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A CONTINUOUS TIME, GENERAL EQUILIBRIUM, INVENTORY-SALES MODEL

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A. Introduction

Application of the stock adjustment model to the study of inventory behavior frequently produces parameter estimates which imply implausibly low speeds of adjustment of actual to target inventories. For example, Feldstein and Auerbach's estimated parameters [1976] imply that firms take almost 19 years to close 95 percent of the gap between actual and desired inventory stocks. Application of the stock adjustment model to other areas--for example, money demand--also yields implausibly low estimates of speeds of adjustment.

One explanation of these anomalous results is that they reflect the effects of temporal aggregation bias (for other explanations, see Eichenbaum [1984] and Goodfriend [1985]). The stock adjustment literature typically assumes that the interval of time separating economic decisions corresponds to the interval of time separating the observations available to the econometrician. Zellner [1968] and Mundlak [1961] showed theoretically that if this is longer than appropriate, then the econometrician could be led to understate the speed of adjustment. This is consistent with the experience of Bryan [1967], who applied the stock adjustment model to bank demand for excess reserves. Bryan found that when the model was applied to weekly data, the estimated time to close 95 percent of the gap between desired and excess reserves was 5.2 weeks. When the model was applied to monthly time aggregated data, the 95 percent closure time was estimated to be 28.7 months. Bryan's results, which can only be due to temporal aggregation bias, reinforce the view that temporal aggregation could account entirely for the anomalous results reported by Feldstein and Auerbach [1976].

We propose to investigate empirically whether temporal aggregation bias can account for the slow speeds of adjustment typically found in studies of inventory behavior. We plan to do this by estimating a continuous time version of the stock adjustment model, and to compare the speed of adjustment implied by the parameter estimates with those reported in the literature. We plan to formulate a general equilibrium model of employment, inventories, and output which implies a continuous time version of the stock adjustment model studied in the literature. In this way, we will have an explicit economic rationale for the stock adjustment model.

In addition to shedding light on the anomalous findings in the stock adjustment literature, we expect that our project will make several other contributions as well. First, we will supply a completely worked example of estimating a rational expectations equilibrium model in continuous time. We hope that other researchers will find this useful in applying continuous time estimation techniques. Second, if the evidence suggests that the continuous time model performs better than a discrete time equivalent, then we plan to perform a formal non-nested test of the continuous time model versus the discrete time model. Non-nested testing in the empirical/macroeconomics/time series context is in its infancy, and we think it would be useful if more of this were done. (See Singleton [1984] for a contribution.) Third, we plan to use the model to provide a concrete example of some of the pitfalls of interpreting moving average representations obtained by estimating time series models unrestricted by economic theory. In doing so, we will be illustrating points made at the theoretical level by Hansen and Sargent [1982].

Section B presents an example of the kind of model we plan to formulate and estimate. There it is shown in what sense the model implies the stock adjustment model studied in the literature. Also, we indicate how we plan to carry out non-nested testing. In Section C we show how we plan to estimate the parameters of the model. In Section D we indicate how we plan to go about illustrating the pitfalls of interpreting moving average representations obtained from unrestricted time series models.

B. The Model

We consider a model of employment, inventories, and output which is similar to the general equilibrium model of employment that appears in Sargent [1979 Chapter XVI, Section 3]. As shown below, the model implies a continuous time version of the stock adjustment equation for inventories that appear in the literature.

We assume a representative household which chooses $\{s(t), N(t); t \geq 0\}$ to maximize

$$\text{(1a)} \qquad E_t \int_0^\infty e^{-r\tau}[u(t+\tau)s(t+\tau) - \frac{A}{2}s(t+\tau)^2 - N(t+\tau)]d\tau,$$

subject to

$$\text{(1b)} \qquad P(t)s(t) = N(t) + \pi(t).$$

Here,

E_t = linear least squares projection operator, conditional on time t information set,

$u(t)$ = disturbance to marginal utility of consumption, with second moment properties specified in Section C below,

$s(t)$ = consumption of the one commodity,

$N(t)$ = employment,

$P(t)$ = price of the one commodity, denominated in labor units,

$\pi(t)$ = lump sum dividend earnings of the household, denominated in labor units, and

A,r = positive constants.

Solving (1) yields the following inverse demand function:

$$P(t) = -As(t) + u(t). \tag{2}$$

The representative firm's profit function is

$$E_t \int_0^\infty e^{-r\tau}\pi(t+\tau)d\tau, \tag{3a}$$

where

$$\pi(t) = P(t)s(t) - N(t) - \frac{b}{2}[s(t)-cI(t)]^2 - \nu(t)I(t) - \frac{e}{2}I(t)^2. \tag{3b}$$

In (3b), $P(t)s(t)$ represents total revenues at time t, and $N(t)$ is the wage bill incurred in producing time t output, $Q(t)$. The third term to the right of the equality in (3b) reflects the idea that there are costs, denominated in units of labor, allowing inventories to deviate from some proportion of sales. (See, e.g., Blanchard [1983, p. 378].) Finally, the last two terms in (3b) represent costs of holding inventories. There, $\nu(t)$ represents a disturbance to the marginal cost of holding inventories. Its second moment properties are described in Section C.

We assume the following production function for $Q(t)$:

$$Q(t) = [\tfrac{2}{a}N(t)]^{.5} \tag{4}$$

In (3) and (4), r, a, b, c, and e are positive constants. The link between current production, inventories, and sales is given by

$$Q(t) = s(t) + DI(t). \tag{5}$$

Substituting (4) and (5) into (3b), we get

$$\pi(t) = P(t)s(t) - \frac{a}{2}[s(t)+DI(t)]^2 - \frac{b}{2}[s(t)-cI(t)]^2 - \nu(t)I(t) - \frac{e}{2}I(t)^2. \tag{3b$'$}$$

The objective of the representative firm at time t is to choose $DI(t+\tau)$, $Q(t+\tau)$, and $s(t+\tau)$; $\tau > 0$ to maximize (3) subject to (4), (5), (13), $I(t)$ given, and beliefs about the law of motion of aggregate $s(t)$. In a rational expectations equilibrium, these beliefs are self-fulfilling. Sargent [1979, p. 375] describes a simple procedure for finding rational expectations equilibria in the linear quadratic, discrete time context. The discussion in Hansen and Sargent [1980] spells out precisely how Sargent's solution procedure [1979] can be modified to accommodate our continuous time setup. Briefly, the procedure is as follows. Write

$$F[I(t),DI(t),s(t),\nu(t),P(t),t] = e^{-rt}\pi(t), \tag{6}$$

where $\pi(t)$ is defined in (3b)'. Then, the objective of the firm at time t is to

$$\underset{\substack{DI(t+\tau),s(t+\tau)\\ \tau > 0}}{\text{maximize}} \quad E_t \int_0^\infty F(I(t+\tau),DI(t+\tau),s(t+\tau),\nu(t+\tau),P(t+\tau),\tau)d\tau \tag{7}$$

subject to I(t) given. The solution to this problem is simplified by exploiting the property to certainty equivalence. Accordingly, we first solve a certainty version of (7) in which future random variables have been replaced by their conditional mean. Then we use a continuous time version of the Wiener-Kolmogorov forecasting formula to express the conditional expectations in terms of observed variables. Standard control theory results inform us that if boundary conditions can be ignored, then the optimal path for I(t) and s(t) satisfies the following conditions:

(8a) $$\frac{\partial F}{\partial s(t)} = 0$$

(8b) $$\frac{\partial F}{\partial I(t)} = D\,\frac{\partial F}{\partial DI(t)}$$

These imply respectively:

(9a) $$P(t) - (a+b)s(t) - aDI(t) + bcI(t) = 0$$

(9b) $$aD^2I(t) - raDI(t) - (c^2b+e)I(t) + aDs(t) + (cb-ra)s(t) = v(t)$$

In rational expectations equilibrium, P(t) must satisfy (2).

Substituting this into (9a) and rearranging

(10) $$s(t) = \left(\frac{-a}{a+b+A}\right)DI(t) + \left(\frac{bc}{a+b+A}\right)I(t) + \left(\frac{1}{a+b+A}\right)u(t).$$

It is convenient to collapse (9b) and (10) into one differential equation in I(t). Substituting for s(t) and Ds(t) in (9b) from (10) get

(11a) $$\{D^2-rD-k\}I(t) = \frac{a+b+A}{a(b+A)}\left[v(t)+\frac{ra-cb-aD}{a+b+A}\,u(t)\right],$$

where

(11b) $$k = \frac{a+b+A}{a(b+A)}\left\{\frac{bc[c(a+A)+ra]}{a+b+A} + e\right\}.$$

Alternatively,

(12a) $(D-\lambda)[D-(r-\lambda)]I(t) =$

$$\frac{a+b+A}{a(b+A)}\nu(t) - \frac{1}{(b+A)}[\frac{1}{a}(bc-ra)+D]u(t),$$

where

(12b) $\lambda = \frac{1}{2}r + [k + \frac{1}{4}r^2]^{1/2}.$

Since $k > 0$, it follows from (12b) that $\lambda > 0$ is real. Moreover, it is easy to verify that $r - \lambda = \frac{1}{2}r - [k + \frac{1}{4}r^2]^{1/2} < 0$. (To see this, consider $f(k) = \frac{1}{2}r - [k + \frac{1}{4}r^2]^{1/2}$ and note that $f(0) = 0$, and $f'(k) < 0$ for $k > 0$.) Solving the stable root $(r-\lambda)$ backward and unstable root (λ) forward in (12a), get

(13)
$$\begin{aligned} DI(t) &= (r-\lambda)I(t) - \frac{a+b+A}{a(b+A)}\int_0^\infty e^{-\lambda\tau}E_t\nu(t+\tau)d\tau \\ &\quad + \frac{1}{b+A}\int_0^\infty e^{-\lambda\tau}E_t[\frac{1}{a}(cb-ra)+D]u(t+\tau)d\tau \\ &= (r-\lambda)I(t) - \frac{a+b+A}{a(b+A)}\int_0^\infty e^{-\lambda\tau}E_t\nu(t+\tau)d\tau \\ &\quad - \frac{1}{b+a}u(t) + \frac{1}{b+a}[\frac{bc}{a} - (r-\lambda)]\int_0^\infty e^{-\lambda\tau}E_t u(t+\tau)d\tau \end{aligned}$$

Substituting (13) into (10),

(14)
$$\begin{aligned} s(t) &= \frac{bc - a(r-\lambda)}{a+b+A}I(t) + \frac{1}{b+A}\int_0^\infty e^{-\lambda\tau}E_t\nu(t+\tau)d\tau \\ &\quad - \frac{a}{(b+A)(a+b+A)}\int_0^\infty e^{-\lambda\tau}E_t[\frac{1}{a}(cb-ra)+D]u(t+\tau)d\tau \\ &\quad + \frac{1}{a+b+A}u(t) \end{aligned}$$

The solution to the certainty equivalent version of (7) is given by (13) and (14). The solution to the stochastic problem is complete once the forecasting problems in (13) and (14) have been solved. We do this in Section C. First, we derive the stock adjustment model implicit (13) and (14).

Let $I^*(t)$ be the level of industrywide inventories such that if $I(t) = I^*(t)$, then $DI(t) = 0$. $I^*(t)$ is taken to be the time t level of "desired" or "target" inventories. By (13),

$$(15) \qquad I^*(t) = \frac{a + b + A}{(r-\lambda)a(b+A)} \int_0^\infty e^{-\lambda\tau} E_t \nu(t+\tau)d\tau - \frac{1}{(r-\lambda)(b+A)} \int_0^\infty e^{-\lambda\tau} E_t \left[\frac{1}{a}(cb-ra)+D\right] u(t+\tau)d\tau$$

Substituting (15) into (13), we get the stock adjustment model:

$$(16) \qquad DI(t) = \alpha\left(I^*(t)-I(t)\right),$$

where $\alpha = (\lambda-r) > 0$.

We require a measure of "speed of adjustment" which can be compared with similar measures in the literature. In order to make this concept precise we imagine, counterfactually, that movements in $I^*(\tau)$ can be ignored over an interval $\tau\varepsilon(t,t+1)$, i.e., $I^*(\tau) = I^*_t$ for $\tau \varepsilon (t,t+1)$. In this case, the solution to (16) is

$$I^*_t - I(t+1) = e^{-\alpha}\left(I^*_t-I(t)\right),$$

or, after adding $I(t) - I^*_t$ to both sides,

$$(17) \qquad I(t+1) - I(t) = (1-e^{-\alpha})\left(I^*_t-I(t)\right).$$

Thus, the amount of a given gap between target inventories, I_t^*, and $I(t)$ that is closed in one period is $T = 1 - e^{-\alpha} = [I(t+1)-I(t)]/[I_t^*-I(t)]$. Our intention is to obtain an estimate of T by jointly estimating the parameters of the model. Our plan is to compare our estimate of T with those reported in the literature. For example, Feldstein and Auerbach's estimate of T is .06 [1976 p. 366], which implies that firms only reduce 6 percent of a gap between actual and desired inventories in one quarter. As they emphasize, their estimate of the speed of adjustment is implausibly low.

The theoretical argument advanced in Zellner [1968] draws attention to the possibility that Feldstein and Auerbach's anomalous results are due to temporal aggregation bias. They make the assumption that the interval of time separating the economic decisions of agents is one quarter, whereas it seems plausible that decisions of the representative agent are in fact made over a finer interval. If this is the case, and the agent makes decisions in continuous time, then our estimate of the speed of adjustment will not be distorted by temporal aggregation bias.

If Zellner's conjecture is confirmed, then we plan to proceed one step further and carry out a formal hypothesis test of the null hypothesis that the continuous time model is true, against the discrete time alternative. The test we have in mind is a Cox-type test [1961] constructed to take into account our vector time series context. In separate work, we hope to generalize this to the vector case. (Other work, such as that of Pesaran and Deaton [1978], does not apply to the time series context. Walker [1967] developes the theory for the scalar time series context.) Even if the latter effort fails, we can still compute the Cox statistic and present it as a model diagnostic statistic. The computations required for this are described and applied in Christiano [1984].

C. The Reduced Form of the Model

We assume that $u(t)$ and $\{\nu(t)\}$ are covariance stationary with the following continuous time Wold representation:

$$(18)\qquad \begin{pmatrix} u(t) \\ \nu(t) \end{pmatrix} = \begin{bmatrix} h_1 & + & d_1 \\ h_2 & & d_2 \end{bmatrix} t \quad \begin{pmatrix} \psi(D) & 0 \\ 0 & \phi(D) \end{pmatrix} \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix}$$

where $\varepsilon(t) = [\varepsilon_1(t), \varepsilon_2(t)]^T$ is the continuous time vector of linear least squares innovations in $u(t)$, $\nu(t)$ with $E\varepsilon(t)\varepsilon(t-\tau)^T = \delta(\tau)V$, where δ is the Dirac delta function. Also, ψ and ϕ are rational polynomials in the time derivative operator D, and are assumed to be analytic in the (open) right half of the complex plane. Further restrictions will be placed on ψ and ϕ below.

The forecasting problems in (13) and (14) have the following solutions, as proved in Hansen and Sargent [1980]:

$$(19)\qquad \int_0^\infty e^{-\lambda\tau} E_t [\tfrac{1}{a}(cb-ra)+D] u(t+\tau)d\tau =$$

$$F(D)\varepsilon_1(t) + \{[\tfrac{1}{a}(cb-ra)h_1]+d_1\} \frac{1}{\lambda} + \frac{1}{a}(cb-ra)\frac{d_1}{\lambda^2} + \frac{1}{a}(cb-ra)\frac{d_1}{\lambda} t$$

$$(20)\qquad \int_0^\infty E_t e^{-\lambda\tau} \nu(t+\tau)d\tau = [\frac{-\phi(D)+\phi(\lambda)}{D-\lambda}]\varepsilon_2(t) + (\frac{h_2}{\lambda}) + (\frac{d_2}{\lambda_2}) + \frac{1}{\lambda} d_2 t,$$

where

$$F(D) = \frac{-[\frac{1}{a}(cb-ra)+D]\psi(D)+[\frac{1}{a}(cb-ra)+\lambda]\psi(\lambda)}{D - \lambda}$$

Substituting (19) and (20) into (13) and (14), and rearranging yields

$$(21a)\qquad \begin{pmatrix} I(t) \\ s(t) \end{pmatrix} = f + gt + C(D)\varepsilon(t)$$

where, f and g are 2 x 1 element vectors whose elements are functions of a, b, c, e, A, r, h_1, h_2, d_1, d_2. Also,

(21b)

$$C_{11}(D) = \frac{F(D)}{(b+A)[D-(r-\lambda)]}$$

$$C_{12}(D) = -\frac{a+b+A}{a(b+A)[D-(r-\lambda)]}\left(\frac{-\phi(D)+\phi(\lambda)}{D-\lambda}\right)$$

$$C_{21}(D) = \left(\frac{-a}{(b+A)(a+b+A)}\right)\left[\frac{D-\frac{bc}{a}}{D-(r-\lambda)}\right]F(D) + \frac{\psi(D)}{a+b+A}$$

$$C_{22}(D) = \left(\frac{1}{b+A}\right)\left[\frac{D-\frac{bc}{a}}{D-(r-\lambda)}\right]\left[\frac{-\phi(D)+\phi(\lambda)}{D-\lambda}\right].$$

It is easy to verify that there are no restrictions across the elements of f, g, and C(D).

We do not have observations on $\{I(t),s(t)\}$, but on $\{I(t),\bar{s}(t)\}$, where $\bar{s}(t)$ is s(t) averaged over the unit interval. In order for our statistical model to make sense, we require that $\{I(t),\bar{s}(t)\}$ be a physically realizable stochastic process. This places restrictions on ϕ and ψ in (18), which we now discuss. Realizability of $\{I(t),\bar{s}(t)\}$ requires:

$$\lim_{|s|\to\infty} C_{11}(s) = \lim_{|s|\to\infty} C_{12}(s) = 0$$

$$\lim_{|s|\to\infty} |C_{21}(s)| < \infty, \quad \lim_{|s|\to\infty} |C_{22}(s)| < \infty.$$

Examination of (21b) reveals that the latter imply the following restrictions on ϕ and ψ:

$$\phi(D) = \frac{\gamma_0 + \gamma_1 D + \cdots + \gamma_{m+1} D^{m+1}}{\alpha_0 + \alpha_1 D + \cdots + \alpha_{m-1} D^{m-1} + D^m} = \frac{\gamma(D)}{\alpha(D)}$$

(22)

$$\psi(D) = \frac{\delta_0 + \delta_1 D + \cdots + \delta_n D^n}{\beta_0 + \beta_1 D + \cdots + \beta_{n-1} D^{n-1} + D^n} = \frac{\delta(D)}{\beta(D)},$$

where $m, n > 0$. (Earlier, we imposed the restriction that $\alpha(s) = 0$ implies $Re(s) < 0$, $\beta(s) = 0$ implies $Re(s) < 0$.) Since γ_m, γ_{m+1}, and δ_n are permitted to be non-zero in the above expressions, $u(t)$ and $\nu(t)$ are permitted to be non-realizable stochastic processes. This in turn gives rise to the possibility that $s(t)$ and $P(t)$ are not realizable (see, for example, (2) and (10)). In interpreting this note, in any case, $s(t)$, $P(t)$, $u(t)$, and $\nu(t)$ are realizable after they have been integrated over an arbitrarily short interval.

The extensive cross-equation restrictions between the rows of $C(D)$, in addition to the rational form of $C(D)$, can be expected--after some additional restrictions--to result in the model's parameters being identified from discrete data (see, Hansen and Sargent [1983] and Christiano [1982]).

D. Computing the Frequency Domain Approximation to the Likelihood Function

The objective of this section is to provide a computationally convenient strategy for evaluating the frequency domain approximation to the likelihood of $\{\bar{Y}(t), t=1,\ldots,T\}$ where $\bar{Y}(t) \equiv (I(t), \bar{s}(t))^T$. We assume that inventories $I(t)$, are measured point-in-time and at the beginning of the sampling interval. Consequently,

$$(23)\qquad \overline{s}(t) = \int_0^1 s(t+\tau)d\tau.$$

In addition, we make the simplifying assumption that all roots of polynomials are distinct. Finally, the discussion below assumes that $\{\overline{Y}(t)\}$ is covariance stationary and has a zero mean. Equation (21a) indicates that this assumption is approximately satisfied if $\{\overline{Y}(t)\}$ is the disturbance in a least square regression of $\overline{Y}(t)$ on a constant and linear trend.

The outline of this section is as follows. First, we derive an expression for the continuous time spectral density of $Y(t) \equiv (I(t),s(t))^T$. We denote this by $S_y(i\omega)$, where $\omega\varepsilon(-\infty,+\infty)$ and $i^2 \equiv -1$. Define $S_{\overline{y}}(i\omega)$ as the spectral density of $\overline{Y}(t) \equiv (I(t),\overline{s}(t))^T$ at frequency ω. In the second part of this section, we obtain $S_{\overline{y}}$ from S_y and recover the covariance function of $\overline{Y}(t)$ from $S_{\overline{y}}$. Denote this by $R_{\overline{y}}(\tau) = E\overline{Y}(t)\overline{Y}(t-\tau)^T$. The $R_{\overline{y}}(\tau)$ function at integer values of τ is then used to compute the spectral density of $\{\overline{Y}(t),t=0,\pm1,\pm2,\ldots\}$. We denote this by $S^d_{\overline{y}}(e^{-i\omega})$, where $\omega\varepsilon(-\pi,\pi)$. The third and final part of this section shows how to combine $\{\overline{Y}(t),t=1,\ldots,T\}$ and $S^d_{\overline{y}}$ to compute the frequency domain approximation to the likelihood function.

1. <u>The Continuous Time Spectral Density of $\{Y(t)\}$</u>

We begin by providing computationally convenient expressions for $F(D)$ and $[-\phi(D)+\phi(\lambda)]/(D-\lambda)$. Note:

$$(24)\quad \phi(D) = \frac{\gamma(D)}{\alpha(D)} = \frac{\gamma(D)-[\gamma_{m+1}D+\gamma_m-\gamma_{m+1}\alpha_{m-1}]\alpha(D)}{\alpha(D)} + [\gamma_{m+1}D+\gamma_m-\gamma_{m+1}\alpha_{m-1}].$$

Write

(24a) $\alpha(D) = (D-\rho_1) \dots (D-\rho_m)$,

and suppose that $\rho_i \neq \rho_j$ for $i \neq j$, $\rho_i \neq r - \lambda$ for all i. Then (24) can be written in partial fractions expansion form as follows:

(25a) $$\phi(D) = \begin{cases} \left(\sum_{j=1}^{m} \frac{B_j}{D-\rho_j}\right) + \left(\gamma_{m+1}D+\gamma_m-\gamma_{m+1}\alpha_{m-1}\right) & m>0 \\ \gamma_0+\gamma D & m=0 \end{cases}$$

Here,

(25b) $$B_j = \frac{\gamma(\rho_j)}{\prod_{\substack{k=1 \\ k\neq j}}^{m} (\rho_j-\rho_k)} \qquad j = 1, \dots, m \text{ (if } m > 0)$$

Using (25), it is straightforward to verify that

(26) $$-\frac{\phi(D) - \phi(\lambda)}{D-\lambda} = \begin{cases} \sum_{j=1}^{m} \left(\frac{B_j}{\lambda-\rho_j}\right)\left(\frac{1}{D-\rho_j}\right) - \gamma_{m+1} & m>0 \\ -\gamma_1 & m=0 \end{cases}$$

We now turn to $F(D)$. Note:

(27a) $$(D-\kappa)\psi(D) = \frac{(D-\kappa)\delta(D)}{\beta(D)}$$

$$= \frac{(D-\kappa)\delta(D) - \delta_n\beta(D)D - [-\kappa\delta_n+\delta_{n-1}-\delta_n\beta_{n-1}]\beta(D)}{\beta(D)}$$

$$+ \delta_n D + (-\kappa\delta_n+\delta_{n-1}-\delta_n\beta_{n-1}),$$

where

(27b) $\kappa = -\frac{1}{a}(cb-ra)$

Then,

$$(28a)\qquad (D-\kappa)\psi(D) = \begin{cases} \left(\sum_{j=1}^{n} \frac{A_j}{D-\mu_j}\right) + \delta_n D + (-\kappa\delta_n+\delta_{n-1}-\delta_n\beta_{n-1}) & n>0 \\ \delta_o(D-\kappa) & n=0 \end{cases}$$

where

$$(28b)\qquad A_j = \frac{(\mu_j-\kappa)\delta(\mu_j)}{\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j-\mu_k)} \qquad j = 1, \ldots, n \text{ (if } n > 0)$$

$$(28c)\qquad \beta(D) = (D-\mu_1) \ldots (D-\mu_n).$$

Here, we assume that $\mu_i \neq \mu_j$ if $i \neq j$, $\mu_i \neq r - \lambda$, $\mu_i \neq \rho_j$ for all i, j. Finally,

$$(29)\qquad F(D) = \begin{cases} \left[\sum_{j=1}^{n}\left(\frac{A_j}{\lambda-\mu_j}\right)\left(\frac{1}{D-\mu_j}\right)\right] - \delta_n & n>0 \\ -\delta_o & n=0 \end{cases}$$

Substituting (26) and (29) into (21) and rearranging, we get

$$(30a)\qquad \alpha(D)\beta(D)[D-(r-\lambda)]\begin{pmatrix} I(t) \\ s(t) \end{pmatrix} = \tilde{C}(D)\varepsilon(t)$$

where,

$$\tilde{C}_{11}(D) = \left(\frac{1}{b+A}\right)\left\{\left[\sum_{j=1}^{n} \frac{A_j}{\lambda-\mu_j} \prod_{\substack{k=1\\k\neq j}}^{n} (D-\mu_k)\right] - \delta_n \beta(D)\right\}\alpha(D)$$

$$\tilde{C}_{12}(D) = -\frac{a+b+A}{a(b+A)}\,\beta(D)\left\{\sum_{j=1}^{m}\frac{B_j}{\lambda-\rho_j}\prod_{\substack{k=1\\k\neq j}}^{m}(D-\rho_k)-\gamma_{m+1}\alpha(D)\right\}$$

(30b)

$$\tilde{C}_{21}(D) = \frac{-a}{a+b+A}\left(D-\frac{bc}{a}\right)\tilde{C}_{11}(D) + \frac{1}{a+b+A}\,\delta(D)\alpha(D)[D-(r-\lambda)]$$

$$\tilde{C}_{22}(D) = \left(\frac{-a}{a+b+A}\right)\left(D-\frac{bc}{a}\right)\tilde{C}_{12}(D)$$

and

$$\tilde{C}(D) = \begin{bmatrix} \tilde{C}_{11}(D) & \tilde{C}_{12}(D) \\ \tilde{C}_{21}(D) & \tilde{C}_{22}(D) \end{bmatrix}. \tag{30c}$$

Here,

$$\tilde{C}(D) = \tilde{C}_o + \tilde{C}_1 D + \dots + \tilde{C}_{n+m+1} D^{n+m+1}, \tag{30d}$$

where

$$\tilde{C}_{n+m+1} = \begin{bmatrix} 0 & 0 \\ \dfrac{\delta_n}{b+A} & 0 \end{bmatrix}. \tag{30e}$$

Also,

$$\det \tilde{C}(D) = [D-(r-\lambda)]\,\delta(D)\alpha(D)\beta(D)\frac{1}{a(b+A)}\sum_{j=1}^{m}\frac{B_j}{\lambda-\rho_j}\prod_{\substack{k=1\\k\neq j}}^{m}(D-\rho_k). \tag{30f}$$

Define

$$(31)\qquad \begin{aligned} R_Y(\tau) &= EY(t)Y(t-\tau)^T \qquad\qquad \tau\varepsilon(-\infty,+\infty). \\ S_Y(s) &= \int_{-\infty}^{+\infty} R_Y(\tau)e^{-\tau s}d\tau, \end{aligned}$$

where $s = i\omega$, $\omega\varepsilon(-\infty,+\infty)$. It is well known that

$$(32a)\qquad S_Y(s) = \frac{\tilde{C}(s)V\tilde{C}(-s)^T}{\theta(s)\theta(-s)} ,$$

where

$$(32b)\qquad \begin{aligned} \theta(s) &= \alpha(s)\beta(s)[s-(r-\lambda)] \\ &= \prod_{j=1}^{m} (s-\rho_j) \prod_{j=1}^{n} (s-\mu_j)[s-(r-\lambda)], \end{aligned}$$

if $m, n > 0$. (The modification to (32b) for the case $n = 0$ or $m = 0$ is obvious.)

The continuous time spectral density of $Y(t)$ at frequency $\omega\varepsilon(-\infty,+\infty)$ is $S_Y(i\omega)$. If $\delta_n \neq 0$, so that $s(t)$ is a generalized stochastic process, then $R_Y(\tau)$, $\tau\varepsilon(-\infty,+\infty)$, defined by (31), is a generalized function.

The partial fractions expansion of $S_Y(s)$ is

$$(32a)\qquad \begin{aligned} S_Y(s) &= \frac{\tilde{C}(s)V\tilde{C}(-s)^T}{\theta(s)\theta(-s)} \\ &= \frac{\tilde{C}(s)V\tilde{C}(-s)^T - \tilde{C}_\ell V\tilde{C}_\ell^T\theta(s)\theta(-s)^T}{\theta(s)\theta(-s)} + \tilde{C}_\ell V\tilde{C}_\ell^T \\ &= \sum_{j=1}^{\ell} \frac{W_j}{s-r_j} + \sum_{j=1}^{\ell} \frac{W_j^T}{-s-r_j} + \tilde{C}_\ell V\tilde{C}_\ell^T, \end{aligned}$$

where

$$\text{(32b)} \qquad W_j = \frac{\tilde{C}(r_j)V\tilde{C}(-r_j)^T}{-2r_j \prod\limits_{\substack{k=1 \\ k\neq j}}^{\ell} (r_j-r_k)(-r_j-r_k)}, \qquad j = 1, \ldots, \ell$$

$$\text{(32c)} \qquad \begin{aligned} &\ell = m + n + 1 \\ &(r_1,\ldots,r_\ell) \equiv (\rho_1,\ldots,\rho_m,\mu_1,\ldots\mu_n,r-\lambda). \end{aligned}$$

2. <u>The Spectral Density of Unit Sampled $\{\overline{Y}(t)\}$</u>

Our first step is to obtain an expression for $S_{\overline{Y}}(i\omega)$, the spectral density of the continuous time $\{\overline{Y}(t)\}$ process at frequency $\omega\varepsilon(-\infty,+\infty)$. We then deduce the covariance function of $\overline{Y}(t)$ from $S_{\overline{Y}}$ and use it to compute the spectral density of sampled $\overline{Y}(t)$. The latter is denoted by $S^d_{\overline{Y}}(z)$ for $z=e^{-i\omega}$, $\omega\varepsilon(-\pi,\pi)$.

In operator notation, the link between $\overline{Y}(t)$ and $Y(t)$ is the following:

$$\text{(33)} \qquad \overline{Y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{e^D-1}{D} \end{bmatrix} Y(t).$$

Consequently,

$$\text{(34)} \qquad S_{\overline{Y}}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{e^s-1}{s} \end{bmatrix} S_Y(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{e^{-s}-1}{-s} \end{bmatrix}$$

$$= \begin{bmatrix} S^{11}_{\overline{Y}}(s) & S^{12}_{\overline{Y}}(s) \\ S^{21}_{\overline{Y}}(s) & S^{22}_{\overline{Y}}(s) \end{bmatrix}.$$

Similarly, define $S_Y = [S_Y^{ij}]$. Then, from (34)

(35)
$$S_{\bar{Y}}^{11}(s) = S_Y^{11}(s)$$

$$S_{\bar{Y}}^{12}(s) = S_Y^{12}(s)\left(\frac{e^{-s}-1}{-s}\right)$$

$$S_{\bar{Y}}^{21}(s) = S_{\bar{Y}}^{12}(-s)$$

$$S_{\bar{Y}}^{22}(s) = \left(\frac{e^{-s}-1}{-s}\right)\left(\frac{e^{s}-1}{s}\right) S_Y^{22}(s).$$

We seek now to recover $R_{\bar{y}}$ from $S_{\bar{y}}$. In doing so, we make use of the fact that $R_{\bar{y}}$ and $S_{\bar{Y}}$ are related as follows:

$$S_{\bar{Y}}(s) = \int_{-\infty}^{+\infty} R_{\bar{Y}}(\tau)e^{-s\tau}d\tau, \tag{36}$$

where $s = i\omega$, $\omega\varepsilon(-\infty,+\infty)$. Write $R_{\bar{Y}} = [\bar{R}_{ij}]$, $S_{\bar{Y}} = [S_{\bar{Y}}^{ij}]$, $S_Y = [S_Y^{ij}]$, $W_k = [W_k^{ij}]$; $k = 1, \ldots, \ell$. Then,

$$S_{\bar{Y}}^{22}(s) = \left(\frac{e^{s}-1}{s}\right)\left(\frac{e^{-s}-1}{-s}\right)\left\{\sum_{j=}^{\ell}\frac{W_j^{22}}{s-r_j} + \sum_{j+1}^{\ell}\frac{W_j^{22}}{-s-r_j}\right\} \tag{37}$$

$$+ \left(\frac{e^{s}-1}{s}\right)\left(\frac{e^{-s}-1}{-s}\right)\left[\frac{\delta_n}{b+A}\right]^2 V_{11}$$

$$= \int_{-\infty}^{+\infty} R_{\bar{Y}}^{22}(\tau)e^{-\tau s}d\tau,$$

for $s = i\omega$. The R_Y^{22} function that solves (37) is

$$
(38) \qquad R_Y^{22}(\tau) = \begin{cases}
\sum_{j=1}^{\ell} \dfrac{(e^{-r_j}-1)(e^{r_j}-1)}{-r_j^2} W_j^{22} e^{r_j \tau} & \tau < 1. \\[2em]
(1-\tau)\left[\dfrac{\delta_n}{b+A}\right]^2 V_{11} + & \\
\sum_{j=1}^{\ell} \dfrac{W_j^2}{r_j^2}\{2r_j(\tau-1)-(2-e^{r_j})e^{r_j\tau}+e^{(1-\tau)r_j}\} & 0 < \tau < 1 \\[2em]
R_Y(-\tau) & \tau < 0.
\end{cases}
$$

That (38) solves (37) may be verified by checking that (38) satisfies (37) and taking into account the fact that the inverse Fourier transform is unique. Next,

$$
S_Y^{12}(s) = \left(\frac{e^{-s}-1}{-s}\right)\left\{\sum_{j=1}^{\ell} \frac{W_j^{12}}{s-r_j} + \sum_{j=1}^{\ell} \frac{W_j^{21}}{-s-r_j}\right\}
$$

$$
= \int_{-\infty}^{+\infty} R_Y^{12}(\tau)e^{-s\tau}d\tau
$$

$$
S_Y^{21}(s) = \left(\frac{e^{s}-1}{s}\right)\left\{\sum_{j=1}^{\ell} \frac{W_j^{21}}{s-r_j} + \sum_{j=1}^{\ell} \frac{W_j^{12}}{-s-r_j}\right\}
$$

$$
= \int_{-\infty}^{+\infty} R_Y^{21}(\tau)e^{-s\tau}d\tau,
$$

for $s = i\omega$. The unique $R_{\bar{Y}}^{12}$ and $R_{\bar{Y}}^{21}$ functions that solve this are

$$(39) \qquad R_{\bar{Y}}^{12}(\tau) = \begin{cases} \sum_{j=1}^{\ell} W_j^{12}\left(\frac{1-e^{-r_j}}{r_j}\right)e^{\tau r_j} & \tau > 1 \\ \sum_{j=1}^{\ell} \frac{1}{r_j}\{W_j^{12}(e^{\tau r_j}-1)+W_j^{21}(e^{(1-\tau)r_j}-1)\} & 0 < \tau < 1 \\ R_{\bar{Y}}^{21}(-\tau) & \tau < 0 \end{cases}$$

$$(40) \qquad R_{\bar{Y}}^{21}(\tau) = \begin{cases} \sum_{j=1}^{\ell} W_j^{21} \frac{1}{r_j}(e^{r_j}-1)e^{r_j\tau} & \tau > 0 \\ R_{\bar{Y}}^{12}(-\tau) & \tau < 0. \end{cases}$$

We summarize our results about $R_{\bar{Y}}$ as follows:

$$(41a) \qquad R_{\bar{Y}}(0) = \left(\sum_{j=1}^{\ell} \tilde{W}_j\right) + \tilde{C}_\ell V \tilde{C}_\ell^T$$

$$(41b) \qquad R_{\bar{Y}}(\tau) = \sum_{j=1}^{\ell} \overline{W}_j e^{\tau r_j} \qquad \tau = 1, 2, \ldots$$

where

$$(41c) \qquad \tilde{W}_j = \begin{bmatrix} W_j^{11} & W_j^{21} \frac{1}{r_j}(e^{r_j}-1) \\ W_j^{21} \frac{1}{r_j}(e^{r_j}-1) & W_j^{22} \frac{2}{-r_j^2}(r_j+1-e^{r_j}) \end{bmatrix}$$

$$(41d) \qquad \overline{W}_j = \begin{bmatrix} W_j^{11} & W_j^{12} \frac{1}{-r_j}(e^{-r_j}-1) \\ W_j^{21} \frac{1}{r_j}(e^{r_j}-1) & W_j^{22} \frac{1}{-r_j^2}(e^{-r_j}-1)(e^{r_j}-1) \end{bmatrix},$$

for $j = 1, \ldots, \ell$.

Now, the spectral density of $\{\overline{Y}(t), t=1,\pm1,\pm2,\ldots\}$ is defined by

$$(42) \qquad S_{\overline{Y}}^{d}(z) = \sum_{\tau=-\infty}^{+\infty} R_{\overline{Y}}(\tau) z^{\tau}$$

for $z = e^{-i\omega}$, $\omega\varepsilon(-\pi,\pi)$. Substituting (41) into (42), get

$$(43a) \qquad S_{\overline{Y}}^{d}(z) = M(z) + M(z^{-1})^{T} - K,$$

where

$$(43b) \qquad M(z) = \sum_{j=1}^{\ell} \frac{\overline{W}_j}{1-e^{r_j}z}$$

$$(43c) \qquad K = \sum_{j=1}^{\ell} (\overline{W}_j + \overline{W}_j^{T}) - R_{\overline{Y}}(0).$$

Equation (43) represents a simple formula for obtaining $S_{\overline{Y}}^{d}$ from V-defined after (18)--and the parameters of C(D), defined in (21). First, one obtains the A's, B's, ρ's, and μ's defined in (25) and (28). These are then used to compute the W's and r's in (32). Then the W's are modified to get the $\tilde{W}$'s and $\overline{W}$'s. With these objects in hand, the $S_{\overline{Y}}^{d}(e^{-i\omega})$ function can be evaluated at any desired frequency $\omega\varepsilon(-\pi,\pi)$.

3. The Frequency Domain Approximation to The Likelihood Function

We assume that T observations, $\overline{Y}(1), \ldots, \overline{Y}(T)$ are available. Up to an additive constant, the frequency domain approximation to the log of the Gaussian likelihood function of this sample is given by

$$(44) \qquad \mathcal{L} = -\frac{1}{2}\sum_{j=1}^{T} \log \det [S_{\overline{Y}}^{d}(e^{-i\omega_j})]$$

$$-\frac{1}{2}\sum_{j=1}^{T} \operatorname{trace} [S_{\overline{Y}}^{d}(e^{-i\omega_j})^{-1} I(\omega_j)],$$

$\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, T$. The computation of $S_{\overline{Y}}^{d}$ was described in Section 2 above. The expression $I(\omega)$ is a function of the data only.

$$(45a) \qquad I(\omega) = \frac{1}{T}\,\overline{Y}(\omega)\overline{Y}(\omega)^{H},$$

where H denotes the Hermetian transform and

$$(45b) \qquad \overline{Y}(\omega) = \sum_{t=1}^{T} \overline{Y}(t)e^{-\omega it}.$$

The expression in (44) is maximized over the vector of free parameters:

$$\zeta = (a,b,c,e,A,r,\delta_o,\ldots,\delta_n,\beta_o,\ldots,\beta_{n-1},$$

$$\gamma_o,\ldots,\gamma_{m+1},\alpha_o,\ldots,\alpha_{m-1},V_{11},V_{22},V_{12}).$$

Given a value for ζ, one computes $\mathcal{L}$ in the following sequence of steps:

Step 1: Compute λ from r, a, b, A, c, e using (11) and (12).

Step 2: Compute $\rho_1, \ldots, \rho_m$ from $\alpha_0, \ldots, \alpha_{m-1}$ using (24a).

Step 3: Compute $B_1, \ldots, B_m$ from $\gamma_0, \ldots, \gamma_{m+1}$ and $\rho_1, \ldots, \rho_m$ using (25b).

Step 4: Compute $\mu_1, \ldots, \mu_n$ from $\beta_0, \ldots, \beta_{n-1}$ using (28c).

Step 5: Compute $A_1, \ldots, A_n$ from a, c, b, r, $\mu_1, \ldots, \mu_n$, $\delta_0, \ldots, \delta_n$ using (28b).

Step 6: Compute $W_1, \ldots, W_\ell$ from $(r_1,\ldots,r_\ell) = (\rho_1,\ldots,\rho_m, \mu_1,\ldots,\mu_n, r-\lambda)$, $\ell = m + n + 1$, ζ, using (30) and (32b).

Step 7: Compute $\tilde{W}_j$, $\overline{W}_j$, $j = 1, \ldots, \ell$ from $W_1, \ldots, W_\ell$, $r_1, \ldots, r_\ell$ using (41c) and (41d).

Step 8: Evaluate $S^d_{\overline{Y}}(e^{-i\omega_k})$ at $\omega_k = \frac{2\pi k}{T}$, $k = 1, \ldots, T$, from $\tilde{W}_j$, $\overline{W}_j$, r_j, $j = 1, \ldots, \ell$, δ_n, V_{11}, using (30e), (41a), and (43).

Step 9: Substitute $S^d_{\overline{Y}}$ into (44). Also use I computed from (45a). In evaluating (44), one would want to exploit the symmetry properties of $S^d_{\overline{Y}}$ and I. In addition, if $I(0) \equiv 0$, then frequency zero should be omitted in order to avoid $\mathcal{L}$ being unbounded above.

E. The Sampled Representation of $\{\overline{Y}(t)\}$

This section describes algorithms for computing two kinds of time series representation for $\{\overline{Y}(t), t=0,\pm1,\pm2,\ldots3\}$. The first produces a set (K^+,W^+,Ω^+) that satisfy

$$(46a) \qquad S^d_{\overline{Y}}(z) = K^+(z)^{-1}W^+(z)\Omega^+W^+(z^{-1})^T[K^+(z^{-1})^{-1}]^T$$

$$(46b) \qquad \det K^+(z) = 0 \text{ implies } |z| > 1$$

(46c) $\quad \det W^{+}(z) = 0$ implies $|z| > 1$

(46d) $\quad K^{+}(z) = I + K_1^{+}z + \ldots + K_p^{+}z^{p}$

(46e) $\quad W^{+}(z) = I + W_1^{+}z + \ldots + W_q^{+}z^{q}.$

Here, $K^{+}(z)$ and $W^{+}(z)$ are 2 x 2 matrix polynomials in z and Ω^{+} is a 2 x 2 positive semidefinite matrix. The matrix polynomial $S_{\overline{Y}}^{d}(z)$ is defined in (43). Conditions under which a (K^{+},W^{+},Ω^{+}) exists and is unique are discussed in Hannan [1969].

This section also presents an algorithm for obtaining $(\theta^{+},\tilde{C}^{+},V^{+})$ where

(47a) $$S_{\overline{Y}}^{d}(z) = \frac{\tilde{C}(z)V^{+}\ \tilde{C}^{+}(z^{-1})^{T}}{\theta^{+}(z)\ \theta^{+}(z^{-1})}$$

(47b) $\quad \theta^{+}(z) = 0$ implies $|z| > 1$

(47c) $\quad \det \tilde{C}^{+}(z) = 0$ implies $|z| > 1$

(47d) $\quad \theta^{+}(z) = 1 + \theta_1^{+}z + \ldots + \theta_p^{+}z^{p}$

(47e) $\quad \tilde{C}^{+}(z) = I + \tilde{C}_1^{+}z + \ldots + \tilde{C}_q^{+}z^{q}.$

Here, $\theta^{+}(z)$ is a scalar polynomial in z and $\tilde{C}^{+}(z)$ is a 2 x 2 matrix polynomial in z, while V^{+} is positive semidefinite. Hannan [1970, Chapter 3] discusses the existence and uniqueness of $(\theta^{+},\tilde{C}^{+},V^{+})$. Sufficient conditions are that $S_{\overline{Y}}(i\omega)$ is a positive matrix for almost all $\omega \in (-\infty,+\infty)$.

The first part of this section describes the calculation of (K^{+},W^{+},Ω^{+}). We then consider the calculation of $(\theta^{+},\tilde{C}^{+},V^{+})$.

1. <u>Calculating (K^+, W^+, Ω^+)</u>

Below we show how to compute (K^+, W^+, Ω^+). The strategy involves first obtaining K^+ using a technique suggested by Phillips [1959]. We then find W^+ and Ω^+ using methods described in Whittle [1983] or Rozanov [1963].

Write

$$(48) \qquad M(z) = U(z)^{-1}V(z),$$

where $U(z)$ and $V(z)$ are (as yet unknown) square, finite-ordered polynomial matrices in non-negative powers of z. The expression $M(z)$ is defined in (43b). Substituting from (43b) into (48), and premultiplying by $U(z)$, get

$$(49) \qquad U(z) \sum_{j=1}^{\ell} \frac{\bar{W}_j}{1 - \tilde{r}_j z} = V(z),$$

where $\tilde{r}_j = e^{r_j}$, $j = 1, \ldots, \ell$. Multiply both sides of (49) by the scalar polynomial $(1-\tilde{r}_k z)$:

$$U(z)\{\bar{W}_k + (1-\tilde{r}_k z) \sum_{\substack{j=1 \\ j \neq k}}^{3} \frac{\bar{W}_j}{1 - \tilde{r}_j z}\} = (1-\tilde{r}_k z)V(z),$$

$k = 1, \ldots, \ell$. Evaluating this expression at $z = \tilde{r}_k^{-1}$ for each k gives

$$(50) \qquad U(\tilde{r}_k^{-1})\bar{W}_k = 0,$$

$k = 1, \ldots, \ell$. Using the result in (30f), note that $\bar{W}_j$ has rank 1 for all j. (Note: $\det \tilde{C}(r_k) = 0$, $k = 1, \ldots, \ell$.)

Expression (50) for $k = 1, \ldots, \ell$ represents at most 2ℓ independent equations. Consequently, we can determine as many as 2ℓ parameters for $U(z)$. Write

$$(52) \qquad U(z) = I + U_1 z + \ldots + U_c z^c$$

Here, $c = \frac{1}{2}\ell$ if ℓ is even and $c = \frac{1}{2}(\ell+1)$ if ℓ is odd. Consider the even case first. In this case, there are 2ℓ unknown parameters in $U(z)$, so that a necessary condition for (50) to determine $U(z)$ is satisfied. Suppose now that ℓ is odd. In this case, $U(z)$ contains $4c = 2\ell + 2$ parameters--two more than we can hope to solve for using (50). If, for example, we arbitrarily set the left column of U_c to zero, then a necessary condition for (50) to determine the remaining parameter of $U(z)$ is satisfied. In the remainder of this section, we assume that sufficient conditions for (50) to uniquely identify $U(z)$ are satisfied. In this case, (50) represents a set of linear equations in the unknown elements of $U(z)$ and can readily be solved.

With the matrix $U(z)$ in hand, $V(z)$ in (49) is easy to compute. Write

$$(54) \qquad V(z) = \sum_{j=1}^{\ell} \frac{U(z)\bar{W}_j}{1 - \tilde{r}_j z}.$$

The fact that $U(\tilde{r}_j^{-1})\bar{W}_j$ is the zero matrix indicates that the scalar polynomial $1 - \tilde{r}_j z$ cancels with every element of $U(z)\bar{W}_j$, for $j = 1, \ldots, \ell$. Therefore, $V(z)$ is a matrix polynomial of order $c - 1$. Taking this into account and expanding the right hand side in (54), get

$$V_0 + V_1 z + \ldots + V_{c-1} z^{c-1} = (I+U_1 z+\ldots+U_c z)[\sum_{j=1}^{\ell} \overline{W}_j \sum_{k=0}^{\infty} (\tilde{r}_j z)^k].$$

Matching coefficients:

$$(55) \qquad V_k = \sum_{s=0}^{k} U_s \sum_{j=1}^{\ell} \overline{W}_j (\tilde{r}_j)^{(k-s)}, \qquad k = 0, 1, \ldots, c-1$$

where $U_0 = I$. Using the given $U(z)$ and $V(z)$ matrices, substitute from (48) into (43a) to get

$$(56) \qquad S_Y^d(z) = U(z)^{-1}V(z) + V(z^{-1})^T[U(z^{-1})^{-1}]^T - K.$$

Pre- and post-multiplying by $U(z)$ and $U(z^{-1})^T$ respectively get

$$(57) \qquad U(z)S_Y^d(z)U(z^{-1})^T = V(z)U(z^{-1})^T + U(z)V(z^{-1})^T - U(z)KU(z^{-1})^T$$

$$= G(z).$$

For example, when $\ell = 3$ so that $c = 2$,

$$(58a) \qquad G(z) = G_0 + G_1 z + G_2 z^2 + G_1^T z^{-1} + G_2^T z^{-2},$$

where

$$G_0 = V_0 + V_1 U_1^T + V_0^T + U_1 V_1^T - [K+U_1 K U_1^T + U_2 K U_2^T]$$

$$(58b) \qquad G_1 = V_1 + U_1 V_0^T + U_2 V_1^T - [U_1 K + U_2 K U_1^T]$$

$$G_2 = U_2 \sum_{j=1}^{3} (\tilde{W}_j - \overline{W}_j).$$

Obviously, we can identify $K^+(z) \equiv U(z)$. The objects $W^+(z)$ and Ω^+ are then found as the solution to the following matrix factorization problem:

$$(59) \qquad W^+(z)\Omega^+ W^+(z^{-1})^T = G(z).$$

It is easily confirmed that $S_{\overline{Y}}(i\omega)$ positive for almost all $\omega\varepsilon(-\infty,+\infty)$ guarantees that $G(e^{-i\omega})$ is positive for almost all $\omega\varepsilon(-\pi,\pi)$. Results in Hannan [1970, Chapter 3] then guarantee that the factorization in (59), together with the conditions (46c) and (46e) with $q = c$, is unique. Algorithms described by Whittle [1983] or Rozanov [1963] may be used to solve (59). (Whittle rules out det $G(1) = 0$.)

Consider the case $n = m = \gamma_0 = \delta_0 = 1$, $\gamma_1 = \gamma_2 = \delta_1 = 0$, so that $\ell = 3$ and $c = 2$. This case, because $p = q = 2$ in (46), neatly illustrates some effects of sampling and averaging from a continuous time process. In the present case, (30f) becomes

$$\det \tilde{C}(D) = \frac{\alpha(D)\beta(D)[D-(r-\lambda)]}{(\lambda-\rho_1)a(b+A)}$$

Therefore, premultiplying (30a) by $\tilde{C}(D)^{-1} = \tilde{C}(D)^a/\det \tilde{C}(D)$, we get

$$(60)\qquad (\lambda-\rho_1)a(b+A)\tilde{C}(D)^a\, Y(t) = \varepsilon(t).$$

Here, $\tilde{C}(D)^a$ denotes the adjoint matrix of $\tilde{C}(D)$. Thus, in this case, $\{Y(t)\}$ is a pure vector second order autoregression in continuous time. Sampled and averaged $\{Y(t)\}$, denoted by $\{\overline{Y}(t), t=0,\pm1,\pm2,...\}$ on the other hand, is a discrete time ARMA(2,2) process. One moving average term is due to sampling and the other to averaging.

The following numerical example illustrates the points made above. We assume the parameterization studied in section F, in which $n = m = \gamma_0 = \delta_0 = 1$, $\gamma_1 = \gamma_2 = \delta_1 = 0$, so that (60) is the relevant representation. The following continuous time parameterization was chosen:

$$[I+A_1D+A_2D^2]Y(t) = \varepsilon(t),$$

where

$$A_1 = \begin{bmatrix} 11.23 & 5.57 \\ .286 & 2.0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 13.906 & 0 \\ .571 & 0 \end{bmatrix}$$

$$E\varepsilon(t)\varepsilon(t)^T = \begin{bmatrix} 9 & 18 \\ 18 & 17 \end{bmatrix}.$$

Also, det $(I+A_1s+A_2s^2) = 0$ implies $s = -.1, -.5$, or $-.812$.

Using the calculations described in this section, the representation for $\{\bar{Y}(t), t=0,\pm1,\pm2,...\}$ is:

$$(I+K_1^+L+K_2^+L^2)\bar{Y}(t) = (I+W_1^+L+W_2^+L^2)u(t)$$

where

$$K_1^+ = \begin{bmatrix} -.814 & -8.776 \\ -.003 & -1.141 \end{bmatrix}, \qquad K_2^+ = \begin{bmatrix} 0 & 4.931 \\ 0 & .318 \end{bmatrix}$$

$$W_1^+ = \begin{bmatrix} .209 & -9.035 \\ 1.942 & -.290 \end{bmatrix}, \qquad W_2^+ = \begin{bmatrix} -17.25 & -2.620 \\ -1.113 & -.169 \end{bmatrix}$$

$$Eu(t)u(t)^T = \Omega^+ = \begin{bmatrix} 16.870 & -111.068 \\ -111.068 & 7884.57 \end{bmatrix}.$$

Also, the roots of det $K^+(z)$ are 1.105, 1.649, 2.252 and the roots of det $W^+(z)$ are $.641 \pm 3.935i$.

This example illustrates some of the comments on identification made above. In particular, the sampled VARMA (2,2) representation for $\{\bar{Y}(t)\}$ is not identified without the indicated zero restrictions on K_2^+. Without these restrictions (but imposing $K_0^+ \equiv W_0^+ \equiv I$) there is a two dimensional infinity of ways of choosing

K_i^+, W_i^+, $i = 1, 2$ which satisfy (46). This can be seen by noting that the null spaces of K_2^+ and W_2^+ have a non-empty intersection and then applying the argument in Hannan [1969 p. 224, second to last paragraph].

2. <u>Calculating $(\theta^+, \tilde{C}^+, V^+)$</u>

Obtaining $(\theta^+, \tilde{C}^+, V^+)$ is straightforward. One method of doing this is to execute the calculations just described and set $\theta^+(z) = \det K^+(z)$, $\tilde{C}^+(z) = [\text{adjoint}(K^+(z))]W^+(z)$. This (indirect) procedure suffers from the shortcoming that sufficient conditions for the existence and uniqueness of $U(z)$ in (52) are difficult to establish, even though we know that $S_{\bar{y}}(i\omega)$ is positive for almost all $\omega \in (-\infty, +\infty)$. The latter is sufficient for the calculations described below to work.

Define

$$\theta^+(z) = \prod_{j=1}^{\ell} (1-\tilde{r}_j z) \tag{61}$$

Multiplying $S_{\bar{Y}}^d$ in (43a) by $\theta^+(z)\theta^+(z^{-1})$, get

$$\theta^+(z)S_{\bar{Y}}^d(z)\theta^+(z^{-1}) = \theta^+(z^{-1}) \sum_{j=1}^{\ell} \bar{W}_j \prod_{\substack{k=1 \\ k\neq j}}^{\ell} (1-\tilde{r}_k z) + \theta^+(z) \sum_{j=1}^{\ell} \bar{W}_j^T \prod_{\substack{k=1 \\ k\neq 1}}^{\ell} (1-\tilde{r}_k z^{-1}) - \theta^+(z)K\theta^+(z^{-1}) \tag{62}$$

$$= G(z),$$

say, where $G(z) = G(z^{-1})^T$. The matrix polynomial $G(z)$ will be of order ℓ $\tilde{C}^+$ V^+ are found with the properties

$$(63) \qquad \tilde{C}^+(z)V^+\tilde{C}^+(z^{-1})^T = G(z),$$

(47c), and (47e). A sufficient condition for the solution to (63) to exist and be unique is that $S_y(i\omega)$ be positive for almost all $s\varepsilon(-\infty,+\infty)$.

F. Parameter Identification and Estimation With AR(1) Disturbances

In this section we assume $n = m = \gamma_0 = \delta_0 = 1$, $\gamma_1 = \gamma_2 = \delta_1 = 0$. In this case, (30) becomes

$$(64a) \qquad \theta(D)\binom{I(t)}{s(t)} = [\tilde{C}_0+\tilde{C}_1D+\tilde{C}_2D^2]\varepsilon(t),$$

where $\theta(s) = (s+\alpha_0)(s+\beta_0)[s-(r-\lambda)]$ and

$$(64b) \qquad \tilde{C}_0 = \begin{bmatrix} q_1\alpha_0 & q_2\beta_0 \\ \dfrac{\alpha_0(q_1bc-(r-\lambda))}{a+b+A} & \dfrac{q_2bc\beta_0}{a+b+A} \end{bmatrix}$$

$$(64c) \qquad \tilde{C}_1 = \begin{bmatrix} q_1 & q_2 \\ \dfrac{-aq_1(\alpha_0-\frac{bc}{a})+\alpha_0-(r-\lambda)}{a+b+A} & \dfrac{-aq_2}{a+b+A}[\beta_0-\frac{bc}{a}] \end{bmatrix}$$

$$(64d) \qquad \tilde{C}_2 = \begin{bmatrix} 0 & 0 \\ \dfrac{1-aq_1}{a+b+A} & \dfrac{-aq_2}{a+b+A} \end{bmatrix}.$$

In (64),

$$(64e) \qquad q_1 = \frac{\frac{1}{a}(cb-ra) - \beta_0}{(\lambda+\beta_0)(b+A)} \qquad q_2 = \frac{-(a+b+A)}{a(b+A)(\lambda+\alpha_0)}.$$

Write $\tilde{\varepsilon}(t) = \tilde{C}_0\varepsilon(t)$, $\tilde{\tilde{C}}(D) = \tilde{C}(D)\tilde{C}_0^{-1} = I + \tilde{\tilde{C}}_1 D + \tilde{\tilde{C}}_2 D^2$. Here,

$$(65a) \qquad \tilde{\tilde{C}}_1 = \begin{bmatrix} \dfrac{q_1 bc(\beta_0-\alpha_0) + \alpha_0(r-\lambda)}{\beta_0\alpha_0(r-\lambda)} & \dfrac{q_1(\alpha_0-\beta_0)(a+b+A)}{\beta_0\alpha_0(r-\lambda)} \\ \dfrac{(bc)^2 q_1(\beta_0-\alpha_0) + bc\alpha_0\beta_0(r-\lambda)[bc(\alpha_0-\beta_0)-a\alpha_0\beta_0]}{(a+b+A)\beta_0\alpha_0(r-\lambda)} & \dfrac{q_1 bc(\alpha_0-\beta_0) + \beta_0(r-\lambda-\alpha_0)}{\beta_0\alpha_0(r-\lambda)} \end{bmatrix}$$

$$(65b) \qquad \tilde{\tilde{C}}_2 = \begin{bmatrix} 0 & 0 \\ \dfrac{(1-aq_1)bc\beta_0 + a[q_1 bc-(r-\lambda)]\alpha_0}{(a+b+A)\beta_0\alpha_0(r-\lambda)} & \dfrac{-\beta_0(1-aq_1) - aq_1\alpha_0}{\beta_0\alpha_0(r-\lambda)} \end{bmatrix}$$

Because the upper row of $\tilde{\tilde{C}}_2$ contains zeroes, there are six parameters in $\tilde{\tilde{C}}(D)$. The scalar polynomial $\theta(D)$ supplies three more. Finally, there are three parameters in $\tilde{V} \equiv E\,\tilde{\varepsilon}(t)\tilde{\varepsilon}(t)^T$. Thus, the continuous time Wold representation of $\{Y(t)\}$ has twelve "reduced form" parameters. There is no aliasing identification problem in the present case because the poles of the spectrum of $\{Y(t)\}$ (e.g., $-\alpha_0, -\beta_0, r-\lambda$) are real. Consequently, the necessary and sufficient condition for parameter identification is that the mapping from the structural parameters to θ, $\tilde{\tilde{C}}_1$, $\tilde{\tilde{C}}_2$, $\tilde{V}$ have a unique inverse in the space of admissible structural parameters. The structural parameters are

$$\xi = (a,b,c,e,A,r,\beta_0,\alpha_0,V_{11},V_{12},V_{22}).$$

Since there are 11 of them, a necessary condition for them to be identified is met. Unfortunately, since the reduced form parameters are not independent, identification does not occur.

The restrictions of the model imply that no more than 9 of the 12 reduced form parameters of θ, $\tilde{\tilde{C}}_1$, $\tilde{\tilde{C}}_2$, $\tilde{V}$ are free to vary independently. There are two ways to see this. One is to recall from (60) that in the present case, $\{Y(t)\}$ has a pure vector second order autoregressive representation in continuous time. In that equation, there are at most six identifiable autoregressive parameters and three variance-covariance terms. (Of the eight autoregressive parameters, two are zero.) An alternative way to see this is to study (65). Simple algebra yields:

$$\text{(66a)} \qquad q_1 bc = \frac{[\tilde{\tilde{C}}_1^{11}\beta_0-1]\,\alpha_0(r-\lambda)}{(\beta_0-\alpha_0)}$$

$$\text{(66b)} \qquad q_1(a+b+A) = \frac{\tilde{\tilde{C}}_1^{12}\beta_0\alpha_0(r-\lambda)}{\alpha_0-\beta_0}$$

$$\text{(66c)} \qquad aq_1 = \frac{\beta_0[\alpha_0(r-\lambda)\tilde{\tilde{C}}_2^{22}+1]}{\beta_0-\alpha_0}.$$

Dividing (66c) into (66a) and (66b) respectively yields,

$$\text{(67a)} \qquad \frac{bc}{a} = \frac{[\tilde{\tilde{C}}_1^{11}\beta_0-1]\,\alpha_0(r-\lambda)}{\beta_0[\alpha_0(r-\lambda)\tilde{\tilde{C}}_2^{22}+1]}$$

$$\text{(67b)} \qquad \frac{a + b + A}{a} = \frac{-\tilde{\tilde{C}}_1^{12}\beta_0\alpha_0(r-\lambda)}{\beta_0[\alpha_0(r-\lambda)\tilde{\tilde{C}}_2^{22}+1]}.$$

According to (64e),

$$\text{(67c)} \qquad r = \frac{\frac{cb}{a} - \beta_0 + (r-\lambda-\beta_0)(\frac{a+b+A}{a} - 1)q_1 a}{(\frac{a+b+A}{a} - 1)q_1 a + 1}.$$

By (66c) and (67a-b), the parameters r, $\frac{bc}{a}$, $\frac{a+b+A}{a}$ can be recovered uniquely from

$$\tilde{\tilde{C}}_2^{22},\ \tilde{\tilde{C}}_1^{11},\ \tilde{\tilde{C}}_1^{12},\ \alpha_0,\ \beta_0,\ (r-\lambda).$$

We maintain that $\tilde{\tilde{C}}_2^{21}$ and the bottom row of $\tilde{\tilde{C}}_1$ are determined from θ and the remaining elements of $\tilde{\tilde{C}}_1$ and $\tilde{\tilde{C}}_2$. This is seen as follows:

$$\text{(68a)} \qquad \tilde{\tilde{C}}_1^{22} = \frac{(\alpha_0+\beta_0)(r-\lambda) - \beta_0\alpha_0}{\beta_0\alpha_0(r-\lambda)} - \tilde{\tilde{C}}_1^{11},$$

$$\text{(68b)} \qquad \tilde{\tilde{C}}_1^{21} = \frac{(\frac{bc}{a})bcq_1(\beta_0-\alpha_0) + (\frac{bc}{a})\alpha_0\beta_0 + (r-\lambda)[(\frac{bc}{a})(\alpha_0-\beta_0)-\alpha_0\beta_0]}{(\frac{a+b+A}{a})\beta_0\alpha_0(r-\lambda)}$$

$$= \frac{(\frac{bc}{a})bcq_1(\beta_0-\alpha_0) - (r-\lambda)\alpha_0\beta_0 + (\frac{bc}{a})[\beta_0(\alpha_0-(r-\lambda)) + \alpha_0(r-\lambda)]}{(\frac{a+b+A}{a})\ \beta_0\alpha_0(r-\lambda)}$$

$$\text{(68c)} \qquad \tilde{\tilde{C}}_2^{21} = \frac{(1-aq_1)(\frac{bc}{a})\beta_0 + [q_1 bc-(r-\lambda)]\alpha_0}{(\frac{a+b+A}{a})\beta_0\alpha_0(r-\lambda)}$$

$$= \frac{(\frac{bc}{a})\beta_0 + q_1 bc(\alpha_0-\beta_0) - (r-\lambda)\alpha_0}{(\frac{a+b+A}{a})\beta_0\alpha_0(r-\lambda)}.$$

Equations (68b) and (68c) are obtained by dividing the numerator and denominator terms of the corresponding elements in (65) by a. Note that everything in the right hand sides of (68) can be obtained from the roots of θ, equations (66) and (67), and from $\tilde{\tilde{C}}_1^{11}$, $\tilde{\tilde{C}}_1^{12}$, $\tilde{\tilde{C}}_2^{22}$.

Thus, the twelve elements of θ, $\tilde{\tilde{C}}$, $\tilde{\tilde{V}}$ are spanned by the following nine element vector:

$$(69)\qquad (\alpha_0,\beta_0,\lambda-r,\frac{bc}{a},\frac{a+b+A}{a},r,\tilde{\tilde{V}}_{11},\tilde{\tilde{V}}_{12},\tilde{\tilde{V}}_{22}).$$

This vector is restricted by the conditions that all its elements, except perhaps V_{12}, be positive. The restriction on V_{12} is $V_{11}V_{22} - V_{12}^2 > 0$. The implications for the vector in (69) are as follows. First, (12b) and the line thereafter show that $\lambda > 0$, $r - \lambda < 0$. Also, r, $(bc/a) > 0$, $(a+b+A)/a > 1$. Finally, $[\tilde{\tilde{V}}_{ij}]$ is positive semi-definite. Consider the vector Γ:

$$(70)\qquad \Gamma \equiv (\alpha_0,\beta_0,\lambda-r,\frac{bc}{a},\frac{a+b+A}{a},r,L_{11},L_{22},L_{12}),$$

where the first seven elements are restricted to be non-negative and $\tilde{\tilde{V}}_{11} = L_{11}^2$, $\tilde{\tilde{V}}_{12} = L_{21}L_{11}$, $\tilde{\tilde{V}}_{22} = L_{21}^2 + L_{22}^2$. The preceding discussion shows that if α_0, β_0, $\lambda - r$ can be identified, then the whole of Γ is identified. Unfortunately, identifiability of Γ is tenuous when we drop the assumption that α_0, β_0, $\lambda - r$ are identified. The basic difficulty is that there are six ways to allocate the labels "$-\alpha_0$", "$-\beta_0$," and "$\lambda - r$" to the roots of $\theta(s)$. This in turn creates the possibility that corresponding to a given Γ, there are five other points in R^9 which give rise to the same values for $\theta(s)$, $\tilde{\tilde{C}}_1$, $\tilde{\tilde{C}}_2$, $\tilde{\tilde{V}}$ as does Γ. These five points differ from Γ in their implied values for α_0, β_0, $\lambda - r$, $\frac{bc}{a}$, $\frac{a + b + A}{a}$, r. Conditional on the assigned values of α_0, β_0, and $\lambda - r$, the latter three are uniquely chosen to satisfy (66c) and (67). If parameter identification is to obtain at Γ, there must be some way to rule out all five alternatives to Γ. It may be shown that the restrictions implied by (68) are of no help in this respect. On the other hand, the restrictions $(bc/a) > 0$, $\frac{a + b + A}{a} > 1$, and $r > 0$, not "too" large, may do the job. Consider the following example:

$$\Gamma = (.00840,.00761,6.06,102.,44.2,.01,L_{11},L_{22},L_{12})$$

Values have not been assigned to L_{ij}, i, $j = 1, 2$ since these are not relevant to the discussion. The given parameters imply

$$\tilde{\tilde{C}}_1 = \begin{bmatrix} 211.522 & -34.8344 \\ 212.616 & 38.9635 \end{bmatrix}, \quad \tilde{\tilde{C}}_2 = \begin{bmatrix} 0 & 0 \\ -49.7959 & 20.4174 \end{bmatrix}$$

$$\theta(s) = -.000388 + .0972s + 6.078s^2 + s^3.$$

The five other parameterizations which give rise to this reduced form are

$$(.00761,6.06,.00840,-.0135,-.00223,-3.71465,L_{11}, L_{22}, L_{12})$$
$$(.00840,6.06,.00761,-.0135,-.00223,-3.506,L_{11},L_{22},L_{12})$$
$$(.00761,.00840,6.06,-74.3,-28.0,-1.997,L_{11},L_{22},L_{12})$$
$$(6.06,.00840,.00761,-74.3,-28.0,-74.0936,L_{11},L_{22},L_{12})$$
$$(6.06,.00761,.00840,102,44.2,101.558,L_{11},L_{22},L_{12}).$$

Among the above five alternatives, all but the fifth are inconsistent with the a priori restrictions. On the other hand, the fifth parameterization implies an extremely large discount rate: 10155.8 percent. This parameterization, which is observationally equivalent to Γ, may therefore be ruled out as implausible.

A modification to the procedure in Section D is required in order to evaluate the frequency domain approximation to the likelihood function at a value of Γ. In particular, replace steps 1 through 6 with the following:

Step 1: Compute $\tilde{V}_{11}$, $\tilde{V}_{12}$, $\tilde{V}_{22}$ from L_{11}, L_{21}, L_{22}.

Step 2: Set $r_1 = -\alpha_0$, $r_2 = -\beta_0$, $r_3 = r - \lambda$.

Step 3: Compute $\tilde{\tilde{C}}_1$, $\tilde{\tilde{C}}_2$ from (65), and set $\tilde{\tilde{C}}_3 = 0$.

Step 4: Compute $W_j = \dfrac{\tilde{\tilde{C}}(r_j)\tilde{V}\tilde{\tilde{C}}(-r_j)^T}{-2r_j \prod\limits_{\substack{k\neq 1 \\ k\neq j}}^{3} (r_j-r_k)(-r_j-r_k)}$, $j = 1, \ldots, 3$.

Using the W's in step 4,

$$S_y(s) = \sum_{j=1}^{3} \frac{W_j}{s-r_j} + \sum_{j=1}^{3} \frac{W_j^T}{-s-r_j}. \tag{71}$$

Expression (71) is (32a) with $\ell = 3$. To finish evaluating the likelihood function, proceed with Steps 7, 8, and 9 in Section D.

G. Temporal Aggregation Bias

In this section we describe an algorithm for computing the probability limit of the estimator of an econometrician who understands that the data are generated by the model of Section B, but who mistakenly assumes that the economy evolves in discrete time, with a timing interval equal to the data sampling interval. We assume that the data are generated by the continuous time model, and denote their spectral density by $S_y^d(e^{i\omega},\zeta)$, $\omega \in (0,2\pi)$. The parameter vector ζ is defined after (45b) and S_y^d is defined in (43).

The econometrician posits that the inventory-sales data are generated as the solution to the following problem:

$$(72)\qquad \max_{\{s_{t+j},I_{t+j+1}\}_{j=0}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j \{P_{t+j}s_{t+j} - \frac{a}{2}[s_{t+j}+I_{t+j}-I_{t+j-1}]^2 - \frac{b}{2}[s_{t+j}-cI_{t+j}]^2 - \nu_{t+j}I_{t+j} - \frac{e}{2}I_{t+j}^2\},$$

subject to I_t given. All variables are defined as before. The only new variable is β, the unit interval discount factor, which is expected to lie in the unit interval.

The first order necessary conditions corresponding to s_t and I_{t+1} respectively, are

$$(73a)\qquad s_t = -\left(\frac{a-bc}{a+b+A}\right)I_t + \frac{a}{a+b+A}I_{t-1} + \frac{1}{a+b+A}u_t$$

$$(73b)\qquad \beta a I_{t+1} - [a(1+\beta)+bc^2+e]I_t + aI_{t-1} + \beta a s_{t+1} - (a-bc)s_t = \nu_t.$$

In (73) we have made use of $P_t = -As_t + u_t$.

Substituting (73a) into (73b) and rearranging, get

$$(74a)\qquad (1-\lambda L)(1-\frac{1}{\lambda\beta}L)I_{t+1} = \frac{a+b+A}{\beta a[b(c+1)+A]}\left[\frac{a-bc}{a+b+A}u_t - \frac{\beta a}{a+b+A}u_{t+1} + \nu_t\right]$$

$$(74b)\qquad \lambda + \frac{1}{\lambda\beta} = -\frac{a+b+A}{\beta a[b(c+1)+A]}\left[\frac{\beta a^2+(a-bc)^2}{a+b+A} - (a+bc^2+e+\beta a)\right]$$

and $|\lambda| < 1$. The solution to (74a) which solves (72) is

$$(75)\qquad I_t = \lambda I_{t-1} - \frac{\lambda\beta}{\beta a[b(c+1)+A]} \sum_{i=0}^{\infty} E_t[(a-bc)-\beta a L^{-1}](\lambda\beta)^i u_{t+i}$$

$$- \frac{a+b+A}{\beta a[b(c+1)+A]}\lambda\beta \sum_{i=0}^{\infty}(\lambda\beta)^i E_t \nu_{t+i}$$

$$= \lambda I_{t-1} - \frac{\lambda(a-bc)}{a[b(c+1)+A]} u_t - \frac{[(a-bc)\lambda - a]}{a[b(c+1)+A]} \sum_{i=1}^{\infty}(\lambda\beta)^i E_t u_{t+i}$$

$$- \frac{(a+b+A)\lambda}{a[b(c+1)+A]} \sum_{i=0}^{\infty}(\lambda\beta)^i E_t \nu_{t+i}.$$

As in the continuous time case, given a time series model for $\{u_t,\nu_t\}$, (75) and (73a) imply a time series representation for $Y_t \equiv (I_t,s_t)^T$. This in turn implies a spectral density, $S^e(e^{i\omega},\xi)$, for $\omega \in (0,2\pi)$. (The superscript "e" denotes "econometrician".) Here, ξ denotes a vector of parameters which includes β, a, b, c, e, A, and the parameters of the statistical representation of $\{u_t,\nu_t\}$.

Define

$$\hat{\xi}(\zeta) = \text{P lim } \hat{\xi}_T.$$

The above Plim is computed on the assumption that the data are generated by the continuous time model, with parameter values ζ. Here, $\hat{\xi}_T$ is the (misspecified) maximum likelihood estimator of ξ. Asymptotically, this is equivalent with the following frequency domain estimator:

$$(77)\qquad \hat{\xi}_T = \underset{\xi}{\text{argmax}}\ \frac{1}{T}\{-\frac{1}{2}\sum_{j=1}^{T} \log\det[S^e(e^{-i\omega_j},\xi)]$$

$$-\frac{1}{2}\sum_{j=1}^{T}\text{trace}\,[S^e(e^{-i\omega_j},\xi)^{-1}I(\omega_j)\},$$

where $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, T$, and I is defined in (45). It can be shown that Plim $\hat{\xi}_T$ is the argmax of (77) as $T \to \infty$ with I replaced by the true spectrum of the data, which is S_y^d. Thus,

$$(78) \qquad \hat{\xi}(\zeta) = \underset{\xi}{\operatorname{argmax}} \{-\int_0^{2\pi} \log \det [S^e(e^{-i\omega},\xi)]d\omega$$

$$- \int_0^{2\pi} \operatorname{trace}[S^e(e^{-i\omega},\xi)^{-1} S_y^d(e^{-i\omega},\zeta)]d\omega\}$$

In (77) and (78) the maximization is carried out over the range of admissible values of ξ.

Our objective is to compute $\hat{\xi}(\zeta)$ for a variety of values of ζ. In each case, we plan to compare the speed of adjustment implied by ζ with that implied by $\hat{\xi}(\zeta)$. The difference is due to temporal aggregation bias.

As an illustration, we consider the case in which u_t and ν_t have AR(1) representations and fail to be Granger caused by each other, or any other model variable. This is the discrete time version of the model considered in Section F. Accordingly, suppose

$$(79a) \qquad \begin{aligned} u_t &= \mu u_{t-1} + \varepsilon_{1t} \\ \nu_t &= \rho \nu_{t-1} + \varepsilon_{2t} \end{aligned}$$

where $|\mu| < 1$, $|\rho| < 1$. Also, $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})^T$ is a vector white noise with

$$(79b) \qquad E\,\varepsilon_t\,\varepsilon_{t-\tau} = \begin{matrix} \Omega & \tau=0 \\ 0 & \tau\neq 0 \end{matrix} .$$

Substituting from (79a) into (75),

$$(80)\qquad I_t = \lambda I_{t-1} + hu_t + g\nu_t,$$

where

$$h = \frac{-1}{a[b(c+1)+A]}\{\lambda(a-bc)+[(a-bc)\lambda-a]\frac{\lambda\beta\mu}{1-\lambda\beta\mu}\}$$

$$g = -\frac{(a+b+A)\lambda}{a[b(c+1)+A]}\,\frac{1}{1-\lambda\beta\rho}.$$

Equations (73a), (79a), and (80) imply

$$(1-\rho L)(1-\mu L)(1-\lambda L)I_t = h(1-\rho L)\varepsilon_{1t} + g(1-\mu L)\varepsilon_{2t}$$

$$(1-\rho L)(1-\mu L)(1-\lambda L)s_t$$

$$= \frac{1}{a+b+A}\{(1-\rho L)[1-(a-bc)h+(ah-\lambda)L]\varepsilon_{1t}$$

$$+ g(1-\mu L)[bc-a+aL]\varepsilon_{2t}\}$$

Writing this in matrix form,

$$(81a)\qquad (1-\rho L)(1-\mu L)(1-\lambda L)\begin{pmatrix} I_t \\ s_t \end{pmatrix} = (C_0+C_1L+C_2L^2)\varepsilon_t,$$

where

$$(81b)\qquad C_0 = \begin{bmatrix} h & g \\ \frac{1-(a-bc)h}{a+b+A} & \frac{g(bc-a)}{a+b+A} \end{bmatrix}$$

$$(81c)\qquad C_1 = \begin{bmatrix} -h\rho & -g\mu \\ \frac{(ah-\lambda) - \rho[1-(a-bc)h]}{a+b+A} & \frac{g[a-\mu(bc-a)]}{a+b+A} \end{bmatrix}$$

$$(81d)\qquad C_2 = \begin{bmatrix} 0 & 0 \\ \frac{-\rho(ah-\lambda)}{a+b+A} & \frac{-g\mu a}{a+b+A} \end{bmatrix}$$

In order to obtain an identifiable reduced form, compute $\tilde{C}(L) = C(L)C_0^{-1}$ and $\tilde{\Omega} = C_0\ \Omega\ C_0^T$. The Wold representation of (I_t, s_t) then possesses 12 non-zero reduced form parameters. However, only nine of these are free to vary independently. This is fewer than the number of structural parameters of the problem:

$$\zeta = (\beta, a, b, c, e, A, \mu, \rho, \Omega_{11}, \Omega_{22}, \Omega_{12}).$$

Like in the continuous time case, only nine of these reduced form parameters are free to vary independently. This will be proved in the discussion that follows.

Consider the following nine element parameter vector:

$$\Gamma = \left(\mu, \rho, \lambda, \frac{bc}{a}, \frac{a+b+A}{a}, \beta, \tilde{\Omega}_{11}, \tilde{\Omega}_{22}, \tilde{\Omega}_{12}\right).$$

The admissible region for Γ is $\underline{P}$, where

$$\underline{P} \equiv \{x \varepsilon R^9:\ x_i \neq x_k, i,k=1,2,3;\ |x_i| < 1,\ i = 1,2,3,6;$$

$$x_i > 0,\ i=4,7,8;\ x_5 > 1;\ x_7 x_8 - x_9^2 > 0\}.$$

We will show that for all $\Gamma \ \varepsilon \ \underline{P}$, the mapping from Γ to $\theta(L)$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{\Omega}$ has a locally unique inverse in $\underline{P}$. We will also show that given $\theta(L)$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{\Omega}$ in the range of the mapping $\Gamma \ \varepsilon \ \underline{P} \rightarrow \{\theta(L), \tilde{C}_1, \tilde{C}_2, \tilde{\Omega}\}$, there are no more than six isolated elements in the associated inverse mapping that belong to $\underline{P}$. For any given $\theta(L)$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{\Omega}$, we provide an algorithm for determining how many elements of the inverse mapping belong to $\underline{P}$. If there is only one, then the model is said to be globally identified at $\theta(L)$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{\Omega}$.

Note first that

$$\tilde{C}_1 = \begin{bmatrix} \tilde{C}_1^{11} & \tilde{C}_1^{12} \\ \tilde{C}_1^{21} & \tilde{C}_1^{22} \end{bmatrix}, \qquad \tilde{C}_2 = \begin{bmatrix} 0 & 0 \\ \tilde{C}_2^{21} & \tilde{C}_2^{22} \end{bmatrix},$$

where

$$\tilde{C}_1^{11} = h(a-bc)(\mu-\rho) - \mu$$

$$\tilde{C}_1^{21} = \frac{-(ah-\lambda)(\frac{bc}{a} - 1) + \rho[1-(a-bc)h](\frac{bc}{a} - 1) + [1-\mu(\frac{bc}{a} - 1)][1-(a-bc)h]}{[(a+b+A)/a]}$$

$$\tilde{C}_1^{12} = h(\mu-\rho)(a+b+A)$$

$$\tilde{C}_1^{22} = -(\lambda+\rho) + h(\rho-\mu)(a-bc)$$

$$\tilde{C}_2^{21} = \frac{\rho(\frac{bc}{a} - 1)(ah-\lambda) + \mu[(a-bc)h-1]}{[(a+b+A)/a]}$$

$$\tilde{C}_2^{22} = ha(\mu-\rho) + \rho\lambda$$

There are six ways to infer μ, ρ, λ from the coefficients of $\theta(L) = (1-\rho L)(1-\mu L)(1-\lambda L)$. For every such way, it is possible to uniquely infer $\frac{bc}{a}$, $\frac{a + b + A}{a}$, β, as we show below. First, note

$$ah = \frac{\tilde{C}_2^{22} - \rho\lambda}{\mu - \rho}$$

$$(a-bc)h = \frac{\tilde{C}_1^{11} + \mu}{\mu - \rho}$$

$$(a+b+A)h = \frac{\tilde{C}_1^{12}}{\mu - \rho}$$

Consequently,

$$\frac{a + b + A}{a} = \frac{\tilde{C}_1^{12}}{\tilde{C}_2^{22} - \rho\lambda}$$

$$\frac{bc}{a} = 1 - \frac{\tilde{C}_1^{11} + \mu}{\tilde{C}_2^{22} - \rho\lambda}$$

Next taking into account the formula for h, note that

$$ah = -\frac{\{\lambda(1 - \frac{bc}{a}) + [(1 - \frac{bc}{a})\lambda - 1]\lambda\mu\beta\}}{[\frac{bc}{a} + \frac{b+A}{a}](1-\lambda\beta\mu)} \quad \lambda\left\{1 - \frac{bc}{a} - \rho\mu\right\}$$

Hence,

$$\cancel{\beta = \frac{\{[\frac{bc}{a} + \frac{b+A}{a}]\lambda\mu ah - [(1 - \frac{bc}{a})\lambda-1]\lambda\mu\}^{-1}}{\{[\lambda(1 - \frac{bc}{a}) + [\frac{bc}{a} + \frac{b+A}{a}]ah\}}}$$

$$\beta = \frac{1}{\lambda\mu}\left\{1 + ah\left[\frac{bc}{a} + \frac{b+A}{a}\right]\right\}^{-1}\left\{ah\left[\frac{bc}{a} + \frac{b+A}{a}\right] + \lambda\left(1 - \frac{bc}{a}\right)\right\}$$

Since--for the given values of λ, μ, ρ--ah, $\frac{b + A}{a}$ and $\frac{bc}{a}$ are identified, it follows that β is too. In fact, ρ, μ and λ are locally identifiable from $\theta(L) = (1-\rho L)(1-\mu L)(1-\lambda L)$. This establishes that all $\Gamma \varepsilon \underline{P}$ are locally identifiable from the reduced form coefficients $\theta(L)$, $\tilde{C}_2^{22}$, $\tilde{C}_1^{11}$, $\tilde{C}_1^{12}$ and $\tilde{\Omega}$.

It is of interest to note that $\tilde{C}$, $\theta(L)$ and $\tilde{\Omega}$ do not permit inferring more than the nine elements of Γ. To see this, note that $\tilde{C}_1^{21}$, $\tilde{C}_1^{22}$, and $\tilde{C}_2^{21}$ are exact functions of the remaining elements of $\tilde{C}$ and of $\theta(L)$.

We have shown local, but not global, identification. In fact, global identification does not obtain for all $\Gamma \varepsilon \underline{P}$. It is straightforward to check global identification for any particular $\Gamma \varepsilon \underline{P}$. First, execute the mapping $\Gamma \rightarrow \theta(L)$, $\tilde{C}_1$, $\tilde{C}_2$ ($\tilde{\Omega}$ can be ignored). Then compute the six different ways of allocating the labels "μ," "ρ," and "λ" to the roots of $\theta(L)$. Then compute the associated six values of $\frac{a + b + A}{a}$, $\frac{bc}{a}$, and β using the formulas provided above. The result will be six vectors in R^9, where one of these is by construction the $\Gamma \varepsilon \underline{P}$ we started out with. If none of the alternatives to Γ belong to $\underline{P}$, then global identification obtains at the point $\Gamma \varepsilon \underline{P}$.

At this point, the sources of temporal aggregation bias in estimates of speed of adjustment are clear. The restrictions implied for the continuous and discrete time Wold representation by the continuous and discrete version, respectively, are similar. However, the continuous time model loses the property of being a pure AR(2) process upon sampling the data and averaging sales. This sampling and averaging results in the continuous and discrete models having different implications for measured data. The experiment outlined in this section can be undertaken in such a way as to isolate the contributions to bias of sampling and of averaging.

H. The Effect of Averaging on Covariances

In a recent paper, Blinder [1984] studies the so-called production smoothing model of inventories and sales. In that model, the short run production function is concave and the underlying shocks are dominated by a serially uncorrelated demand shock. The model of this paper, with the variance of $\nu(t)$ small and $u(t)$ serially uncorrelated, has these properties. The short run production function clearly is concave, since (with b=0) it takes the form

$$\tilde{f}(N) = \frac{2}{a}[N+z]^{1/2},$$

where z is a function of the existing stock of inventories. (Note that the degree of concavity in $\tilde{f}$ is inversely related to the size of a. The variable z is guaranteed to be positive if the stock of inventories is at a level where the marginal product of inventories is positive.)

The production smoothing model has the implication, according to Blinder, that the covariance of sales, $s(t)$, and inventory investment, $DI(t)$, is negative. In this model, inventories act as a buffer to smooth production in the face of disturbances to sales. Blinder observes that the covariance of measured sales and investment is in fact positive, thus contradicting the implication of the production smoothing model. Blinder argues that the model can be reconciled with the data by introducing the right kind of serial correlation in $u(t)$ (it needs a "hump shaped" moving average representation), or by raising the variance on the production shock, $\nu(t)$. The analysis of this paper suggest a third possibility, namely that the positive covariance between measured sales and investment is an artifact of averaging. (A fifth possibility, not discussed here, is b large and positive.)

Recall that measured sales, $\bar{s}(t)$, are related to actual sales, $s(t)$, by $\bar{s}(t) = \int_0^1 s(t+\tau)d\tau$. Similarly, measured investment, $I(t+1) - I(t)$, is related to actual investment by the averaging operator: $I(t+1) - I(t) = \int_0^1 DI(t+\tau)d\tau$. It seems in principle possible that $Cov(DI(t), s(t)) < 0$ and $Cov(I(t+1) - I(t), \bar{s}(t)) > 0$. To see this, consider two random variables, x_t and y_t, and suppose that $Cov(x_t,y_t) < 0$, but that $Cov(x_t,y_{t-\tau})$ is large and positive for $\tau = \pm 1$. Then it is the case that $Cov(\frac{1}{2}(x_t+x_{t+1}), \frac{1}{2}(y_t+y_{t+1})) = Cov(x_t,y_t) + \frac{1}{2}Cov(x_{t+1},y_t) + \frac{1}{2}Cov(x_t,y_{t+1}) > 0$. Evidently, the condition required for averaging to reverse the sign of a covariance is a special one. It requires that the contemporaneous covariance and lagged covariance be opposite in sign.

In a (failed) attempt to find an example of the case $Cov(s(t),DI(t)) < 0$ and $Cov(\bar{s}(t),I(t+1)-I(t)) > 0$ we considered the following model:

$$Dx(t) = Ax(t) + \varepsilon(t),$$

where

$$x(t) = \begin{pmatrix} s(t) \\ DI(t) \end{pmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix},$$

$\varepsilon(t)$ is white, and $a_{ii} < 0$ $i = 1, 2$. then,

$$e^{A\tau} = \begin{bmatrix} e^{a_{11}\tau} & \left(\frac{a_{12}}{a_{22}-a_{11}}\right)\left(e^{a_{22}\tau} - e^{a_{11}\tau}\right) \\ 0 & e^{a_{22}\tau} \end{bmatrix}$$

Now, $EY(t)Y(t-\tau)^T = e^{A\tau}\mathrm{Var}(Y_t)$, where

$$\mathrm{Var}(Y_t) = \begin{bmatrix} -\frac{1}{2a_{11}} - k^2\left(\frac{1}{2a_{22}} + \frac{1}{2a_{11}} - \frac{2}{a_{11}+a_{22}}\right) & \frac{a_{12}}{2a_{22}(a_{11}+a_{22})} \\ \frac{a_{12}}{2a_{22}(a_{11}+a_{22})} & -\frac{1}{2a_{22}} \end{bmatrix}$$

Here, $k = \frac{a_{12}}{a_{22} - a_{11}}$. Thus, in particular,

$$\mathrm{Cov}(s(t),\ DI(t)) = \frac{a_{12}}{2a_{22}(a_{11}+a_{22})}$$

$$\mathrm{Var}(DI(t)) = -\frac{1}{2a_{22}}$$

Now,

$$\mathrm{Cov}(\bar{s}(t),\ I(t+1) - I(t))$$

$$= E\int_0^1 s(t+\tau)d\tau \int_0^1 DI(t+\nu)d\nu$$

$$= \int_0^1\int_0^1 \mathrm{Cov}[s(t+\tau),\ DI(t+\nu)]d\tau d\nu$$

$$= \left(\frac{2\rho}{a_{22}-a_{11}}\right) \mathrm{Var}(DI(t))\left\{\left[\frac{e^{a_{22}}-1-a_{22}}{a_{22}}\right]-\left[\frac{e^{a_{11}}-1-a_{11}}{a_{11}}\right]\right\}$$

$$\mathrm{Var}(I(t+1) - I(t)) = 2\ \mathrm{Var}(DI(t))\left[\frac{e^{a_{22}}-1-a_{22}}{a_{22}^2}\right],$$

where

$$\rho = \frac{\text{Cov}(s(t),DI(t))}{\text{Var}(DI(t))} = \frac{-a_{12}}{a_{11} + a_{22}}.$$

Define

$$\bar{\rho} = \frac{\text{Cov}(\bar{s}(t),I(t+1)-I(t))}{\text{Var}(I(t+1)-I(t))}.$$

Then,

$$\frac{\bar{\rho}}{\rho} = \left(\frac{1}{a_{22}-a_{11}}\right)\left\{a_{22}\left[\frac{e^{a_{11}}-1-a_{11}}{e^{a_{22}}-1-a_{22}}\right]\left(\frac{a_{22}}{a_{11}}\right)^2 - a_{11}\right\}$$

$$= \left(\frac{a_{22}}{a_{22}-a_{11}}\right)\left[\frac{\frac{1}{2} + \frac{1}{3!}\,a_{11} + \frac{1}{4!}\,a_{11}^2 + \cdots}{\frac{1}{2} + \frac{1}{3!}\,a_{22} + \frac{1}{4!}\,a_{22}^2 + \cdots}\right]\left(\frac{a_{22}}{a_{11}}\right)^2 - \left(\frac{a_{11}}{a_{22}-a_{11}}\right).$$

If it were possible to choose a_{11} and a_{22} so that $(\bar{\rho}/\rho) < 0$, then we'd have an example in which averaging produces a switch in the sign of a covariance. Unfortunately, this seems not to be possible, at least for extreme values of a_{11} and a_{22}. The following results may be confirmed:

$$a_{22} \to 0 \Rightarrow \frac{\bar{\rho}}{\rho} \to 1$$

$$a_{22} \to -\infty \Rightarrow \frac{\bar{\rho}}{\rho} \to \infty$$

$$a_{11} \to 0 \Rightarrow \frac{\bar{\rho}}{\rho} \to \text{positive constant}$$

$$a_{11} \to -\infty \Rightarrow \frac{\bar{\rho}}{\rho} \to 1$$

We do not have an analytic result for the case $a_{11} \to a_{22}$. However, numerical simulations suggest that in this case $(\bar{\rho}/\rho) \to$ a positive constant.

I. Interpreting Fundamental Moving Average Representations

A current widespread practice is to compute and report the fundamental moving average representation of unconstrained estimated discrete time vector ARMA models. Frequently, errors in this representation ("innovations") are interpreted as surprise shocks to the utility or production functions of agents. In addition, the moving average coefficients are interpreted as reflecting the transmission mechanisms whereby these surprises dynamically influence the variables observable to the econometrician.

A basic difficulty associated with the above procedure was pointed out by Hansen and Sargent [1982]. They note that dynamic economic theory does not always imply that the innovations in the sampled data observed by the econometrician coincide with shocks to technology and preferences. A divergence can arise for two reasons. First, assuming a correct model timing specification, the model may imply a nonfundamental representation. In this case, the innovations are a square sumable linear combination of shocks to technology and preferences going into the infinite past. (See Hansen and Sargent [1980, ftn. 12] for a simple illustration of this possibility.) The second potential source of divergence arises if agents are making decisions over a finer interval than the data sampling interval, for example, in continuous time. In this case the first possibility mentioned above can also arise (e.g., detC(D) in (21) may not be invertible). However, even when this does not occur, there can be a divergence between the innovations in the discrete time sampled representation and the continuous time disturbances to preferences and technology. In addition, the discrete time moving average coefficients may deviate sharply from resembling a sampled version of the continuous time moving average representation.

We propose to supply evidence on the empirical importance of both of the above difficulties in the context of our bivariate model of inventories and sales. As a byproduct of the estimation we plan to carry out, we can compute the moving average representation that appears in operator form in (21), evaluated at the estimated parameter values. We plan to compare this with the moving average coefficients of the fundamental representation of sampled averages from our continuous time model. Finally, we plan to use discrete time data on sales and inventories to estimate the one step ahead prediction errors that agents are making in continuous time (Hansen and Sargent [1982]) describe projection procedures for doing this). We will compare these with the innovations computed from our discrete time fundamental sample representation.

As an example, consider the case $n = m = \delta_0 = \delta_0 = 1$, $\delta_1 = \delta_2 = \delta_1 = 0$, which was studied in Section F.

There we report

$$Y(t) = \frac{\tilde{\tilde{C}}(D)}{\theta(D)} \tilde{\varepsilon}(t)$$

$$= \int_0^\infty f(\tau)\tilde{\varepsilon}(t-\tau)d\tau,$$

where

$$f(\tau) = \begin{pmatrix} f_1(\tau) \\ f_2(\tau) \end{pmatrix} = A_1 e^{-\alpha_o \tau} + A_2 e^{-\beta_o \tau} + A_3 e^{(r-\lambda)\tau}, \ \tau > 0,$$

and

$$A_1 = \frac{-\tilde{C}(-\alpha_o)}{(\beta_o-\alpha_o)(\alpha_o+r-\lambda)} \equiv \begin{pmatrix} A_1^1 \\ A_1^2 \end{pmatrix}$$

$$A_2 = \frac{-\tilde{C}(-\beta_o)}{(\alpha_o-\beta_o)(\beta_o+r-\lambda)} \equiv \begin{pmatrix} A_2^1 \\ A_2^2 \end{pmatrix}$$

$$A_3 = \frac{\tilde{C}(r-\lambda)}{(r-\lambda+\alpha_o)(r-\lambda-\beta_o)} \equiv \begin{pmatrix} A_3^1 \\ A_3^2 \end{pmatrix} ,$$

Where A_i^j are two element row vectors, i, j = 1, 2. Note that $f_1(0) = 0$ and $f_2(0) \neq 0$. This is an implication of the fact that I(t) is, and s(t) is not, differentiable. In addition to $\{f(\tau)\}$, it is also of interest to compute $\{\overline{f}(\tau)\}$, the moving average representation of $\{\overline{Y}(t)\}$, where

$$\overline{Y}(t) = \begin{pmatrix} I(t) \\ \int_0^1 s(t+\tau)d\tau \end{pmatrix}.$$

The function $\{\overline{f}(\tau)\}$ is defined by

$$\overline{Y}(t) = \int_{-1}^{\infty} \overline{f}(\tau)\tilde{\varepsilon}(t-\tau)d\tau,$$

where

$$\overline{f}(\tau) = \begin{cases} f_1(\tau) \\ \overline{f}_2(\tau) \end{cases} ,$$

and $f_1(\tau) = 0$ $\tau < 0$, $\overline{f}_2(\tau) = 0$ $\tau < -1$. Here,

$$\overline{f}_2(\tau) = \int_0^1 f_2(\tau+k)dk \quad -1 < \tau.$$

Straightforward calculations show that

$$\overline{f}_2(\tau) = \begin{cases} A_1^2\left(\frac{1-e^{-\alpha_o}}{\alpha_o}\right)e^{-\alpha_o\tau} + A_2^2\left(\frac{1-e^{-\beta_o}}{\beta_o}\right)e^{-\beta_o\tau} \\ \quad + A_3^2\left(\frac{1-e^{(r-\lambda)}}{\lambda-r}\right)e^{(r-\lambda)\tau} & \tau > 0 \\ \\ A_1^2\left[\frac{1-e^{-\alpha_o(\tau+1)}}{\alpha_o}\right] + A_2^2\left[\frac{1-e^{-\beta_o(\tau+1)}}{\beta_o}\right] \\ \quad + A_3^2\left[\frac{1-e^{(r-\lambda)(\tau+1)}}{\lambda-r}\right] & -1 < \tau < 0 \end{cases}$$

It is easy to verify that $\overline{f}(-1) = 0$, and that $\overline{f}_2(\cdot)$ is differentiable for all $\tau > -1$. However, $\int_{-1}^{\infty} \overline{f}(\tau)\tilde{\varepsilon}(t-\tau)d\tau$ is not the fundamental representation for $\{\overline{Y}(t)\}$. To get this requires first factorizing the spectral density of $\{\overline{Y}(t)\}$, which is

$$S_{\overline{Y}}(s) = G(s)S_Y(s)G(-s)^{\tau},$$

where $S_Y(s)$ is the spectral density of $\{Y(t)\}$ and

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{e^s - 1}{s} \end{bmatrix}.$$

It appears that this factorization is difficult to accomplish, since $G(s)$ is not rational. If inventories were measured end-of-period, rather than beginning of period, then the problem vanishes. To see this, suppose that the data are $\breve{Y}(t)$, where

$$\breve{Y}(t) = G(-D)y(t).$$

In this case, $[G(-D)\tilde{C}(D)/\theta(D)]\tilde{\varepsilon}(t) = \int_0^\infty g(\tau)\tilde{\varepsilon}(t-\tau)d\tau$, where

$$g_1(\tau) = f_1(\tau)$$

$$g_2(\tau) = \int_0^1 f_2(\tau-k)dk,$$

for $\tau > 0$. This is a fundamental representation for $\{\tilde{Y}(t)\}$.

A number of normalization questions arise when comparing matrix continuous time and discrete time moving average representations. To see this, suppose

$$Y(t) = \int_0^\infty f(\tau)\varepsilon(t-\tau)d\tau$$

$$Y(t) = \sum_{j=0}^\infty C_j u(t-j).$$

The first model is that of $\{y(t), t \text{ real}\}$, while the second is that of $\{Y(t), t \text{ integer}\}$. Since the above representations involve products of moving averages and errors, there is considerable latitude in how to measure these, while leaving the model substantively unaffected. Before undertaking comparisons of objects like $\{f(t), \tau>0\}$ and $\{C_j, j=0,1,2,\ldots\}$, some "natural" normalization for these must be found.

References

Bergstrom, A. R., 1983, "Gaussian Estimation of Structural Parameters in Higher Order Continuous Time Dynamic Models, "Econometrica, Vol. 51, no. 1, January.

__________, 1984, "The Estimation of Parameters in Non-Stationary Higher Order Continuous Time Dynamic Models," unpublished.

Blanchard, Olivier, 1983, "The Production and Inventory Behavior of the American Automobile Industry," Journal of Political Economy, Vol. 91, no. 3, June.

Blinder, A. S., 1984, "Can the Production Smoothing Model of Inventory Behavior be Saved," manuscript.

Bryan, W. R., 1967, "Bank Adjustment to Monetary Policy: Alternative Estimates of the Lag," American Economic Review, 57:855-864.

Chatterjee, Sangit, 1985, "Bootstrapping ARIMA Models," Northeastern University, unpublished.

Christiano, Lawrence J., 1982, "Estimating the Parameters of Continuous Time Rational Expectations Models, with an Application to the Term Structure of Interest Rates," Ph.D. thesis, Columbia University.

__________, 1984, "The Effects of Aggregation Over Time on Tests of the Representative Agent Model of Consumption," December.

Durbin, James, 1961, "Efficient Fitting of Linear Models for Continuous Stationary Time Series From Discrete Data," Bulletin of the International Institute of Statistics, Vol. 48, no. 4, pp. 273-282.

Eichenbaum, Martin, 1984, "Rational Expectations and the Smoothing Properties of Inventories and Finished Goods," Journal of Monetary Economics, 14:71-96.

Feldstein, M. S., and Auerbach, A. 1976, "Inventory Behavior in Durable Goods Manufacturing: The Target Adjustment Model, "Brookings Papers on Economic Activity, 2:351-408

Goodfriend, Marvin, 1985, "Reinterpreting Money Demand Regressions," in Brunner and Meltzer, editor, Volume 22, Carnegie-Rochester Conference Series on Public Policy.

Hannan, E. J., 1970, Multiple Time Series, Wiley.

________, 1969, "The Identification of Vector Mixed Autoregressive-Moving Average Systems," Biometrika.

Hansen, Lars P., and Thomas J. Sargent, 1983, "The Dimensionality of the Aliasing Problem in Models with Rational Spectral Densities," Econometrica, March.

________, 1981a, "Identification of Continuous Time Rational Expectations Models from Discrete Time Data," unpublished manuscript.

________, 1981b, "Formulating and Estimating Continuous Time Rational Expectations Models," Federal Reserve Bank of Minneapolis Staff Report #75, October.

________, 1980, "Methods for Estimating Continuous Time Rational Expectations Models From Discrete Time Data," Federal Reserve Bank of Minneapolis.

________, 1982, "Two Difficulties with Interpreting Vector Autoregressions," unpublished.

Harvey, A. C., and J. H. Stock, 1985, "The Estimation of Higher Order Continuous Time Autoregressive Models," Econometric Theory, 1.

Hendry, David, 1983, "Comment," Econometric Reviews, 2(1), pp. 111-114.

Mundlak, Y., 1961, "Aggregation Over Time In Distributed Lag Models," International Economic Review, 2:154-63.

Nobel, Ben, 1969, Applied Linear Algebra, Prentice-Hall.

Pesaran, M. H., and A. S. Deaton, 1978, "Testing Non-nested, Nonlinear Regression Models," Econometrica, Vol. 46, no. 3, May.

Phillips, A. W., 1959, The Estimation of Parameters in Systems of Stochastic Differential Equations, Biometrika, Vol. 46.

Rozanov, Yu A., 1967, Stationary Random Processes, San Francisco, Holden-Day.

Sargent, Thomas J., 1979, Macroeconomic Theory, Academic Press.

Singleton, Kenneth, 1984, "Testing Specifications of Economic Agents' Optimum Problems Against Non-Nested Alternatives," unpublished.

Walker, A. M., 1967, "Some Tests of Separate Families of Hypotheses in Time Series Analysis," Biometrika, 54.

Whittle, Peter, 1983, Prediction and Regulation by Linear Least-Square Methods, Second edition, revised, University of Minnesota Press.

Zellner, Arnold, 1968, "Note on Effect of Temporal Aggregation on Estimation of a Stock Adjustment Equation," manuscript.