Federal Reserve Bank of Minneapolis
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WALRAS' LAW AND NONOPTIMAL EQUILIBRIA IN OVERLAPPING GENERATIONS MODELS

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Working Paper 393

Revised June 1991

ABSTRACT

This paper demonstrates a connection between failure of Walras' Law and nonoptimal equilibria in a quite general overlapping generations model. Consider the following implication of Walras' Law in finite economies. Suppose that all prices are positive and that all agents are on their budget lines. Then, no matter how the set of goods is partitioned, there cannot be an excess supply (in value terms) for some other set in the partition with excess demand (in value terms) for some other set in the partition. We use the Cass (1972), Benveniste (1976, 1986), Balasko and Shell (1980), and Okuno and Zilcha (1980) price characterization of optimality of equilibria in pure exchange overlapping generations models to show the following link between the above implication of Walras' Law and optimality of a competitive equilibrium. A competitive equilibrium is nonoptimal if and only if the above implication of Walras' Law fails in its neighborhood.

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I. Introduction

The purpose of this paper is to demonstrate a connection between failure of Walras' Law and nonoptimal equilibria in a quite general overlapping generations (hereafter, OLG) model. Consider the following implication of Walras' Law in finite economies. Fix a partition of the set of goods. Suppose that preferences satisfy local nonsatiation so that for any strictly positive price vector all agents are on their budget lines. Define the excess supply (in value terms) for a subset of goods as the sum of values of excess supplies for each good in the subset. Then, for any strictly positive price vector there cannot be an excess supply (in value terms) for some set in the partition without excess demand (in value terms) for some other set in the partition. We use the Cass (1972), Benveniste (1976, 1986), Balasko and Shell (1980), and Okuno and Zilcha (1980) price characterization of optimality of equilibria in pure exchange OLG models to show the following link between the above implication of Walras' Law and optimality of a competitive equilibrium. A competitive equilibrium is nonoptimal if and only if the above implication of Walras' Law fails in its neighborhood[1]. More specifically, define an ε-excess supply allocation as an allocation together with a set of positive prices such that: (i) each

[1] It is important to emphasize that the implication of Walras' Law we are focusing on is that in finite economies it is not possible to generate excess supply for some set without excess demand for some other set. It is also, obviously, true that it is not possible to generate excess demand for some set without excess supply for some other set. We will show by example in section 2 that the failure of this latter implication in an OLG model has no necessary connection with the nonoptimality of a competitive equilibrium. Further, throughout this paper, failure of Walras' Law in an OLG economy is to be understood to mean that the previously noted implication of Walras' Law in finite economies does not hold.

agent's allocation is on his/her demand curve (generated with the competitive equilibrium allocation as endowments), (ii) there is positive excess supply at some date (i.e., for the set of date t goods in value terms) with no excess demand (in value terms) at any date, and (iii) the allocation is uniformly within ε of the endowment (also the competitive equilibrium allocation). We then show that there exist ε-excess supply allocations for all sufficiently small ε, if and only if the competitive equilibrium allocation is nonoptimal.

The possible failure of Walras' Law in economies with a double infinity of goods and agents (Shell 1971) is well-known. As Wilson (1981, p. 96) notes, "In particular, as is already known from the study of simple models of overlapping generations, the equilibrium price sequence need not assign a finite valuation to the aggregate endowment. In these cases, the usual implications of Walras' Law may no longer be valid, and as a consequence a competitive equilibrium may not exist."[2] As is again well-known, the condition that the aggregate endowment have a finite valuation is sufficient, though not necessary for optimality (Wilson 1981, Section 5). The purpose of this paper is to develop the connection between the failure of Walras' Law in the sense described in the first paragraph and nonoptimal equilibria with positive prices. As the statement of the result indicates, the endowments that lead to a given equilibrium play no role in this relationship and therefore, we may confine attention to interior weakly pareto optimal (WPO) allocations (Balasko and Shell 1980) and the prices

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[2]See, Geanakoplos and Brown [1986], and Geanakoplos [1987] for an illuminating discussion of the various puzzles of overlapping generations economies as arising from a lack of market clearing at infinity.

that support such allocations. Excess demands are then defined relative to endowments which are given by the WPO allocation.

In order to motivate the particular statement of the result, we will discuss some other possible interpretations of the connection between nonoptimality and failure of Walras' Law. First, we need to define excess demand functions relative to some endowments. Second we need to define a failure of Walras' Law. This could be a weak definition in terms of some partition of the set of goods (the partition by dates being a natural choice) or a strong definition which requires excess supply of some good with no excess demand for any good. The strong definition is equivalent to requiring that the demands generated be inefficient in the usual sense that more of some good can be provided with no less of any good. Third, we need to define the allocation whose nonoptimality is related to the failure of Walras' Law. In the following, we consider some of the possibilities and suggest that our original statement of the connection may be the most reasonable.

It is probably not useful to focus on the optimality or nonoptimality of an arbitrarily given endowment allocation. If the endowment allocation is not WPO then it is not optimal. Thus, the natural allocations to be focusing on are either: (i) an endowment allocation which is WPO, or (ii) the set of equilibria associated with an arbitrary endowment allocation. Corresponding to (ii), a possible result might be the following--"There exist prices which violate Walras' Law (either in value terms for some partition of goods or in terms of each good) if and only if there exists a nonoptimal equilibrium." We do indeed demonstrate such a result (formulated in terms of excess supplies for each good) for a simple OLG economy with only one good at each date and two period lived agents. Whether it holds more generally is not known.

This brings us to our formulation of the connection which is (i). Note that when the endowment allocation is WPO, the two formulations (i) and (ii) are identical. This follows because if there exists a nonoptimal equilibrium, then the endowment allocation which is WPO is not optimal since such an equilibrium (if distinct from the endowment allocation) must Pareto dominate the endowment allocation. Conversely, if the endowment allocation is not optimal then there trivially exists an equilibrium (i.e., the endowment allocation which is WPO together with the support prices) which is not optimal.

Thus, the only question that remains is whether to use the weak or the strong form of the failure of Walras' Law. If the strong form is used, then in one direction the connection is trivial. If there exist prices that generate excess supply of some good with no excess demand for any good, then the resulting allocation is feasible and hence Pareto superior to the endowment allocation. However, we have not been able to establish the converse proposition, namely, "If a WPO endowment allocation not optimal, then there exist prices that result in excess supply of some good with no excess demand for any good," or construct a counterexample. We are, therefore, led to consider the weak version of the violation of Walras' Law with a given WPO endowment allocation.

There is some connection between our notion of the failure of Walras' law and when monetary equilibria do or do not exist. Starting from a WPO endowment allocation, if a monetary equilibrium existed then the endowment allocation is clearly nonoptimal. At the monetary equilibrium prices, if the extra income was removed from agents who held fiat money and all goods were normal goods then there would be excess supplies of some goods but no excess demands for any goods. However, in this paper we do not make any assumptions regarding the normality of goods. Even if we did, the proposition that, "If a WPO endowment allocation is nonoptimal, then a monetary equilibrium exists," is not known to be generally true[3].

The connection between the failure of Walras' Law (in the weak sense) and the nonoptimality of WPO allocations is not obvious. The failure of Walras' Law has to do with allocations (not necessarily feasible) that lie on the demand curves of agents for some prices different from the prices supporting the WPO endowment allocation. Nonoptimality has to do with the existence of utility improving allocations which are feasible but may not necessarily lie on agents' demand curves. That is, there may not exist any Pareto superior plan that lies on the demand curve of each agent.

The rest of this paper is organized as follows. In section II we first describe a simple example that motivates the result and then exhibit the result in a quite general setting. Section III contains a proof of version (ii) of the result for the special case of two period lived agents and only one good at each date. A concluding summary is given in section IV. The Appendix contains the proof of a lemma which is crucial to our result.

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[3]There is a fair amount of literature on the existence of an optimal equilibrium (with or without valued fiat money) in the OLG model. See, Cass, Okuno and Zilcha [1980], Millan [1981], Okuno and Zilcha [1983], Benveniste and Cass [1986] and Burke [1987].

II. An Example

Consider the following OLG economy with one good at each date $t \geq 1$, and a sequence of two period lived agents indexed by their dates of birth $t \geq 0$. For $t \geq 1$, agent t has endowments (w_1, w_2) of goods dated t and (t+1), respectively, and maximizes utility given by $x_1(t)x_2(t+1)$ where $x_1(t)$ and $x_2(t+1)$ are agent t's consumption of date t and (t+1) goods, respectively. Agent 0 has endowment w_2 of date 1 good and maximizes consumption of date 1 good, $x_2(1)$. Suppose $w_1 > w_2$. Then, $p(t) = \beta^t$, $1 < \beta < w_1/w_2$, is a sequence of positive prices at which there is excess supply of every dated good. If $\beta = 1$, then there is excess supply of date 1 good but no excess demands. If $\beta > w_1/w_2$, then there is excess demand for every dated good. If $w_1 \leq w_2$, then for $\beta > 1$ there is excess demand for every dated good. However, in this case there is no β which can generate excess supply of some good with no excess demands.

The above example suggests several things. Firstly, it is likely that there will always exist price vectors that generate positive excess demands (with no excess supplies) whether or not the given WPO allocation is nonoptimal. However, it seems that there can exist price vectors generating positive excess supplies (with no excess demands) only if the WPO allocation is nonoptimal. Secondly, in the example, having excess supply of some dated good with no excess demands implies feasibility of the implied demands. Therefore, the demand allocations are automatically Pareto superior and hence demonstrate the nonoptimality of the WPO allocation. With many goods at each date, it may only be possible to generate excess supplies in value terms for the set of date t goods for each t. Since the resulting allocations are not necessarily feasible (though obviously utility improving), proving nonoptimality is not automatic. Thirdly, there may exist prices and corresponding demand allocations which are infeasible and can still be dominated in utility terms by means of feasible allocations. This can happen when $w_1 > w_2$ and β is only slightly larger than w_1/w_2. The resulting infeasible demand allocation can be dominated in utility terms by a feasible demand allocation corresponding to some β such that $1 \leq \beta < w_1/w_2$. We will, in fact, make use of this observation later on to construct Pareto superior allocations by improving on possibly infeasible demand allocations which themselves dominate the WPO (but, nonoptimal) endowment allocation.

We now proceed to establish the main result in a general setting.

A General Pure Exchange OLG Model

At each date $t \geq 1$, a finite set of two period lived agents G_t is born. These agents are alive at dates t and (t+1) only, and are referred to as members of generation t. At date t = 1, in addition to G_1 there is a finite set of initial old agents G_0 who are only alive during t = 1. At each date $t \geq 1$, there is a finite number n_t of perishable and freely disposable goods. We assume that the sequences $\{n_t\}$ and $\{\#G_t\}$ are each bounded above. The set of date t goods is denoted N_t. Further, and for specificity, we assume that any one period lived agents at date t (those who desire date t goods only) are counted as belonging to G_{t-1}.

We assume that the utility function of each agent, defined over the set of goods that are actually desired[5], satisfies the following assumptions: (i) twice continuously differentiable, (ii) strictly quasi-concave (iii) strictly monotonic. These assumptions will yield well-defined and continuously differentiable excess demand functions at positive prices. In addition, we will assume that preferences satisfy uniform curvature conditions that enable us to use the price characterization of optimality, as in Benveniste (1976,1986) and Okuno and Zilcha (1980). We start with an interior WPO endowment allocation that is bounded and bounded away from zero[6]. Let $\{\bar{p}_t\}$ be the associated sequence of support prices, where $\bar{p}_t \in R^{n_t}_{+}$ for $t \geq 1$ is the price vector of date t goods. Given our assumptions, $\bar{p}_t >> 0$ and hence we may define, $\bar{\theta}_t = \bar{p}_t/\|\bar{p}_t\|$ and $\bar{\beta}_t = \|\bar{p}_{t+1}\|/\|\bar{p}_t\|$ for all $t \geq 1$, where $\|.\|$ is the Euclidean norm. For $t \geq 1$, let $z^{ht1}(\theta_t, \beta_t\theta_{t+1}) \in R^{n_t}$ and $z^{ht2}(\theta_t, \beta_t\theta_{t+1}) \in R^{n_{t+1}}$ be the excess demand functions of agent $h \in G_t$ for goods dated t and (t+1), respectively. Further, let $z^{h0}(\theta_1) \in R^{n_1}$ be the excess demand function of agent $h \in G_0$ for date 1 goods. These excess demand functions are defined by taking the endowment of each agent to be his/her allocation as determined by the given WPO allocation. By summing over $h \in G_t$ for $t \geq 0$, we obtain the following aggregate excess demand

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[5]A particular good, say x, is desired by an agent if there exist two consumption bundles in the agent's consumption set which differ only in the amounts of good x, and such that the agent strictly prefers one over the other.

[6]Specifically, let $\bar{x}^{\alpha}_i$ be agent α's endowment of good i. Then $\bar{x}^{\alpha}_i \geq \varepsilon > 0$, for all i such that agent α desires good i, and for all agents α. Further, $\Sigma_{\alpha}\, \bar{x}^{\alpha}_i \leq W$ for all i. Note that since the endowment allocation is WPO, it assigns positive amounts of only those goods that are actually desired by an agent and zero of all other goods. Further, the generational structure makes it clear that the number of agents desiring any good is uniformly (across goods) bounded.

functions.

$$(2.1a)\quad z^{t1}(\theta_t,\beta_t\theta_{t+1}) = \sum_{h\in G_t} z^{ht1}(\theta_t,\beta_t\theta_{t+1}),$$

$$(2.1b)\quad z^{t2}(\theta_t,\beta_t\theta_{t+1}) = \sum_{h\in G_t} z^{ht2}(\theta_t,\beta_t\theta_{t+1}),$$

$$(2.1c)\quad z^{0}(\theta_1) = \sum_{h\in G_0} z^{h0}(\theta_1).$$

By summing the budget constraints (at equality, by virtue of monotonicity of preferences) of individual agents, we obtain the following.

$$(2.2a)\quad \theta_t z^{t1}(\theta_t,\beta_t\theta_{t+1}) + \beta_t\theta_{t+1}z^{t2}(\theta_t,\beta_t\theta_{t+1}) \equiv 0,\ t \geq 1,$$

$$(2.2b)\quad \theta_1 z^{0}(\theta_1) \equiv 0.$$

The general result we are aiming for is stated in the following proposition which will be proved after obtaining a preliminary (and more special) result.

Proposition 1:

An interior WPO endowment allocation that is bounded and bounded away from zero is nonoptimal if and only if for each positive and sufficiently small ε, there exist positive prices such that: (i) the resulting demand allocation is uniformly within ε of the endowment, and (ii) there is positive excess supply (in value terms) for some date t set of goods and no excess demand (in value terms) for any date t set of goods.

Proof. Later. □

A Preliminary Result

We start with the price characterization of optimality of WPO allocations as contained in Benveniste (1976, 1986) and Okuno and Zilcha (1980). Assume that for each agent, the closure of each indifference surface is contained in the strictly positive orthant of the consumption set and that the indifference surfaces satisfy the assumptions of strictness and smoothness. Further, the coefficients of strictness are bounded away from zero whereas the coefficients of smoothness are bounded above. Let,

$$(2.3a) \qquad \gamma_t = \left(\prod_{j=1}^{t} \bar{\beta}_j \right)^{-1}$$

$$(2.3b) \qquad \mu_t = \sum_{j=1}^{t} \gamma_j$$

Then, the WPO allocation with support prices $(\bar{\theta}_t, \bar{\beta}_t)$ is nonoptimal if and only if, $\lim \mu_t < \infty$.

Let $\lambda_t > 0$ and define

$$(2.4) \qquad \nu^t(\lambda_t) \equiv -\bar{\theta}_t z^{t1}(\bar{\theta}_t, \lambda_t \bar{\beta}_t \bar{\theta}_{t+1}), \; t \geq 1.$$

Since the sequence of prices $(\bar{\theta}_t, \bar{\beta}_t)$ support the endowment allocation for the excess demand functions, it follows that

$$(2.5) \qquad \nu^t(1) = 0, \; t \geq 1.$$

Further, it is easy to see from (2.2) that when prices are given by $\{\bar{\theta}_t, \lambda_t \bar{\beta}_t\}$ the value of excess demand of the date t set of goods is given by (actually, proportional to) the following.

(2.6a) $\bar{\theta}_1[z^0(\bar{\theta}_1) + z^{11}(\bar{\theta}_1,\lambda_1\bar{\beta}_1\bar{\theta}_2)] = -\nu^1(\lambda_1)$

(2.6b) $\bar{\theta}_{t+1}[z^{t2}(\bar{\theta}_t,\lambda_t\bar{\beta}_t\bar{\theta}_{t+1}) + z^{t+1,1}(\bar{\theta}_{t+1},\lambda_{t+1}\bar{\beta}_{t+1}\bar{\theta}_{t+2})] =$
$[\nu^t(\lambda_t)/\lambda_t\bar{\beta}_t] - \nu^{t+1}(\lambda_{t+1}),\ t \geq 1.$

Therefore, if we can find a sequence $\{\lambda_t\}$ such that

(2.7a) $\nu^1(\lambda_1) \geq 0,$

(2.7b) $\nu^{t+1}(\lambda_{t+1}) - [\nu^t(\lambda_t)/\lambda_t\bar{\beta}_t] \geq 0,\ t \geq 1,$

(2.7c) $\nu^t(\lambda_t) > 0$ for some $t \geq 1$

then the sequence of prices $\{\bar{\theta}_t,\lambda_t\bar{\beta}_t\}$ yield excess supply (in value terms) for some date t with no excess demand (in value terms) at any date. We now define,

(2.8a) $\eta^t(\lambda_t) \equiv \nu^t(\lambda_t)/(1-\lambda_t)$ for $\lambda_t > 0,\ \lambda_t \neq 1,\ t \geq 1$

(2.8b) $\eta^t(1) \equiv [\partial\nu^t(\lambda_t)/\partial\lambda_t]_{\lambda_t=1},\ t \geq 1.$

Note that $\eta^t(\lambda_t) > 0$ for $\lambda_t > 0$. Therefore, without loss of generality we may restrict $\lambda_t \in (0,1]$ if we are to satisfy the inequalities (2.7). Now let,

(2.9) $x_t \equiv \prod_{j=1}^{t} \lambda_j.$

We now state the following implication of earlier assumptions on preferences and endowments. An outline of the proof is given in the Appendix.

Lemma 1:

(2.10a) $\eta^t(\lambda_t) \leq b < \infty$ for $\lambda_t \in [0,1]$, $t \geq 1$

(2.10b) $0 < a \leq \eta^t(\lambda_t)$ for $\lambda_t \in [1-\delta,1]$ and some $\delta \in (0,1)$, $t \geq 1$.

Proof: In the Appendix. Conditions (2.10) arise essentially from the assumptions of uniform strictness and smoothness. □

Next, we state and prove a preliminary result.

Proposition 2

The WPO endowment allocation is nonoptimal (i.e., $\lim \mu_t < \infty$) if and only if there exists a sequence $\{\lambda_t\}$ with $\lambda_t \in (0,1]$ for all t, satisfying inequalities (2.7).

Proof:

(i) Necessity:

Suppose that $\lim \mu_t < \infty$. Define the sequence $\{\tilde{\nu}^t\}$ as follows.

(2.11) $$1/\{\tilde{\nu}^t\} \equiv (1/a)[1 + \sum_{k=0}^{\infty} (\prod_{j=0}^{k} \bar{\beta}_{t+j})^{-1}] = \sum_{s=t-1}^{\infty} \gamma_s/(a\gamma_{t-1}).$$

Given that $\lim \mu_t < \infty$, it is easy to show that $\lim\inf (1/\tilde{\nu}^t) = \infty$. Therefore, given $\varepsilon > 0$, there exists T large enough such that

(2.12) $$\tilde{\nu}^t < \varepsilon,\ t > T.$$

Now choose $\{\lambda_t\}$ such that

(2.13a) $\lambda_t = 1$, $t \leq T$

(2.13b) $\nu^t(\lambda_t) = \tilde{\nu}^t$, $\lambda_t < 1$, $t > T$.

We can choose ε in such a way that $\lambda_t \in [1-\delta,1]$ where δ is as given in (2.10b). The inequalities (2.7) are obviously satisfied for $t \leq T$. For $t > T$ we have from (2.11) that

$$(2.14) \qquad (\gamma_{t-1}/\tilde{\nu}^{t}) - (\gamma_{t}/\tilde{\nu}^{t+1}) = \gamma_{t-1}/a.$$

Dividing through by γ_{t-1} and using (2.3a), (2.8a), and (2.10b) we have

$$(2.15) \qquad (1/\tilde{\nu}^{t}) - (1/\bar{\beta}_t\tilde{\nu}^{t+1}) = (1/a) \geq 1/\eta^{t}(\lambda_t) = (1-\lambda_t)/\nu^{t}(\lambda_t) = (1-\lambda_t)/\tilde{\nu}^{t}.$$

Therefore,

$$(2.16) \qquad \tilde{\nu}^{t+1} \geq (\tilde{\nu}^{t}/\lambda_t\bar{\beta}_t)$$

and the result follows using (2.13b).

(ii) Sufficiency:

Suppose that a sequence $\{\lambda_t\}$ exists that satisfies (2.7) with $\lambda_t \in (0,1]$ for all t. Then $\lambda_t < 1$ for some t. So, let s be the first date such that $\lambda_s < 1$. From equations (2.3), (2.7), and (2.9) we have

$$(2.17a) \qquad \nu^{s}(\lambda_s) > 0$$

$$(2.17b) \qquad \nu^{t}(\lambda_t) \geq \nu^{s}(\lambda_s)x_{s-1}\gamma_{t-1}/(x_{t-1}\gamma_{s-1}) > 0, \ t \geq s.$$

Therefore, $\lambda_t < 1$ for all $t \geq s$. Now using (2.8) and (2.10a) we get,

$$(2.18)\quad 1-\lambda_t \geq \nu^s(\lambda_s)x_{s-1}\gamma_{t-1}/[x_{t-1}\gamma_{s-1}\eta^t(\lambda_t)] \geq \nu^s(\lambda_s)x_{s-1}\gamma_{t-1}/(bx_{t-1}\gamma_{s-1}),$$
$$t \geq s.$$

Multiplying both sides by x_{t-1}, we obtain,

$$(2.19)\quad x_{t-1} - x_t \geq d_s\gamma_{t-1},\ t \geq s$$

where,

$$(2.20)\quad d_s = \nu^s(\lambda_s)x_{s-1}/(b\gamma_{s-1}).$$

Summing (2.19) from t = s+1 through t = T+1, we have,

$$(2.21)\quad x_s - x_{T+1} \geq d_s(\mu_T - \mu_{s-1}).$$

Therefore,

$$(2.22)\quad \mu_T \leq \mu_{s-1} + (x_s - x_{T+1})/d_s.$$

Since $\lambda_t \in (0,1]$ for all t, $\{x_t\}$ is bounded and hence $\{\mu_T\}$ is bounded. Therefore, $\lim \mu_T < \infty$. □

The "only if" part of the above proposition (necessity) is quite satisfactory in terms of the general result. However, the "if" part (sufficiency) is not quite satisfactory because it assumes that excess supplies can be generated for each collection of dated goods by price vectors of the special form $(\bar{\theta}_t, \lambda_t\bar{\beta}_t)$. That is, the new price vector involves changing the interest factors but not the relative prices within each collection of dated goods. Thus, the above proposition is a special case of proposition 1 if the price vector is restricted to be of the form $(\bar{\theta}_t, \lambda_t \bar{\beta}_t)$. We now proceed to remedy this difficulty.

A More General Result

Let $(x_2(t,i), x_1(t,i))$ be the aggregate allocation of good $i \in N_t$ for the old and the young, respectively, at date t. We denote the vector of these allocations over all $i \in N_t$ by $(x_2(t), x_1(t))$. Now, let $(\hat{x}_2(t), \hat{x}_1(t))$ be the aggregate demand vector for the old and the young, respectively, at date t under the prices $(\hat{\theta}_t, \hat{\beta}_t)$. We denote by $\hat{\theta}_t(i)$ the price of good $i \in N_t$. Similarly, let $(\bar{x}_2(t), \bar{x}_1(t))$ be the WPO aggregate endowment allocation for the old and the young, respectively, at date t. For each $\varepsilon > 0$, the prices $(\hat{\theta}_t(\varepsilon), \hat{\beta}_t(\varepsilon))$ and the aggregate demands $(\hat{x}_2(t,\varepsilon), \hat{x}_1(t,\varepsilon))$ constitute on ε-excess supply allocation if

(2.23a) $\hat{\theta}_1(\varepsilon)[\bar{x}_1(1) - \hat{x}_1(1,\varepsilon)] \geq 0$

(2.23b) $\hat{\theta}_{t+1}(\varepsilon)[\bar{x}_1(t+1) - \hat{x}_1(t+1,\varepsilon)] \geq \hat{\theta}_t(\varepsilon)[\bar{x}_1(t) - \hat{x}_1(t,\varepsilon)]/\hat{\beta}_t(\varepsilon)$ for all $t \geq 1$

(2.23c) $\hat{\theta}_t(\varepsilon)[\bar{x}_1(t) - \hat{x}_1(t,\varepsilon)] > 0$ for some $t \geq 1$

(2.23d) $\sup_{t \geq 1} \sup_{i \in N_t} |\bar{x}_1(t,i) - \hat{x}_1(t,i,\varepsilon)| < \varepsilon.$

We can now prove the following.

Proposition 3.

Suppose that there is an $\bar{\varepsilon} > 0$ such that there exists an ε-excess supply allocation for any positive $\varepsilon < \bar{\varepsilon}$. Then the endowment allocation (assumed to be WPO) is not optimal.

Proof. Without loss of generality, we will assume that,

$$(2.24) \qquad \hat{\theta}_1(\varepsilon)[\bar{x}_1(1) - \hat{x}_1(1,\varepsilon)] > 0.$$

Therefore, it follows that

$$(2.25) \qquad \hat{\theta}_t(\varepsilon)[\bar{x}_1(t) - \hat{x}_1(t,\varepsilon)] > 0 \text{ for all } t \geq 1.$$

Since the endowment allocation $\{(\bar{x}_2(t),\bar{x}_1(t)),t\geq 1\}$ is bounded and bounded away from zero, and $\|\hat{\theta}_t(\varepsilon)\| = 1$ for all $t \geq 1$ and ε, and by virtue of (2.23d), we can always find a sequence of goods $\{i_t\}$ one at each date t and some ε sufficiently small such that,

$$(2.26) \qquad \bar{x}_1(t,i_t) > \hat{\theta}_t(\varepsilon)[\bar{x}_1(t) - \hat{x}_1(t,\varepsilon)]/\hat{\theta}_t(\varepsilon,i_t), \ t \geq 1.$$

Now consider the alternative allocation, $(x_2(t),x_1(t))$ where,

$$(2.27a) \qquad x_1(t,i) = \bar{x}_1(t,i) \text{ for } i \neq i_t, \ t \geq 1$$

$$(2.27b) \qquad x_1(t,i_t) = \bar{x}_1(t,i_t) - \hat{\theta}_t(\varepsilon)[\bar{x}_1(t) - \hat{x}_1(t,\varepsilon)]/\hat{\theta}_t(\varepsilon,i_t), \ t \geq 1.$$

$$(2.27c) \qquad x_2(t,i) = \bar{x}_2(t,i) + \bar{x}_1(t,i) - x_1(t,i), \ t \geq 1, \text{ all } i$$

This alternative allocation is clearly feasible. We will show that in terms of utilities it dominates the (possibly infeasible) allocation $(\hat{x}_2(t,\varepsilon),\hat{x}_1(t,\varepsilon))$ which, in turn, clearly dominates the endowment allocation in utility terms. This will establish the nonoptimality of the WPO endowment allocation.

It follows from (2.27) and (2.23) that

$$\hat{\theta}_t(\varepsilon)x_1(t) = \hat{\theta}_t(\varepsilon)\hat{x}_1(t,\varepsilon) \tag{2.28}$$

$$\begin{aligned}\hat{\theta}_{t+1}(\varepsilon)[x_2(t+1) - \hat{x}_2(t+1,\varepsilon)] &= \hat{\theta}_{t+1}(\varepsilon)[\bar{x}_2(t+1) - \hat{x}_2(t+1,\varepsilon) + \\ &\quad \bar{x}_1(t+1) - \hat{x}_1(t+1,\varepsilon) + \hat{x}_1(t+1,\varepsilon) - x_1(t+1)] \\ &= \hat{\theta}_{t+1}(\varepsilon)[\bar{x}_1(t+1) - \hat{x}_1(t+1,\varepsilon)] - \\ &\qquad \hat{\theta}_t(\varepsilon)[\bar{x}_1(t) - \hat{x}_1(t,\varepsilon)]/\hat{\beta}_t(\varepsilon) \\ &\geq 0.\end{aligned} \tag{2.29}$$

Following Benveniste (1976, 1986) and Okuno and Zilcha (1980) we define the aggregate upper contour set of the t^{th} generation at the "^" allocation as $P_t(\hat{x}_1(t),\hat{x}_2(t+1))$. An allocation to the t^{th} generation, $(x_1(t),x_2(t+1))$ is in the set $P_t(\hat{x}_1(t),\hat{x}_2(t+1))$ if:

(i) $(x_1(t),x_2(t+1)) \leq (\bar{x}_1(t)+\bar{x}_2(t),\ \bar{x}_1(t+1)+\bar{x}_2(t+1))$

(ii) there exist $(x_1^h(t),x_2^h(t+1))$ for all $h \in G_t$ such that,

$$\sum_{h\in G_t}(x_1^h(t),x_2^h(t+1)) \leq (x_1(t),x_2(t+1))$$

and,

$$u_t^h(x_1^h(t),x_2^h(t+1)) \geq u_t^h(\hat{x}_1^h(t),\hat{x}_2^h(t+1)) \text{ for all } h \in G_t,$$

where $u_t^h(.)$ is the utility function of $h \in G_t$.

The set $P_t(.)$ is smooth at $(\hat{x}_1(t),\hat{x}_2(t+1))$ with respect to $(\hat{\theta}_t,\hat{\beta}_t\hat{\theta}_{t+1})$ if there is $M_t > 0$ such that, if $(x_1(t),x_2(t+1))$ satisfies,

$$(2.30)\qquad \hat{\beta}_t\hat{\theta}_{t+1}(x_2(t+1)-\hat{x}_2(t+1)) \geq \hat{\theta}_t(\hat{x}_1(t)-x_1(t)) + M_t[\hat{\theta}_t(\hat{x}_1(t)-x_1(t))]^2/\|\hat{\theta}_t\hat{x}_1(t)\|,$$

then,

$$(x_1(t),x_2(t+1)) \in P_t(.).$$

By virtue of the curvature assumptions on preferences mentioned previously (see, for example, Benveniste 1986) each of the sets $P_t(.)$ is smooth. Further, if we define the coefficient of smoothness for $P_t(.)$ as, $\hat{M}_t \equiv \inf \{M_t | M_t$ satisfies (2.30)$\}$, then the sequence $\{\hat{M}_t\}$ is bounded above, by some $M < \infty$.

It is now obvious from (2.28) and (2.29) that the feasible allocation $(x_2(t),x_1(t))$ satisfies (2.30) for all t. Therefore, there exist $(x_1^h(t),x_2^h(t+1))$ for $h \in G_t$, $t \geq 1$ such that,

$$(2.31)\qquad \sum_{h\in G_t \cup G_{t-1}} (x_1^h(t) + x_2^h(t)) = x_1(t) + x_2(t) = \bar{x}_1(t) + \bar{x}_2(t),\quad t \geq 1$$

$$(2.32)\qquad u_t^h(x_1^h(t),x_2^h(t+1)) \geq u_t^h(\hat{x}_1^h(t),\hat{x}_2^h(t+1)) > u_t^h(\bar{x}_1^h(t),\bar{x}_2^h(t+1))$$

for all $h \in G_t$, $t \geq 1$.

Furthermore, for generation G_0, $x_2(1) > \bar{x}_2(1)$ and hence there exist $x_2^h(1)$ for $h \in G_0$ such that,

$$(2.33)\qquad \sum_{h\in G_0} x_2^h(1) = x_2(1) > \bar{x}_2(1)$$

$$(2.34)\qquad u^h(x_2^h(1)) \geq u^h(\bar{x}_2^h(1))$$

This proves the nonoptimality of the WPO endowment allocation $(\bar{x}_2(t),\ \bar{x}_1(t))$. □

We can now give a proof of proposition 1.

Proof of Proposition 1:

The proof of the sufficiency part follows from Proposition 3, while the proof of the necessity part follows from the necessity part of Proposition 2. □

III. One Good Models with Arbitrary Endowments--A Stronger Result

In this section we consider the following alternative version of the relationship between the failure of Walras' Law and nonoptimal equilibria: "Given an arbitrary endowment vector (not necessarily WPO), there exist positive prices that lead to excess supply (in value terms) for some date t set of goods with no excess demand for any date t set of goods, if and only if there exists a nonoptimal competitive equilibrium." We are able to prove the above proposition for the one good (at each date) case. The result will be proved as a corollary to Proposition 2.

Let R_t be the gross interest rate from t to (t+1) and let $S_t(.)$ be the aggregate savings function for generation t. In terms of our previous notation, $\theta_t = 1$, $R_t = 1/\beta_t$ and $S_t(.) = \nu^t(.)$. The sequence of interest rates R_t generate excess supplies, if

(3.1a) $S_1(R_1) \geq 0$

(3.1b) $S_{t+1}(R_{t+1}) \geq R_t S_t(R_t)$, $t \geq 1$

(3.1c) $S_t(R_t) > 0$ for some $t \geq 1$.

The sequence of interest rates $\{R_t\}$ is an equilibrium if

(3.2) $S_t(R_t) = 0, \ t \geq 1.$

In general, there may be many R_t values satisfying (3.2) for each t. For each t, we will focus on the smallest R_t that satisfies (3.2). This will be denoted as $\bar{R}_t$ and will play the role of $1/\bar{\beta}_t$ in Proposition 2.

Our assumptions on preferences and endowments imply the following

Lemma 2.

For each $t \geq 1$, $S_t(R)$ is continuously differentiable for $R > 0$ and satisfies,

$S_t(R) \leq \bar{S} < \infty, \ R > 0, \ t \geq 1$

$S_t(R) \to -\infty$ as $R \to 0, \ t \geq 1.$

Proof. Straightforward. □

Now, let

(3.3a) $\lambda_t \equiv \beta_t/\bar{\beta}_t = \bar{R}_t/R_t$

(3.3b) $\nu^t(\lambda_t) \equiv S_t(R_t) = S_t(\bar{R}_t/\lambda_t)$

(3.3c) $\eta^t(\lambda_t) \equiv \nu^t(\lambda_t)/(1-\lambda_t) = R_t S_t(R_t)/(R_t-\bar{R}_t), \ \lambda \neq 1.$

It then follows that $\nu^t(1) = 0$ for all $t \geq 1$. Further from Lemma 2 and the fact that $\bar{R}_t$ is the smallest R_t satisfying (3.2), we have that

(3.4) $\eta^t(1) \equiv [\partial\nu^t(\lambda_t)/\partial\lambda_t]_{\lambda_t=1} \geq 0, \ t \geq 1.$

We now make an assumption that is a strengthening of (3.4) and parallels conditions (2.10) of Lemma 1.

Assumption 1. There is a $\delta \in (0,1)$ such that

$$(3.5) \qquad 0 < a \leq \eta^t(\lambda_t) \leq b < \infty \text{ for } \lambda_t \in [1-\delta,1],\ t \geq 1.$$

We can now prove the following

Proposition 4.

There exists a nonoptimal equilibrium $\{\hat{R}_t\}$ if and only if there exists a sequence of interest rates $\{R_t\}$ that generates excess supply.

Proof.

(i) Necessity:

Suppose there exists a nonoptimal equilibrium. Then, the equilibrium sequence $\bar{R}_t$ is also nonoptimal because $\bar{R}_t \leq \hat{R}_t$, $t \geq 1$. Now, we can simply apply part (i) of the proof of Proposition 2 to construct a $\{R_t\}$ sequence that generates excess supplies.

(ii) Sufficiency:

Let $\{R_t\}$ be a sequence that generates excess supplies and let $\lambda_t = \bar{R}_t/R_t$. From Lemma 2, we have that $\lambda_t < 1$ for all $t \geq 1$. If $\lambda_t \in [0,1-\delta]$ where δ is as given in assumption 1, then

$$(3.6) \qquad \eta^t(\lambda_t) = \nu^t(\lambda_t)/(1-\lambda_t) \leq \bar{S}/\delta.$$

Putting the above together with assumption 1, we have

$$(3.7) \qquad \eta^t(\lambda_t) = \max[b,\bar{S}/\delta] \text{ for all } \lambda_t \in [0,1].$$

We can now apply the sufficiency part of the proof of Proposition 2 to show that the sequence $\{\bar{R}_t\}$ is nonoptimal. Note that this proof only utilizes the upper bound on $\eta^t(.)$. □

IV. Conclusion

We have shown that a weakly Pareto optimal endowment allocation in a general pure exchange overlapping generations model is nonoptimal if and only if there is a failure of Walras' Law in its neighborhood. That is, there exist positive prices such that: (i) there is excess supply (in value terms) for some date t set of goods without any excess demand (in value terms) for any date t set of goods, and (ii) the demand allocation is uniformly and arbitrarily close to the endowment allocation. Excess supplies are defined relative to the given WPO allocation as endowments. For a more special model consisting of only one good at each date, we were able to prove the following stronger result. Given an arbitrary endowment allocation (not necessarily WPO) there exist prices generating excess supply at some dates with no excess demands if and only if there exists a nonoptimal equilibrium.

Appendix

Here we show that conditions (2.10) of Lemma 1 on the excess demand function for first period consumption (in value terms) are implied by the curvature assumptions of uniform strictness and smoothness on indifference surfaces. First, we do this for the case of a single consumer in each generation and one good at each date. Then, we indicate the generalization to many goods at each date and then many consumers in each generation.

Let $x_1(t)$, $x_2(t+1)$ be consumption when young and when old, respectively. We will represent preferences by the following equation of a typical indifference curve, parameterized by the level of utility, u(t),

(A1) $$x_2(t+1) = f^t(x_1(t),u(t)).$$

Let $(\bar{x}_1(t),\bar{x}_2(t+1))$ and $\bar{u}(t)$ be the endowment point and the associated utility level, respectively. Let $\bar{X}$ be an upper bound for the sequence of aggregate endowments $\{\bar{x}_1(t)+\bar{x}_2(t)\}$ and let $\underline{X}$ be such that, $f^t(\underline{X},\bar{u}(t) > \bar{X}$ for all t.

We assume the following

(A2.a) $f^t(x,u) > 0$ for all u, and $x > 0$

(A2.b) $\lim_{x\to 0} f^t(x,u) = \infty$, $\lim_{x\to\infty} f^t(x,u) = 0$

(A2.c) $f_1^t(x,u) < 0$ and $f_2^t(x,u) > 0$ for all u, and $x > 0$

(A2.d) $\sup_{t\geq 1} \sup \{-f_1^t(x,u) \mid x \in [\underline{X},\bar{X}],\ u \geq \bar{u}(t)\} < \infty$

$\inf_{t\geq 1} \inf \{-f_1^t(x,u) \mid x \in [\underline{X},\bar{X}],\ u \geq \bar{u}(t)\} > 0$

(A2.e) $f^t(x,u)$ is twice continuously differentiable w.r.t. x and satisfies the bounds, $0 < m \leq f^t_{11}(x,u) \leq M < \infty$ for all $x \in [\underline{X},\bar{X}]$ and $u \geq \bar{u}(t)$.

Let $\bar{\beta}_t$ be the discount factor supporting the endowment allocation $(\bar{x}_1(t),\bar{x}_2(t+1))$. Therefore,

$$\text{(A3)} \qquad \bar{\beta}_t = -1/f^t_1(\bar{x}_1(t),\bar{u}(t)).$$

By (A2.d), $\bar{\beta}_t$ is bounded and bounded away from zero.

By applying Taylor's theorem to (A1) and using (A3), we obtain

$$\text{(A4)} \qquad f^t(x_1(t),\bar{u}(t)) = f^t(\bar{x}_1(t),\bar{u}(t)) - (x_1(t)-\bar{x}_1(t))/\bar{\beta}_t + (1/2)(x_1(t)-\bar{x}_1(t))^2 f^t_{11}(\hat{x}_1(t),\bar{u}(t))$$

where $\hat{x}_1(t)$ is between $x_1(t)$ and $\bar{x}_1(t)$.

Now, let $\lambda_t < 1$ and let $(x_1(t),x_2(t+1))$ be the demand point corresponding to $\lambda_t\beta_t$. Obviously, $x_1(t) < \bar{x}_1(t)$, $x_2(t+1) > \bar{x}_2(t+1)$, and $u^t(x_1(t),x_2(t+1)) > \bar{u}(t)$. For λ_t sufficiently close to unity, $x_1(t) \in [\underline{X},\bar{X}]$. Therefore, from (A4) and (A2.e), we obtain

$$\text{(A5)} \qquad x_2(t+1) > f^t(x_1(t),\bar{u}(t)) > \bar{x}_2(t+1) - (x_1(t)-\bar{x}_1(t))/\bar{\beta}_t + (m/2)(x_1(t)-\bar{x}_1(t))^2.$$

The budget constraint is,

$$\text{(A6)} \qquad x_1(t)-\bar{x}_1(t) + \lambda_t\bar{\beta}_t(x_2(t+1) - \bar{x}_2(t+1)) = 0.$$

Using (A6) to substitute for $(x_2(t+1) - \bar{x}_2(t+1))$ in (A5) and manipulating, we get

$$0 < (\bar{x}_1(t)-x_1(t))/(1-\lambda_t) < 2/(m\lambda_t\bar{\beta}_t).$$

Thus, the slope of the demand curve for first period consumption at $\lambda_t = 1$ is bounded above. Further, for sufficiently small $\delta \in (0,1)$ the expression $(\bar{x}_1(t)-x_1(t))/(1-\lambda_t)$ is bounded above by $2/[m(1-\delta)\bar{\beta}_t]$ for all $\lambda_t \in [1-\delta,1]$. Moreover, if $\lambda_t \in [0,1-\delta]$, then $(\bar{x}_1(t)-x_1(t))/(1-\lambda_t) \le \bar{x}_1(t)/\delta \le \bar{X}/\delta$. Therefore, $(\bar{x}_1(t)-x_1(t))/(1-\lambda_t)$ is bounded above for all $\lambda_t \in [0,1]$.

Now, if we let $x_1^c(\lambda_t,t)$ be the compensated demand curve for first period consumption corresponding to the utility level $\bar{u}(t)$ then we have,

$$f_1^t(x_1^c(\lambda_t,t),\bar{u}(t)) \equiv -1/(\lambda_t\bar{\beta}_t).$$

The slope of the ordinary demand curve for first period consumption at $\lambda_t = 1$ equals the slope of the compensated demand curve at $\lambda_t = 1$. Therefore, we have,

$$[\partial x_1(t)/\partial\lambda]_{\lambda=1} = [\partial x_1^c(t)/\partial\lambda]_{\lambda=1} = 1/[\bar{\beta}_t f_{11}^t(\bar{x}_1(t),\bar{u}(t))] > 1/(\bar{\beta}_t M) > 0.$$

Consequently, there is some neighborhood $[1-\delta,1]$ with $\delta \in [0,1]$ such that $(\bar{x}_1(t)-x_1(t))/(1-\lambda_t)$ is bounded away from zero for all $\lambda_t \in [1-\delta,1]$.

We now show how to generalize the above conclusion to the case of many goods at each date. The young consumer maximizes his/her utility, $u^t(x_1(t),x_2(t+1))$ subject to:

$$\bar{\theta}_1(t)(x_1(t)-\bar{x}_1(t)) + \lambda_t\bar{\beta}_t\bar{\theta}_2(t+1)(x_2(t+1) - \bar{x}_2(t+1)) = 0.$$

where $\bar{\theta}_1(t)$ and $\bar{\theta}_2(t+1)$ are the (normalized) price vectors of goods consumed when young and when old, respectively. Since the prices $(\bar{\theta}_1(t),\bar{\beta}_t\bar{\theta}_2(t+1))$ are support prices for the endowment allocation $(\bar{x}_1(t),\bar{x}_2(t+1))$ we know that

$$[(x_1(t),x_2(t+1))]_{\lambda=1} = (\bar{x}_1(t),\bar{x}_2(t+1)).$$

The utility maximization problem can be solved in two steps. First, let $v_1(t)$ and $v_2(t+1)$ be nonnegative scalars and define, $v_1(t) = \bar{\theta}_1(t)\bar{x}_1(t)$ and $v_2(t+1) = \bar{\theta}_2(t+1)\bar{x}_2(t+1)$. Further, let

$$W^t(v_1(t),v_2(t+1),\bar{\theta}_1(t),\bar{\theta}_2(t+1)) = \max u^t(x_1(t),x_2(t+1)) \text{ subject to: }$$
$$\bar{\theta}_1(t)x_1(t) \le v_1(t),\ \bar{\theta}_2(t+1)x_2(t+1) \le v_2(t+1).$$

Then, solve

$$\max W^t(v_1(t),v_2(t+1),\bar{\theta}_1(t),\bar{\theta}_2(t+1)) \text{ subject to: }$$
$$v_1(t)-\bar{v}_1(t) + \lambda_t\bar{\beta}_t(v_2(t+1) - \bar{v}_2(t+1)) = 0.$$

For fixed $(\bar{\theta}_1(t),\bar{\theta}_2(t+1))$ the intermediate utility function $W^t(.)$ can be represented by the equations of the indifference contours,

$$v_2(t+1) = f^t(v_1(t),W^t,\bar{\theta}_1(t),\bar{\theta}_2(t+1))$$

which is the analog of (A1). From now on the analysis proceeds as in the one good at each date case. The excess supply (in value terms) function for the young corresponds to $(\bar{v}_1(t)-v_1(t))$ as a function of λ_t--this is the function $\nu^t(\lambda_t)$ used in the text.

Lastly, the generalization to many consumers in each generation can be carried out exactly as in Benveniste (1986).

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