considered exogenous sources of periodicity in  $A_t(L)$ . Less commonly, perhaps, the periodic components in  $u_t$

correspond to variables that represent agents' endowments. Because these, like periodic components of technology and preferences, are elements of the model's assumed structure, I would term them an endogenous source of periodicity in  $A_t(L)$ .

Case C:  $C_t(L)$  or  $D_t(L)$  Periodic But Other Polynomials Time

Invariant

This is a less common framework for applied economics. It leads to reduced-form time series with seasonal coefficients, a relatively little studied form of economic seasonality.

With  $C_t(L)$  periodic with period  $s$ , the first order condition becomes

$$(2c) \quad [2B(bL^{-1})D(L)]N_t = A(b) - [D_0C_t(L)e_t + bD_1C_{t+1}(L)e_{t+1} + \dots + b^q D_q C_{t+q}(L)e_{t+q}],$$

or

$$(3c) \quad P(bL^{-1})P(L)N_t = A(b) - [D_0C_t(L)e_t + bD_1C_{t+1}(L)e_{t+1} + \dots + b^q D_q C_{t+q}(L)e_{t+q}],$$

or

$$(4c) \quad P(L)N_t = P(bL^{-1})^{-1}A(b) - P(bL^{-1})^{-1}[D_0C_t(L)e_t + bD_1C_{t+1}(L)e_{t+1} + \dots + b^q D_q C_{t+q}(L)e_{t+q}].$$

Reintroducing the expectations operator and exploiting the periodicity of  $C_t(L)$  gives

$$(5c) \quad P(L)N_t = P(bL^{-1})^{-1}A(b) - \{P(bL^{-1})^{-1}[D_0C_t(L)+bD_1C_{t+1}(L)L^{-1} + \dots + b^qD_qC_{t+q}(L)L^{-q}]\}_+e_t,$$

or

$$(6c) \quad P(L)N_t = K + Q_t(L)e_t,$$

where  $Q_t(L) = Q_{t+s}(L)$ , for all  $t$ .

This stochastic process has a constant term and a time-invariant AR polynomial, but its MA polynomial cycles with period  $s$ . Alternatively, if (6c) is rewritten in a purely autoregressive form, the AR polynomial will be periodic. For example, if we assume  $K = 0$ ,  $P(L) = 1$ , and  $Q_{0t} = 1$  for all  $t$ , then repeated substitution gives

$$N_t = e_t - \sum_{j=0}^{q-1} \left( \prod_{k=0}^j Q_{t-k} \right) N_{t-j-1},$$

or

$$\tilde{P}_t(L)N_t = e_t.$$

Note that periodicity in  $C_t(L)$  can arise either in  $W_t(L)$  or  $V_t(L)$ .  $W_t(L)$  describes how variations in the nonchoice process  $u_t$  affect the objective function  $R_t$ .  $W_t(L)$  is thus part of the taste-technology structure of the economy and can be viewed as an endogenous source of periodicity in  $C_t(L)$ . By contrast,  $V_t(L)$  determines how the nonchoice process  $u_t$  evolves; it is generally the reduced form output of an exogenous system and can be viewed as an exogenous source of periodicity in  $C_t(L)$ .

As an example of exogenous periodicity in  $C_t(L)$ , consider a small country producing a good  $X_t$  according to the technology

$$X_t = D_1(L)N_t.$$

If the country faces a world price  $P_{xt}$  governed by

$$P_{xt} = C_0 + C_{1t}(L)e_{1t},$$

the country's revenue from producing  $X_t$  is

$$(8c) \quad P_{xt}X_t = C_0D_1(L)N_t + [C_{1t}(L)e_{1t}][D_1(L)N_t].$$

In the complete planning problem for the small country, the first term on the right side of (8b) would be incorporated in  $A_t(L)N_t$ , but the second would be incorporated in  $C_t(L)$  and  $D(L)$ .

With  $D_t(L)$  periodic with period  $s$ , and all other polynomials time invariant, we get

$$(2c') \quad [2B(bL^{-1})B(L)]N_t = A(b) - [D_{t,0} + bD_{t+1,1}L^{-1} + \dots + b^q D_{t+q,q}L^{-q}]C(L)e_t.$$

Proceeding as before we obtain

$$(4c') \quad P(L)N_t = P(bL^{-1})^{-1}A(b) - P(bL^{-1})^{-1}[D_{t,0} + bD_{t+1,1}L^{-1} + \dots + b^q D_{t+q,q}L^{-q}]C(L)e_t,$$

or, reintroducing expectations and exploiting the periodicity of  $D_t(L)$

$$(5c') \quad P(L)N_t = P(bL^{-1})^{-1}A(b) - \{P(bL^{-1})^{-1}[D_{t,0} + bD_{t+1,1}L^{-1} + \dots + b^q D_{t+q,q}L^{-q}]C(L)\}_+ e_t.$$

This can again be written as  $P(L)N_t = K + Q_t(L)e_t$ , as in (6c).

The previous example can also be easily modified to illustrate an endogenous source of seasonality in  $[C_t(L)e_t][D_t(L)N_t]$ . Let the technology now be

$$X_t = \bar{D}_t(L)N_t$$

and the world price  $P_{xt}$  be governed by

$$P_{xt} = C_0 + \bar{C}(L)\bar{e}_t.$$

Then the country's revenue from producing  $X_t$  is

$$(8c') \quad P_{xt}X_t = C_0\bar{D}_t(L)N_t + [\bar{C}(L)\bar{e}_t][\bar{D}_t(L)N_t].$$

Again, the first term on the right side of (8c') would be incorporated in  $A_t(L)N_t$  and the second would be incorporated in  $[C_t(L)e_t][D_t(L)N_t]$ . The technology  $\bar{D}_t(L)N_t$  might reflect seasonal weather or cultural factors that influence factor productivity, and would typically be an endogenous source of seasonality.

#### Case D: $B_t(L)$ Periodic But Other Polynomials Time Invariant

Like Case C, this is a less common framework for applied economics and leads to reduced-form time series with seasonal coefficients. Here, however, periodicity appears in the characteristic polynomial of the first-order condition. This complicates analysis of the problem but ultimately results in a periodic mean and periodic AR and/or MA polynomials in the reduced form.

Periodicity of period  $s$  in  $B_t(L)$  gives the first-order condition

$$(2d) \quad [2B_{t,0}^B(L)N_t + 2bB_{t+1,1}^B(L)N_{t+1} + \dots + 2b^q B_{t+q,q}^B(L)N_{t+q}] \\ = A(b) - [D(bL^{-1})C(L)]e_t,$$

or

$$(2d) \quad [2B_{t,0}^B(L)L^q + 2bB_{t+1,1}^B(L)L^{q-1} + \dots + 2b^q B_{t+q,q}^B(L)]N_{t+q} \\ = A(b) - [D(bL^{-1})C(L)]e_t.$$

The polynomial on the left side of (2d), call it  $F_t(L)$ , is of order  $2q$ . The appendix shows that, after reintroducing uncertainty, (2d) can be solved either as

$$(3d) \quad \bar{P}(L)N_t = \bar{K}_t + \bar{Q}_t(L)e_t$$

or

$$(3d') \quad P_t(L)N_t = K_t + Q_t(L)e_t,$$

depending on how roots of the characteristic polynomial are shifted between the AR and MA polynomials. The periodicity of the coefficients in this reduced form stems from the periodicity of  $B_t(L)$ . Since  $B_t(L)$  is part of the taste-technology structure of the model, the periodicity of the coefficients in (3d) can be viewed as endogenous.

A wide variety of plausible economic models will lead to reduced forms like (3d). For example, consider a country producing a good  $X_t$  according to the technology

$$X_t = \bar{B}_t(L)N_t,$$

where  $\bar{B}_t(L)$  is periodic of period  $s$ . Assume that the country faces a downward sloping world demand curve with world price,  $P_{xt}$ , given by

$$P_{xt} = A_0 - B_0 X_t.$$

Then the countries revenues will be

$$(4d) \quad P_{xt} X_t = A_0 \bar{B}_t(L) N_t - B_0 [\bar{B}_t(L) N_t]^2.$$

Although the complete planning problem for this country would not have just  $B_t(L)$  but also have  $A_t(L)$  periodic of period  $s$ , the results of this section combined with analysis of Case B imply that the reduced-form for the competitive equilibrium would have to take the form shown in (3d). Given the plausibility of seasonally varying technologies and tastes for many goods and services, it would not be surprising if many economic time series displayed seasonality of the form of (3d).

Appendix:

Solving Models with Periodic Characteristic Equations

In Case D the first-order condition takes the form

$$(1) \quad F_t(L)N_{t+q} = w_t,$$

where  $w_t = A(b) - [D(bL^{-1})C(L)]e_t$  and  $F_t(L)$  is a periodic (period =  $s$ ) polynomial of order  $2q$ . Note that (1) can be rewritten as

$$(2) \quad \begin{aligned} w_t &= [F_{0t} + F_{1t}L + \dots + F_{qt}L^q]N_{t+q} \\ &= F_{0t}N_{t+q} + [F_{1t} + F_{2t}L + \dots + F_{qt}L^{q-1}]LN_{t+q} \\ &= F_{0t}N_{t+q} + \left[\frac{F_t(L)}{L}\right]_+ N_{t+q-1}, \end{aligned}$$

or,

$$(3) \quad N_{t+q} = (1/F_{0t})w_t + f_t(L)N_{t+q-1},$$

where

$$f_t(L) = (1 - 1/F_{0t}) \left[\frac{F_t(L)}{L}\right]_+ N_{t+q-1}.$$

By repeated substitution for lagged values of  $N$  on the right side of 3, we get

$$(4) \quad \begin{aligned} N_{t+q} &= (1/F_{0t})w_t + f_t(L) \left[ \left(\frac{1}{F_{0t-1}}\right)w_{t-1} + f_{t-1}(L) \left[ \left(\frac{1}{F_{0t-2}}\right)w_{t-2} + \dots \right. \right. \\ &\quad \left. \left. + f_{t-s+2}(L) \left[ \left(\frac{1}{F_{0t-s+1}}\right)w_{t-s+1} + f_{t-s+1}(L)N_{t-s} \right] \dots \right] \right] \\ &= G_t(L)w_t + \tilde{H}(L)N_{t-s}, \end{aligned}$$



where

$$G_t(L) = (1/F_{0t}) + (1/F_{0t})(1/F_{0t-1})f_t(L)L + \dots \\ + (1/F_{0t})(1/F_{0t-1})\dots(1/F_{0t-s+1})f_t(L)f_{t-1}(L) \\ \dots f_{t-s+2}(L)L^{s-1}$$

and

$$\tilde{H}(L) = [f_t(L)f_{t-1}(L)\dots f_{t-s+1}(L)].$$

Note that  $f_t(L)$  is also periodic of period  $s$  and that the product of the  $s$  polynomials that define  $\tilde{H}(L)$  is thus time-invariant.

The remainder of the solution proceeds along reasonably familiar lines. Rewrite (4) as

$$(5) \quad H(L)N_{t+q} = v_t,$$

where  $H(L) = [I - \tilde{H}(L)L^s]$  and  $v_t = G_t(L)w_t$ .  $H(L)$  will, in general, have both stable and unstable roots (a stable root being one whose modulus exceeds  $1/b$ ). Let  $P(L)$  be a polynomial formed from its stable roots and  $M(L)$  a polynomial formed from its unstable roots, such that  $M(L)P(L) = H(L)$ . Then rewrite (5) as  $M(L)P(L)N_{t+q} = v_t$  and solve as

$$(6) \quad P(L)N_{t+q} = M(L)^{-1}v_t \\ = [M(L)^{-1}L^q]v_{t+q} \\ = [M(L)^{-1}L^q]G_{t+q}(L)[A(b) - D(bL^{-1})C(L)]e_{t+q}.$$

Reintroducing the expectations operator eliminates future values of  $e_t$  and gives

$$(7) \quad P(L)N_t = K_t + Q_t(L)e_t,$$

where

$$K_t = [M(L)^{-1}L^q]G_t(L)A(b)$$

and

$$Q_t(L) = [-M(L)^{-1}L^qG_t(L)D(bL^{-1})C(L)]_+.$$

This is the form of the solution that arose in Case D, equation (3d). It can be transformed to the form of equation (3d') by multiplying both sides of (7) by factors like  $(1-\rho_{1t}L)$ , where  $\rho_{1t}$  is a stable root of  $F_t(z) = 0$ .

References

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