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THE ADVERSE SELECTION PROBLEM REVISITED: A REACTIVE CORE APPROACH

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I. Introduction

The literature on insurance/signalling economies with private information has employed a number of different equilibrium concepts. Perhaps best known among these are the Nash equilibrium of Rothschild and Stiglitz [1976], the "informationally consistent" equilibrium of Spence [1973], and the "reactive" equilibrium of Wilson [1977]. It is well recognized that there are problems with each of these approaches. Nash equilibria of the Rothschild-Stiglitz type may fail to exist, while with the Spence approach, multiplicity of equilibria occurs. Also, for any of these equilibrium concepts there is no assurance that, when equilibrium allocations do exist, one or more of them will be Pareto Optimal. Although this last feature is not a problem per se, it does suggest that policy interventions are likely to be welfare improving in such environments--ones in which private information creates externalities. As we shall show, however, such a conclusion may be unwarranted.

The problem of nonexistence or multiple equilibria is bothersome, and some have attempted to address it by allowing for strategic behavior on the part of insurance firms. Wilson [1977], for example, has analyzed an insurance economy in which each firm conjectures that other firms will respond to its policies in a prespecified and predictable manner. Specifically, other firms will cancel any insurance contracts that have become unprofitable in the face of competing contract offers. This approach does give existence and uniqueness in a broader class of environments. However, the assumed disequilibrium behavior of insurance firms is difficult to justify, since it need not even result in ceteris paribus profit maximization.

Here, we reconsider the insurance problem using yet another equilibrium concept, a cooperative concept closely related to the core. Since there are externalities in this environment, however, a definition of the core requires a specification of how residual coalitions respond to blocking attempts. The specification we employ is similar in many respects to existing core concepts for economies with public goods. And, as will be explained, the behavior of insurance firms (actually, insurance coalitions here) is similar in some respects to the behavior of firms that is assumed in employing Wilson's notion of equilibrium. Unlike Wilson, however, we define a core arrangement. In a core arrangement, each coalition specifies the allocation its members are to receive--conditional on who its members are, and what terms they can get in other coalitions. As will be made precise in what follows, agents rationally sort among coalitions, and "non-credible threats" are not permitted.

We particularly like the feature that in a core arrangement, agents' allocations depend upon who does--and who does not--join their coalition. This feature corresponds, at least in a stylized way, to the contracts actually offered by mutual life insurance companies. As Rothschild and Stiglitz noted in passing,

> The peculiar provision of many insurance contracts, that the effective premium is not determined until the end of the period (when the individual obtains what is called a dividend), is perhaps a reflection of the uncertainty associated with who will purchase the

> policy, which in turn is associated with the uncertainty about what contracts other insurance firms will offer (p. 276).

Here, this is not a "peculiar provision" of insurance contracts; rather, it is an essential feature of the arrangement.

We view this study as a natural extension (and partial synthesis) of two earlier lines of research. First, Prescott and Townsend [1984] investigated (competitive) mechanisms that produced optimal outcomes for certain kinds of private information economies. The mechanisms they investigated did not produce optimal outcomes for economies with adverse selection problems, however. We investigate a "core mechanism" that does produce optimal outcomes in these settings. Second, Myerson [1983] has used a cooperative approach to mechanism design problems by a principal endowed with private information. We also investigate the properties of a cooperative mechanism taking account of the "externalities" introduced by the presence of private information.

Very briefly, the rest of the paper proceeds as follows. Section II specifies the environment, and Section III specifies the game. Section IV defines a candidate core arrangement. In Section V it is proved that the candidate is indeed in the core. Section VI discusses uniqueness and presents the results we have obtained; not entirely satisfactory. Section VII summarizes and discusses an agenda for future work.

II. The Environment

The economy consists of a nonatomic measure space of agents, A. These agents can be partitioned into a finite number of "types". There are n such types, indexed by $i = 1, 2, \ldots, n$. All agents of a given type are identical, ex ante, and each agent is faced with the possibility of either of two states occurring; $s = 1, 2$. There is a single consumption good, with an agent in state s receiving an endowment e_s of the good. $e_1 > e_2$, so $s = 2$ is what is commonly referred to as "the loss state" in this context. Realizations of s are independent and identically distributed across agents of the same type; with type i agents facing a probability p_i that $s = 1$. We will periodically appeal to the law of large numbers, so that there is no aggregate uncertainty.[1/] Finally, we index types so that $1 > p_n > p_{n-1} > \ldots > p_1 > 0$. It is common in this context to refer to types with higher indexes as "lower risk" types.

Let c_{is} denote the consumption of a type i agent in state s. All agents have identical utility functions U(c) defined on R_+, with $U'(c) > 0$, $U''(c) < 0$ for all $c \in R_+$. Also, let μ_i denote the measure of agents of type i, let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, let $\mu_i > 0$ for all i hold, and let $\sum \mu_i = 1$.

It remains to describe the nature of information in this environment. Each agent knows his type prior to the realization of the state and prior to entering into any risk sharing arrangements. Type is private information, ex ante. All trades or arrangements entered into are observable by all, however. Hence this is a standard adverse selection setting.

III. The Arrangement Space

Our objective is to predict what kinds of insurance arrangements will prevail in the environment specified above. Agents are viewed as forming coalitions for the purpose of pooling risk. Since agents are allowed to form arbitrary coalitions, we adopt an equilibrium concept that is closely related to standard core concepts. However, for reasons that are well understood, an "informational externality" arises in adverse selection settings. Thus core concepts that are appropriate to this setting closely resemble definitions of the core for economies with externalities/public goods. Hence some modifications of standard definitions of the core are required. We regard our definition as something of a synthesis of existing core concepts for environments with public goods.[2/]

A coalition is simply a set K of agents; $K \in P(A)$, where P(A) denotes the power set of A. Here, since agents of a particular type are identical ex ante, we may consider two coalitions with different agents, but with identical measures of agents of each type, as identical for our purposes. Thus, as a shorthand, a coalition D is regarded as a vector of measures $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in R_+^n$, where δ_i is the measure of agents of type i who are members of D. Clearly, $0 \le \delta_i \le \mu_i$ for all i.

To foreshadow, we adopt the approach that a cooperative game begins with all agents belonging to an incumbent coalition (which is also the grand coalition). The incumbent coalition suggests, at the first stage of the game, a scheme for allocating resources. We will refer to this scheme as an arrangement. After an arrangement is specified, a blocking coalition may form. The arrangement specifies what allocation members of the (residual) incumbent coalition receive, given the blocking coalition that forms and the actions it takes.

Suppose a blocking coalition does form after an arrangement is specified. Denote the blocking coalition as B, and the (residual) incumbent coalition as Θ. Denote the vector of measures of agent types in B by β, and of agent types in Θ by θ.

We now wish to define actions that are feasible for an arbitrary coalition, D. We adopt the following notation. Recall that c_{is} is the consumption of a type i agent in state s. Let $c_i \equiv (c_{i1}, c_{i2})$, and let $c \equiv (c_1, c_2, \ldots, c_n)$. When we wish to denote the allocation of a coalition with vector of measures δ, we write $c^\delta \equiv (c_1^\delta, c_2^\delta, \ldots, c_n^\delta)$. Also, we define $V_i(c_j) \equiv p_i U(c_{j1}) + (1-p_i)U(c_{j2})$.

Given these notational conventions, we will say that an allocation c^δ is <u>resource feasible</u> for coalition δ if

$$(1) \qquad \sum_{i=1}^{n} \delta_i [p_i(c_{i1}^\delta - e_1) + (1-p_i)(c_{i2}^\delta - e_2)] \le 0.$$

We denote the set of allocations satisfying (1) for given δ by $R(\delta)$. Further, an allocation c^δ is <u>incentive feasible</u> for coalition δ if

$$(2) \qquad V_i(c_i^\delta) \ge V_i(c_j^\delta)$$

for all i and j such that $\delta_i \delta_j > 0$. We denote the set of allocations satisfying (2) for given δ by $I(\delta)$. An allocation is said to be <u>feasible</u> for δ if it is resource and incentive feasible.

The set of allocations that is feasible for δ is denoted $F(\delta) = R(\delta) \cap I(\delta)$. Also, we require that, if $\delta_i = 0$ for some i, $c_i^\delta = (e_1, e_2)$ for that i.

It is now possible to lay out the cooperative game more explicitly. This game can be viewed as being played in three stages. At stage one, the incumbent (and at this juncture grand) coalition specifies an arrangement. After specification of the arrangement stage two of the game occurs, in which any potential blocking coalition may form. The blocking coalition B announces its membership, or equivalently from our view point, announces $\beta \neq 0$ and an allocation $c^\beta \in F(\beta)$. Upon formation of a blocking coalition and a choice of c^β, stage three of the game occurs in which members of the blocking coalition receive the allocation c^β, and members of the (residual) incumbent coalition receive the allocation specified by the arrangement. In particular, the arrangement specifies an allocation for the incumbent coalition as a function of the blocking coalition that forms, and the actions it takes.

This specification of the actions taken by the incumbent coalition may be viewed as follows. In economies with externalities/public goods, the nature of the core can depend critically on how a residual coalition reacts to the attempted formation of a blocking coalition. Our specification simply requires the incumbent coalition to announce in advance how it will react to any blocking attempt.

Formally, then, let $\Delta^n = \{y \varepsilon R_+^n : \sum_{i=1}^n y_i \leq 1;\ 0 \leq y_i \leq \mu_i\}$. Further, let c^β denote the allocation of the blocking coalition β, and define

$$V_i^\beta = \max_{\{c_j^\beta\}} V_i(c) \ \forall \ i = 1, \ldots, n,$$

so V_i^β is the highest expected utility level a type i agent can obtain by joining the blocking coalition. Let $V^\beta \equiv (V_1^\beta, V_2^\beta, \ldots, V_n^\beta)$, and let W denote the set of vectors V^β generated by feasible allocations $x \in F(\beta)$; $\beta \in \Delta^n$. Then an arrangement is a mapping $a: \Delta^n \times W \to R_+^{2n}$ that specifies an allocation $c^\theta = a(\theta, V^\beta) = a(\mu-\beta, V^\beta)$. As above, we require $c_i^\theta \equiv a_i(\theta, V^\beta) = (e_1, e_2)$ if $\theta_i = 0$. A feasible arrangement specifies an allocation that is feasible for θ; i.e., $a(\theta, V^\beta) \in F(\theta)$ for each $\theta = \mu-\beta$ and for each $V^\beta \in W$.

It will be desirable to place an additional restriction on the kinds of arrangements that can be announced by the incumbent coalition. This restriction prevents the incumbent coalition from announcing arrangements specifying allocations $a(\theta, V^\beta)$ that the incumbent coalition would unanimously wish to change ex post if a blocking coalition $\beta = \mu - \theta$ actually formed, and announced an allocation giving rise to the vector V^β. In particular, we will say that given a blocking coalition β, and given an allocation $c^\beta \in F(\beta)$, an arrangement is <u>credible</u> if there fails to exist an allocation $\tilde{c}^\theta \in F(\theta)$, $\tilde{c}^\theta \neq a(\theta, V^\beta)$, such that

(3) $$V_i(\tilde{c}_i^\theta) \geq V_i^\beta \ \forall \ i \text{ such that } \theta_i > 0$$

(4) $$V_i(c_i^\beta) \geq \max_{\{\tilde{c}_j^\theta\}} V_i(c) \ \forall \ i$$

such that $\beta_i > 0$; and such that

(5) $$V_i(\tilde{c}_i^\theta) \geq V_i[a_i(\theta, V^\beta)]$$

for all i such that $\theta_i > 0$, with strict inequality for at least one such i.[3/] We will say that feasible arrangements that are also credible are admissible. We restrict the incumbent coalition to the announcement of admissible arrangements. This restriction prevents an incumbent from announcing allocations in the event of some defection that the (residual) incumbent coalition would unanimously wish to change if that defection actually occurred.

Core Arrangement: Definition

We are now prepared to define blocking and the core. Prior to doing so, we impose a restriction on the kinds of blocking coalitions that can form. In particular, we require that for a blocking coalition $\beta_i \in \{0, \mu_i\}$; that is, a blocking coalition either has (almost) all or no agents of each type. This restriction is not necessary to the analysis, but does greatly simplify the exposition below.

We say that an admissible arrangement $a(\theta, V^\beta)$ is blocked by a coalition B with associated vector of measures $\beta \neq 0$ if $\beta_i \in \{0, \mu_i\}$ $\forall$ i, and if there exists an allocation $c^\beta \in F(\beta)$ such that

(6) $$V_i(c_i^\beta) \geq V_i[a_j(\mu-\beta, V^\beta)]$$

for all i such that $\beta_i > 0$, for all j;

(7) $$V_i[a_i(\mu-\beta, V^\beta)] \geq V_i^\beta$$

for all i such that $\mu_i - \beta_i > 0$; and such that

(8) $$V_i(c_i^\beta) > V_i[a_i(\mu)]$$ [4/]

for all i such that $\beta_i > 0$. Finally, an admissible arrangement that is unblocked is called a core arrangement.[5/]

A word of explanation is in order about conditions (6) and (7). These conditions require that, if a blocking coalition is to consist of a particular mix of types, then the allocation c^β must provide incentives for the proper set of types to defect, given the arrangement specified by the incumbent coalition. Or, in other words, (6) and (7) require that agents be "rationally sorted" among coalitions. These conditions on blocking may be viewed as an aspect of the feasibility constraints imposed on coalition formation by the presence of private information.

IV. A Candidate Arrangement

The arrangement specified here, which we will prove is unblocked, is closely related (in terms of the allocations it specifies) to the allocations studied by Miyazaki (1977) and Spence (1978). It also reflects some of the ideas embodied in Wilson's (1977) equilibrium concept. It will be easiest to begin with the allocation specified by $a(\mu)$, (that is, the allocation of the incumbent if no blocking coalition forms), which is exactly the Miyazaki-Spence allocation.

Suppose, then, that $B = \phi$; i.e., that no blocking coalition forms in the second stage of the game. We will show that a core arrangement exists that in this case specifies the allocation $c^\mu = a(\mu)$ that solves the following problem (P):

$$(P) \qquad \max_{c \in R_+^{2n}} V_n(c_n)$$

subject to

(9) $c \in F(\mu)$

(10) $V_1(c_1) \geq \overline{V}_1$

(11) $V_i(c_i) \geq \overline{V}_i(\mu)$; $i=2, \ldots, n-1$.

Constraint (9) is simply feasibility, which embodies the full set of incentive constraints faced by the coalition, as well as the resource constraint. In order to describe constraints (10) and (11), it is necessary to define $\overline{V}_1$ and $\overline{V}_i(\mu)$; $2 \leq i \leq n-1$.

The value $\overline{V}_1$ is defined to be the maximum level of expected utility that type 1 agents could obtain in a coalition consisting entirely of type 1 agents. Thus $\overline{V}_1$ is the expected utility level obtained by solving

(12) $\max V_1(c_1)$

subject to $p_1(c_{11}-e_1) + (1-p_1)(c_{12}-e_2) \leq 0$.

Then, given $\overline{V}_1$, the values $\overline{V}_i(\mu)$ are defined recursively as the expected utility levels given by the solutions to the problems, (P.i):

(P.i) $\max\limits_{c \in R_+^{2i}} V_i(c_i)$

subject to

(13) $V_j(c_j) \geq V_j(c_k) \; \forall \; j, k \leq i$

(14) $V_j(c_j) \geq \overline{V}_j(\mu)$; $j=1, \ldots, i-1$

(15) $\sum\limits_{j=1}^{i} \mu_j \left[p_j(c_{j1}-e_1)+(1-p_j)(c_{j2}-e_2)\right] \leq 0,$

where $\bar{V}_1(\mu) = \bar{V}_1$. Thus, $\bar{V}_i(\mu)$ is the highest level of expected utility that can be obtained by type i agents in a coalition consisting only of themselves and agents of lower types, if agents of lower types are to have no incentive to defect [constraint (14)].

The allocation solving the problem (P) has been studied by Miyazaki (1977), Spence (1978), and Judd (1985), and has been associated with a Wilson (1977) equilibrium. For our purposes, however, it is necessary to go further and discuss allocations specified by the arrangement if $\beta \neq 0$, that is, if a blocking coalition forms.

If a blocking coalition forms at stage 2, then remaining members of the incumbent coalition can be divided into three categories, according to their type.

(i) Category 1. Let $\underline{i}$ be the lowest index of type represented in the blocking coalition; i.e., the smallest value of i such that $\beta_i > 0$. Then category 1 agents in the (residual) incumbent coalitions are agents of types 1, 2, ..., $\underline{i} - 1$ (if $\underline{i} > 1$).

(ii) Category 2. Category 2 agents in the (residual) incumbent coalition are agents with indexes i such that $\beta_{i-1} > 0$ (while clearly $\beta_i = 0$). Or, in other words, category 2 consists of agents in the incumbent coalition of type i, while type i - 1 agents are in the blocking coalition.

(iii) Category 3. Category 3 agents are members of the (residual) incumbent coalition in the following situation. Let $\beta_j = 0$ (so type j agents are in the incumbent coalition), and let $\beta_{j+1} > 0$ (if $j \neq n$). Let $j-s-1$ be the largest index in the blocking coalition less than j (if $j-s-1 \geq 1$). Then category 3 consists of agents with indexes i satisfying $j-s < i \leq j$.

Figure 1 gives a schematic representation of how, for one example, members of the (residual) incumbent coalition are divided into categories. The categories are exhaustive.

We will prove below that an arrangement specifying allocations as follows is admissible and unblocked.

(i') Category 1. If a type i agent belongs to category 1, then $a_i(\mu-\beta, V^\beta) = c_i^{\mu-\beta}$ is the i^{th} component of the vector $(c_1, c_2, \ldots, c_{\underline{i}-1})$ solving the problem (P.$\underline{i}$-1).

(ii') Category 2. We temporarily defer specifying the allocation of agents belonging to category 2. However, if a type i agent belongs to category 2, define the vector $\hat{c}_i = (\hat{c}_{i1}, \hat{c}_{i2})$ to be the allocation solving two equations

(16) $$V_{i-1}(\hat{c}_i) = V_{i-1}(c_{i-1}^{\beta})$$

(17) $$p_i(\hat{c}_{i1}-e_1) + (1-p_i)(\hat{c}_{i2}-e_2) = 0.$$

Then define

(18) $$\bar{V}_i(\mu-\beta, V^\beta) \equiv V_i(\hat{c}_i)$$

which will be used in a moment.

(iii') Category 3. If an agent of type i is in category 3, let $j-s-1$ be the highest type less than i in the blocking coalition, and let $j + 1$ be the lowest type greater than i in the blocking coalition (if any type higher than i is in the blocking coalition). Then our arrangement $a(\theta,V^\beta)$ specifies an allocation $(c^{\mu-\beta}_{j-s},\ldots,c^{\mu-\beta}_{j}) = [a_{j-s}(\mu-\beta,V^\beta),\ldots,a_j(\mu-\beta,V^\beta)]$ for both category 2 and category 3 members of the residual coalition that solves the following problem:

(A.j) $\max V_j(c_j)$

subject to

(19) $V_i(c_i) \geq V_i(c_k);\ \forall\ i,\ k = j\text{-}s,\ \ldots,\ j$

(20) $V_{j-s-1}(c_{j-s}) \leq V_{j-s-1}(c^\beta_{j-s-1})$ 6/

(21) $\sum_{i=j-s}^{j} \mu_i[p_i(c_{i1}-e_1)+(1-p_i)(c_{i2}-e_2)] = 0$

(22) $V_{j-s}(c_{j-s}) \geq \bar{V}_{j-s}(\mu-\beta,V^\beta)$

(23) $V_i(c_i) \geq \bar{V}_i(\mu-\beta,V^\beta);\ j\text{-}s + 1 \leq i \leq j\text{-}1.$

Equations (19) and (20) are incentive constraints, while equation (21) is a resource constraint for agents of types j-s, j-s+1, ..., j. The term $\bar{V}_{j-s}(\mu-\beta,V^\beta)$ in equation (22) was defined in equation (18). The terms $\bar{V}_i(\mu-\beta,V^\beta)$ appearing in (23) are defined recursively as the levels of expected utility associated with the solutions to the following problems:

(A.i) $\max V_i(c_i)$

subject to

(24) $V_k(c_k) \geq V_k(c_m);\ \forall\ k,m = j\text{-}s,\ \ldots,\ i$

(25) $V_{j-s-1}(c_{j-s}) \leq V_{j-s-1}(c^\beta_{j-s-1})$

$$(26)\qquad \sum_{k=j-s}^{i} \mu_i\left[p_i(c_{i1}-e_1)+(1-p_i)(c_{i2}-e_2)\right] = 0$$

$$(27)\qquad V_{j-s}(c_{j-s}) \geq \overline{V}_{j-s}(\mu-\beta, V^{\beta})$$

$$(28)\qquad V_k(c_k) \geq \overline{V}_k(\mu-\beta, V^{\beta});\ j-s+1 \leq k \leq i-1.$$

Discussion

While the specification of allocations for members of the residual incumbent coalition is notationally cumbersome, the basic idea is straightforward. Suppose types j-s-1 and j + 1 have joined the blocking coalition, while types j-s, ..., j have not. Consider, then, the allocations of types j-s through j in Figure 1. Their allocations are chosen to be resource and incentive feasible, with no resources flowing from this group to any class lower than j-s, and no resources being received from any class higher than j. This subset of the residual coalition is, by construction of the arrangement, self-sufficient in resources.

The specification of this arrangement corresponds quite closely to the behavioral assumptions employed in defining a Wilson equilibrium. In particular, the behavioral assumption employed by Wilson (1977) is that insurance firms "drop" policies that become unprofitable as a result of actions by other firms. Our specification of an arrangement is related, in that the incumbent coalition behaves as follows. If agents of type j join the blocking coalition, agents of types higher than j do not subsidize agents of types lower than j in the (residual) incumbent coalition. Thus, rather than "dropping policies," the incumbent refuses to "indirectly subsidize" the blocking coalition by subsidizing agents with indexes less than j (which would have the effect of relaxing an incentive constraint from the point of view of the blocking coalition).

IV. Existence of a Core Arrangement

It should be clear that, by construction, the arrangement just presented is feasible for the incumbent coalition. We wish also to show that it is a credible arrangement. Denote the arrangement just described by $a^*(\theta,V^\beta)$. Then we state the following lemma.

Lemma 1. $a^*(\theta,V^\beta)$ is a credible arrangement.

Proof. Since $\beta_i \in \{0,\mu_i\}$ for all i, the set of agent types represented in the (residual) incumbent coalition can be partitioned as follows. Let $\underline{i}_1$ be the smallest index with $\beta_i = 0$ (i.e., the lowest type remaining in the incumbent coalition) for any given β, and let $\bar{i}_1 + 1$ be the smallest index larger than $\underline{i}_1$ which is represented in the blocking coalition. Then types $\underline{i}_1$, ..., $\bar{i}_1$ are in the (residual) incumbent coalition. Similarly let $\underline{i}_2$ be the smallest index larger than $\bar{i}_1$ which is represented in the incumbent coalition and define $\bar{i}_2$ analogously to $\bar{i}_1$, so that types $\underline{i}_2$, ..., $\bar{i}_2$ are in the incumbent coalition, while type $\bar{i}_2 + 1$ is not (if $\bar{i}_2 < n$). Proceeding similarly, there are intervals of types $(\underline{i}_1,\ldots,\bar{i}_1)$; $(\underline{i}_2,\ldots,\bar{i}_2)$; ...; $(\underline{i}_h,\ldots,\bar{i}_h)$, with h depending on β, such that for any index i with $\beta_i = 0$, $i \in (\underline{i}_k,\ldots,\bar{i}_k)$ for some k; $1 \le k \le h$. Of course, an interval may include only one class of agent so that $\underline{i}_k = \bar{i}_k$. Figure 2 may help to explain how these intervals are defined.

Now suppose that $\underline{i}_1 = 1$. If so, it should be clear from the specification of $a^*(\theta, V^\beta)$, that the allocation $[a_1^*(\mu-\beta, V^\beta), \ldots, a_{\bar{i}_1}^*(\mu-\beta, V^\beta)]$ solves the problem $(P.\bar{I}_1)$. If $\underline{i}_1 > 1$, then the allocation $[a_{\underline{i}_1}^*(\mu-\beta, V^\beta), \ldots, a_{\bar{i}_1}^*(\mu-\beta, V^\beta)]$ solves the problem $(A.\bar{I}_1)$. Similarly, the allocation $[a_{\underline{i}_k}^*(\mu-\beta, V^\beta), \ldots, a_{\bar{i}_k}^*(\mu-\beta, V^\beta)]$ solves the problem $(A.\bar{I}_k)$; $1 \le k \le h$.

Next suppose, for the purpose of producing a contradiction, that the arrangement $a^*(\theta, V^\beta)$ is not credible for some blocking coalition β. Then for this β there exists an allocation $\tilde{c} \in F(\mu-\beta)$ such that $V_i(\tilde{c}_i) \ge V_i[a_i^*(\mu-\beta, V^\beta)]$ for all i such that $\beta_i = 0$, with strict inequality for some such i; and such that $V_i(\tilde{c}_i) \ge V_i^\beta$ for all i such that $\beta_i = 0$, while $V_i(\tilde{c}_i) \le V_i^\beta$ for all i such that $\beta_i > 0$.

Now notice that $[a_{\underline{i}_1}^*(-), \ldots, a_{\bar{i}_1}^*(-)]$ solves either the problem $(P.\bar{I}_1)$ or the problem $(A.\bar{I}_1)$. In either case, since $V_i(\tilde{c}_i) \ge V_i[a_i^*(\mu-\beta, V^\beta)]$ for all $i \in (\underline{i}_1, \ldots, \bar{i}_1)$, inspection of the problem $(P.\bar{I}_1)$ [or $(A.\bar{I}_1)$] indicates that

$$(29) \qquad \sum_{j=\underline{i}_1}^{\bar{i}_1} \mu_j[p_j(\tilde{c}_{j1}-e_1)+(1-p_j)(\tilde{c}_{j2}-e_2)] \ge 0,$$

with strict inequality if $V_i(\tilde{c}_i) > V_i[a_i^*(\mu-\beta, V^\beta)]$ for some $i \in (\underline{i}_1, \ldots, \bar{i}_1)$. Similarly, since $[a_{\underline{i}_k}^*(-), \ldots, a_{\bar{i}_k}^*(-)]$ solves the problem $(A.\bar{I}_k)$; $1 < k \le h$, and since $V_i(\tilde{c}_i) \ge V_i[a_i^*(-)]$ for all $i \in (\underline{i}_k, \ldots, \bar{i}_k)$, it must be the case that

$$(30) \qquad \sum_{j=\underline{i}_k}^{\bar{i}_k} \mu_j[p_j(\tilde{c}_{j1}-e_1)+(1-p_j)(\tilde{c}_{j2}-c_2) \ge 0$$

holds for all k; k = 2, ..., h; with strict inequality if $V_i(\tilde{c}_i) > V_i[a_i^*(-)]$ for some $i \in (\underline{i}_k, \ldots, \bar{i}_k)$.

Now sum (30) over all $k = 2, \ldots, h$, and add the result to (29) to obtain

$$(31) \qquad \sum_{j=1}^{n} (\mu_j - \beta_j)\left[p_j(\tilde{c}_{j1} - e_1) + (1-p_j)(\tilde{c}_{j2} - e_2)\right] > 0.$$

The inequality is strict, since there exists (by hypothesis) at least one i such that $\beta_i = 0$ and $V_i(\tilde{c}_i) > V_i[a_i^*(\mu-\beta, V^\beta)]$. But (31) contradicts the assumption that $\tilde{c} \in F(\mu-\beta)$, so $a^*(\theta, V^\beta)$ is credible, and Lemma 1 is proved.

We next show that the arrangement $a^*(\theta, V^\beta)$ is not blocked.

<u>Proposition 1</u>. The arrangement $a^*(\theta, V^\beta)$ is a core arrangement.

<u>Proof</u>. For the purpose of deriving a contradiction, suppose a blocking coalition B does exist, with associated vector of measures $\beta \neq 0$. Then there exists a largest index in the blocking coalition, $\bar{I}_h - 1$. To simplify notation call this index q. Then, $\beta_q > 0$, $\beta_i = 0$ if $q < n$ and $q < i \leq n$.

Now, observe three facts. First, by construction of the arrangement $a^*(\theta, V^\beta)$

$$(32) \qquad \sum_{i=1}^{q-1} (\mu_i - \beta_i)\left[p_i\left(a_{i1}^*(-) - e_1\right) + (1-p_i)\left(a_{i2}^*(-) - e_2\right)\right] = 0.$$

Second,

$$(33) \qquad V_i\left(c_i^\beta\right) > V_i\left[a_i^*(\mu)\right] \geq \bar{V}_i(\mu)$$

for all i such that $\beta_i > 0$. The first inequality in (33) follows from the definition of blocking, and the second follows from equation (11) in the construction of $a^*(\mu)$. Third, and finally,

$$(34) \qquad V_i\left[a_i^*(\mu-\beta,V^\beta)\right] \geq \overline{V}_i(\mu-\beta,V^\beta) \geq \overline{V}_i(\mu),$$

for all i such that $\beta_i = 0$ and $i < q$. The first inequality in (34) is by construction of $a_i^*(-)$. The second inequality in (34) should be clear from (33). In particular, the defection of types with indexes lower than i cannot make types above i worse off, since such a defection relaxes an incentive constraint(s) at no resource cost to the incumbent coalition.

Now consider the problem (P.q). [If q = n, consider the problem (P).] In this problem, by (32), (33), and (34), it is feasible to set

$$(35) \qquad c_i = \begin{cases} a_i^*(\mu-\beta,V^\beta); \text{ if } \beta_i = 0 \\ c_i^\beta; \text{ otherwise,} \end{cases}$$

for all $i \leq q$. Resource feasibility of the allocation (35) is guaranteed by (32) and the resource feasibility of c^β. Incentive feasibility of (35) is guaranteed by the incentive feasibility of the arrangement $a^*(-)$, the incentive feasibility of c^β for β, and the "rational sorting" conditions (6) and (7). Finally, (33) and (34) imply that $V_i(c_i) \geq \overline{V}_i(\mu)$ for all $i < q$ if c_i is chosen according to (35).]

But then, recall that $\overline{V}_q(\mu)$ is the expected utility level given by the solution of (P.q), and therefore, $V_q(\overline{\mu}) \geq V_q(c_q^\beta)$. (If q = n, take $\overline{V}_q(\mu) = \overline{V}_n(\mu) = V_n[a_n^*(\mu)]$.) However, by (8), $V_q(c_q^\beta) > V_q[a_q^*(\mu)] \geq \overline{V}_q(\mu)$ [by equation (11) in the construction of $a^*(\mu)$]. This is a contradiction, so $a^*(\theta,V^\beta)$ is unblocked.

VI. Uniqueness

Since the arrangement $a^*(\theta,V^\beta)$ is a core arrangement, it specifies an equilibrium allocation $a^*(\mu)$ that solves the problem (P) above. However, there may be other admissible, unblocked arrangements that lead to the same equilibrium allocation or, alternatively, to a different allocation also in the core.

We have not made much progress on the question of uniqueness. However, it is useful to know that there are situations of interest in which any core arrangement $\hat{a}(\theta,V^\beta)$, with $\hat{a}(\theta,V^\beta) \neq a^*(\theta,V^\beta)$ for some combination (θ,V^β), must have $\hat{a}(\mu) = a^*(\mu)$. Or, stated otherwise, any unblocked arrangement must select $a(\mu)$ to be the solution to (P) in these situations.

We now produce a set of conditions sufficient for this to be true. First, we restrict there to be only two types ($n = 2$). Second, we define the Rothschild-Stiglitz allocation (for $n = 2$) to be the allocation satisfying

$$\bar{c}_{11} = \bar{c}_{12} = p_1e_1 + (1-p_1)e_2, \tag{36}$$

$$p_2(\bar{c}_{21}-e_1) + (1-p_2)(\bar{c}_{22}-e_2) = 0, \tag{37}$$

and

$$V_1(\bar{c}_1) = V_1(\bar{c}_2), \tag{38}$$

with $\bar{c}_1 = (\bar{c}_{11},\bar{c}_{12})$ given by (36). Then we have the following proposition.

Proposition 2. If $n = 2$, and if the Rothschild-Stiglitz allocation is a Pareto optimum, then any core arrangement selects $a(\mu)$ to be the allocation that solves (P).

Remarks. If the Rothschild-Stiglitz allocation is a Pareto optimum, then it is the (unique) solution to (P). The Rothschild-Stiglitz allocation is a Pareto optimum if and only if

$$(39) \qquad \left(\frac{\mu_1}{1-\mu_1}\right)\left[\frac{p_2 - p_1}{p_2(1-p_2)}\right] \geq \frac{U'(\bar{c}_{11})[U'(\bar{c}_{22})-U'(\bar{c}_{21})]}{U'(\bar{c}_{22})U'(\bar{c}_{21})}$$

holds [see Rothschild and Stiglitz (1976), equation (4)].

Proof. To prove the proposition, we suppose that there exists a core arrangement $\hat{a}(\theta,V^{\beta})$ with $\hat{a}(\mu)$ not equal to the solution to (P). We then derive a contradiction by constructing a blocking coalition.

In particular, suppose that $\hat{a}(\theta,V^{\beta})$ is an admissible arrangement with $\hat{a}(\mu)$ not equal to the solution to (P). Construct a blocking coalition that consists of all type 2 agents (i.e., $\beta_2 = \mu_2$, $\beta_1 = 0$), and let $c_2^{\beta} = \bar{c}_2$. Clearly $c_2^{\beta} \in F(\beta)$. Moreover, since $\hat{a}(-)$ is admissible, and hence credible, for this β (and implied V^{β}), $\hat{a}_1(\mu-\beta,V^{\beta}) = \bar{c}_1$ must hold. We now check that the definition of a blocking coalition is satisfied.

First, since $\hat{a}(\mu)$ does not solve (P), while $(\bar{c}_1,\bar{c}_2)$ does, clearly

$$V_2(c_2^{\beta}) > V_2[\hat{a}_2(\mu)],$$

so (8) holds. Also, as is apparent from (38),

$$V_1[\hat{a}_1(\mu-\beta,V^{\beta})] = V_1(\bar{c}_1) \geq V_1(c_2^{\beta}) = V_1(\bar{c}_2),$$

while

$$V_2(c_2^{\beta}) > V_2(\bar{c}_1) = V_2[\hat{a}_1(\mu-\beta,V^{\beta})],$$

as will be evident from Rothschild and Stiglitz (1976). Thus (6) and (7) hold, and this is in fact a blocking coalition. Then the hypothesis that $\hat{a}(\cdot)$ is unblocked leads to a contradiction, establishing the proposition.

VII. Extending the Results

Since the two type case has been extensively studied, it would seem that this is a nontrivial uniqueness result. Unfortunately, however, we have not been able to produce stronger results than this. We believe that a natural way to proceed would be based on the observation that, in the set of economies examined here, there are typically net resource flows from higher types to lower types. A specification of a game in which higher types can exploit the advantage implicit in dispensing net subsidies to other groups would seem to be a natural extension. It would also seem that such a specification would reduce the set of core arrangements. We believe that extensions of this type would constitute an interesting line for future research.

Notes

1/ The applicability of the law of large numbers in this context is justified by Green (1984) and Judd (1985). An approach using nonstandard analysis appears in Stutzer (1986).

2/ See, e.g., Foley (1970), Richter (1974), and Starrett (1973).

3/ Conditions (3) and (4) require that candidate allocations $\tilde{c}^{\theta}$ not change the incentives of individual agents about which coalition they wish to join. Condition (5) says that $\tilde{c}^{\theta}$ is weakly preferred by all coalition members to the allocation specified by the arrangement a(-). It also bears emphasizing that, in the definition of a credible arrangement, the allocation of the blocking coalition as well as its composition, is taken as fixed.

4/ If $\beta = 0$ so that there is no blocking coalition we write $c^{\mu} = a(\mu)$.

5/ A strict inequality is employed in (8) because this simplifies the statement of the proofs below. However (8) can be relaxed to $V_i(c_i^{\beta}) \geq V_i[a_i(\mu)]$ for all i such that $\beta_i > 0$, with strict inequality for some i.

6/ Recall that c_{j-s-1}^{β} is taken as given by the incumbent coalition.

Figure 1

Type	Residual Incumbent Coalition		Blocking Coalition	Category
J + 1		--------->	X	-
J	X			3
J - s + 1	X			3
J - s	X			2
J - s - 1		--------->	X	-
J - s - 2	X			2
.		--------->	X	-
.	X			1
.	X			1
1.	X			1

Figure 2

Intervals $\underline{i}_k \ldots \bar{i}_k$	Residual Incumbent Coalition		Blocking Coalition
$\bar{i}_h$	X		
.	X		
$\underline{i}_h$	X		
		--------->	X
		--------->	X
$\underline{i}_2 = \bar{i}_2$	X		
		--------->	X
$\bar{i}_1$	X		
.	X		
$\underline{i}_1$	X		

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