The Observational Equivalence of Natural and Unnatural Rate Theories of Macroeconomics

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September 1975

Working Paper #: 48

Rsch. File #: 299.1

THE OBSERVATIONAL EQUIVALENCE OF NATURAL AND UNNATURAL RATE THEORIES OF MACROECONOMICS

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September 1975

Note: John Geweke, Christopher Sims, Gary Skoog, and Neil Wallace contributed valuable comments on an earlier draft of this paper. The Federal Reserve Bank of Minneapolis financed this paper. The views expressed herein are solely those of the author and do not necessarily represent the views of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

The usual proof that Friedman's simple k-percent growth rule for the money supply is suboptimal comes from mechanically manipulating a reduced form equation. Those manipulations in general show that pursuing a rule with feedback from current economic conditions to the money supply is better than following Friedman's advice. To be valid, the proof requires that as written in one particular way the reduced form equation will remain unaltered when the monetary authority departs from the old "rule" used during the estimation period and follows a new one. Here I point out that there are always alternative ways of writing the reduced form, one being observationally equivalent with the other, so that each is equally valid in the estimation period. If one assumes that the first form is invariant when the policy rule is changed, the proof of the superiority of rules with feedback over Friedman's rule goes through. But if one assumes that it is the reduced form as written in the second way that remains unchanged, the proof that Friedman is wrong does not obtain, instead the implication being that Friedman's rule does as well as any other deterministic feedback rule, and better than a stochastic rule. Therefore, estimates of reduced forms alone will not permit one to settle the difference between Friedman and advocates of rules with feedback. Given any set of reduced form estimates, there is an invariance assumption that will permit a member of either camp to make his point. In effect, then, this paper poses the question: Does the view that Friedman's k-percent feedback rule is as good as any other deterministic feedback rule place any restrictions on reduced forms? The answer is no. This is distressing, since for a given sampling interval and estimation period, the reduced form estimates summarize everything that the data can ever tell us. To rule on the policy issue thus requires bringing to bear theoretical considerations or doing empirical work of a kind considerably more subtle than that directed solely at estimating reduced forms. In effect, it is necessary to get some evidence on what sort of invariance assumption is the most realistic one to impose. How one does that is a delicate, though not entirely intractable task, the discussion of which is outside the scope of this paper.

This paper is in the nature of a footnote to Lucas' important critique of econometric policy evaluation [3]. Lucas emphasized the critical invariance assumption behind the usual argument for rules with feedback, and showed how that invariance assumption fails to hold in models with rational expectations. As extreme examples of how wrong the standard invariance assumption could be, Lucas [4] and Sargent and Wallace [6] have constructed particular structural models in which one deterministic rule is as good as another, so that the standard proof of the suboptimality of Friedman's rule fails spectacularly in those examples. Those examples were dependent on particular structural setups. The point of this note is that for any estimated reduced form, there is an invariance assumption which if imposed delivers the conclusion that one deterministic rule is equivalent with any other. In effect, then, this note displays some mechanical equivalencies that force one to stumble upon Lucas' observations about the limits of the usual applications of optimal control theory to macroeconometric models.

A casual reader of Marschak's classic paper [4] would perhaps regard the major contention of this paper, that reduced forms alone cannot settle the policy rules controversy, as being obvious. However, applications of optimal control to macroeconometric models do purport to extract implications about the optimal feedback rule solely from estimated reduced forms.

For simplicity, I deal with a bivariate model. I assume that during the estimation period, two variables, y_t and m_t, were described as a realization from a stationary, indeterministic stochastic process. The variable y_t measures some "goal" variable like unemployment or GNP. The variable m_t represents a potential policy instrument. I assume that during the estimation period m_t was exogenous with respect to y, and that m_t caused y_t, in Granger's sense [1]. This means that the (y, m) process can be represented in the particular (Wold) moving average form[1]

$$y_t = \alpha(L)\ \varepsilon_t + \beta(L)\ \eta_t \tag{1a}$$

$$m_t = \gamma(L)\ \varepsilon_t \tag{1b}$$

where

$$\alpha(L) = \sum_{j=0}^{\infty} \alpha_j\ L^j , \quad \beta(L) = \sum_{j=0}^{\infty} \beta_j\ L^j ,$$

and

$$\gamma(L) = \sum_{j=0}^{\infty} \gamma_j\ L^j ;$$

L is the lag operator $(L^n x_t \equiv x_{t-n})$; and η_t and ε_t are mutually uncorrelated and serially uncorrelated random variables with means of zero and finite variances σ_η^2 and σ_ε^2, respectively. (I have omitted constants and any deterministic terms from representation (1), which can be included without affecting the argument.)[2]

The assumption that the Wold representation has the triangular form of (1), i.e., the assumption that m is exogenous and "causes" y, means that under one pretty general additional condition, a final form regression of y on m can be consistently estimated by least squares. In particular, suppose that $\gamma(L)$ is invertible so that m_t has the autoregressive representation

$$\gamma^{-1}(L)\, m_t = \varepsilon_t \quad ,$$

where $\gamma^{-1}(L)$ is a one-sided polynomial in the lag operator L. Substituting the above equation into (1a) gives

$$y_t = \alpha(L)\, \gamma^{-1}(L)\, m_t + \beta(L)\, \eta_t$$

(2) $$y_t = h(L)\, m_t + \beta(L)\, \eta_t$$

where $h(L) = \alpha(L)\, \gamma^{-1}(L)$ *is a one-sided polynomial in the lag operator* L. Equation (2) is a "final form" for y in terms of m and is consistently estimated by least squares, since the η_t process is orthogonal to the m_t process. Assuming that $\beta(L)$ has a one-sided inverse, the "reduced form" for y_t can be obtained as[3]

$$\beta(L)^{-1}\, y_t = \beta(L)^{-1}\, h(L)\, m_t + \eta_t \quad ,$$

or

(3) $$y_t = \sum_{i=0}^{\infty} a_i\, m_{t-i} + \sum_{i=0}^{\infty} b_i\, y_{t-i-1} + \eta_t \quad ,$$

where

$$a(L) = \sum_{i=0}^{\infty} a_i\, L^i = \beta(L)^{-1}\, h(L)$$

and

$$(1 - L\, b(L)) = (1 - \sum_{i=0}^{\infty} b_i\, L^{i+1}) = \beta(L)^{-1} \quad .$$

The reduced form (3) expresses y in terms of current and lagged m's and lagged y's, with a disturbance that is serially uncorrelated and orthogonal to the variables on the right-hand side of the equation. Thus the reduced form (3) can be consistently estimated by least squares.

Write (3) as

$$y_t = a(L)\, m_t + b(L)\, y_{t-1} + \eta_t \quad . \tag{4}$$

It is now easy to illustrate the elements of the usual argument that it is optimal for the policy authority to set m via a rule with feedback from lagged y's to current m. Suppose that the authority's goal is to set m in order to minimize the variance of y. Suppose that the parameters of the reduced form (4) will remain unaltered when the authority abandons (1b), which described policy during the estimation period, and implements a new feedback rule. (Suppose also that $a(L)$ has a one-sided inverse under convolution, which isn't really necessary but rules out unseemly "instrument instability" problems.) Then it is straightforward to show that the authority would minimize the variance of y by using the feedback rule

$$a(L)\, m_t = - b(L)\, y_{t-1} \quad , \tag{5}$$

which is a rule for setting m_t as a function of lagged m's and lagged y's. In general, some of the b_j's are not zero, so that it is optimal for the authority to incorporate feedback from lagged y's to m, presumably to "lean against the wind." So a rule without feedback is suboptimal.

As Lucas has emphasized [3], a key assumption in the above argument for rules with feedback is that the parameters of the reduced form (4) remain unchanged when the authority abandons the rule used during the estimation period and uses a new one. Lucas argued that if the reduced form incorporates the influence of peoples' expectations and if expectations are formed in a well informed or "rational" way, that assumption is not appropriate. Here I point out that there always seems to be an interpretation of the reduced form (4) which completely vitiates the preceding demonstration of the superiority of rules with feedback.

I begin by noting that from (1b) $\gamma_o \varepsilon_t = m_t - E_{t-1} m_t$ where $E_{t-1} m_t$ is the *mathematical expectation of* m_t conditioned on past m's and past y's. In this case, since m is exogenous in the estimation period, lagged y's don't help explain m once lagged m's are taken into account, so that in the estimation period

$$E_{t-1} m_t \equiv E[m_t | m_{t-1}, m_{t-2}, \ldots, y_{t-1}, y_{t-2}, \ldots]$$

$$= E[m_t | m_{t-1}, m_{t-2}, \ldots].$$

The random variable $\gamma_o \varepsilon_t$ is the "innovation" of m_t, the part that can't be predicted on the basis of past m's (or past y's). Now substituting

$$\varepsilon_t = \frac{1}{\gamma_o} (m_t - E_{t-1} m_t)$$

into (1a) and rearranging gives

$$\beta(L)^{-1} y_t = \frac{1}{\gamma_o} \beta(L)^{-1} \alpha(L) (m_t - E_{t-1} m_t) + \eta_t$$

or

(6)
$$y_t = c(L)(m_t - E_{t-1} m_t) + b(L) y_{t-1} + \eta_t$$

where

$$\frac{1}{\gamma_o} \beta(L)^{-1} \alpha(L) = c(L) = \sum_{i=0}^{\infty} c_i L^i \quad ,$$

and where $b(L)$ is as defined under (3).[4] Equation (6) is an alternative version of the reduced form that is equivalent with the version (3) or (4) from the point of view of representing things during the estimation period. Notice that (4) and (6) have identical residuals and so fit equally well.

If we assume that the reduced form (6) remains unaltered when the authority abandons (1b) and adopts a deterministic feedback rule giving m as an exact function of past m's and past y's, a strong sort of "neutrality" result emerges. For under any deterministic feedback rule, say one of the form

$$(7) \qquad m_t = s_1 m_{t-1} + s_2 m_{t-2} + \dots + r_1 y_{t-1} + r_2 y_{t-2} + \dots \quad ,$$

it is true that

$$(8) \qquad E_{t-1} m_t = m_t \quad ,$$

identically in t.

Substituting (8) into (6) gives

$$y_t = b(L)\, y_{t-1} + \eta_t \quad ,$$

which is an autoregressive representation for y that holds regardless of the values of the particular parameters $s_1, s_2, \dots r_1, r_2, \dots$ of the feedback rule (7) selected by the authority. The assumption that it is the reduced form representation (6) that remains unchanged as the policy rule is altered leads to the conclusion that one deterministic feedback rule is as good as any other. There is thus no reason to expect that the authority can do better than it can by implementing the x-percent growth rule recommended by Friedman.

The preceding "neutrality" demonstration rests on the arbitrary assumption that it is the reduced form representation (6) that remains unaltered when the authority institutes a new policy rule outside the estimation period. Of course, the earlier demonstration of the superiority of a rule with feedback depended on the equally arbitrary but different assumption that it was the reduced form representation (4) that remained unchanged from the estimation period even once the new policy rule was instituted. From the viewpoint of extracting policy implications, assuming invariance for the reduced form representation (4) or (6) thus gives drastically different implications. Yet from the point of view of representing the reduced form during the estimation period, (4) and (6) are exactly equivalent. This is what leads me to the conclusion that the empirical evidence from a single estimation period alone, which can be completely summarized by (4) or (6) and an autoregression for m, can never settle things between advocates of rules with feedback and advocates of rules without feedback.

Perhaps this could be dismissed as a mere curiosity if macroeconomists agreed that as between (4) and (6) one of these ways of writing the reduced form is much more likely to remain unchanged when policy changes. The problem is that there is no such agreement. While in the past most macroeconomists have regarded (4) as invariant under changes in the policy rule, that assumption depends critically on the assumption that peoples' expectations are formed using fixed-weight, autoregressive schemes that in general are not "rational." Lucas [3] has argued forcefully against that assumption, but it remains true that the assumption still underlies most macroeconometric policy evaluation, and is an essential element of most applications of control theory to macroeconometric models. On the other hand, reduced form representations resembling

(6) are supposed to be invariant under changes in the policy rule according to some structural macroeconomic models incorporating rational expectations and Lucas' formulation of the aggregate supply function [2].[5] In such models, current and maybe lagged innovations in the money supply are what agents respond to.

The upshot is that the invariance of neither (4) nor (6) to changes in the policy rule would now command a concensus among macroeconomists. The current state of macroeconomic theory seems to me to be very far from supplying a reliable basis for ruling out one of these invariance assumptions in favor of the other. For that reason, I believe that the observational equivalence of (4) and (6) provides some cause to be circumspect about economists' ability now to be sure that rules with feedback clearly dominate rules without feedback (or vice versa).

The reader may wonder whether my assumptions that m is exogenous with respect to y and that m causes y in effect rig things in the preceding argument. They don't. Those assumptions were made to guarantee that a y-on-m reduced form was identifiable and estimable. The estimability of such a reduced form is a sine qua non for the usual argument that rules with feedback dominate rules without feedback. One alternative set of assumptions would have been that y and m caused each other, so that there was mutual feedback between y and m. Only under special circumstances, an instance of which is analyzed in appendix B, is a y-on-m reduced form identified in a system with mutual feedback. As appendix B illustrates, for systems with mutual feedback that are identifiable through a priori restrictions, there obtains the same observational equivalence as analyzed in the text. Another alternative assumption would have been that y was exogenous with respect to m in the estimation period. I have elsewhere advanced the notion that the hypothesis of exogeneity of certain goal variables y (e.g., unemployment) with respect to certain policy instruments m (e.g., the money supply) is a naive, model-free way of stating the natural rate hypothesis.[6] Exogeneity of y with respect to m can readily be shown to be compatible with the notion that one deterministic rule is as good as any other, on a certain invariance assumption. On the other hand, it is straightforward to show that an alternative invariance assumption could be imposed that would imply that Friedman's rule is suboptimal. This is shown in appendix A.

The present argument only shows how reduced forms estimated for a given sampling interval (i.e., quarterly or monthly) over a given estimation period cannot settle the policy rules controversy. That does not mean that there is no way that empirical evidence can be brought to bear on the question. Presumably, by estimating reduced forms for various subperiods or countries across which policy rules differed systematically, light can be shed on what way of writing the reduced form remains invariant.[7] Alternatively, by studying data more and less finely aggregated over time, different implications of our two invariance assumptions might be extracted and tested. Both of these paths involve considerable subtleties. Very little satisfactory evidence has yet been assembled along either path.

Appendix A

HOW "CLASSICAL" MODELS CAN BE INTERPRETED IN "KEYNESIAN" WAYS

This appendix illustrates how for classical models in which real variables are econometrically exogenous with respect to policy variables there is a way of writing the reduced form which, if invariant under rules changes, implies that rules with feedback are optimal. Therefore, evidence that real variables are econometrically exogenous with respect to policy variables, which I have argued [7] is a strong, model-free way of stating the natural rate hypothesis, has "classical" policy implications on one kind of invariance assumption, but does not on another. The argument in this appendix is thus the other half of the observational equivalence dilemma, since here I start with a model originally thought to be very "neutral" or classical and produce an invariance assumption that rationalizes rules with feedback.

Consider a "structural" model of the form

$$\lambda(L)\ y_t = \gamma(m_t - E_{t-1}\ m_t) + u_t \qquad \gamma > 0 \tag{A1}$$

$$m_t = d(L)\ \varepsilon_t \tag{A2}$$

where

$$\lambda(L) = \sum_{i=0}^{\infty} \lambda_i L^i, \quad d(L) = \sum_{i=0}^{\infty} d_i L^i \ ;$$

$\lambda(L)$ and $d(L)$ both have one-sided inverses under convolution; and u_t and ε_t are mutually and serially independent random variables, with means of zero and finite variances. The model (A1), (A2) is a two-variable example of the "classical" model described by Sargent [7]. Here innovations in m (money) produce sympathetic movements in y. In this model, y is exogenous with respect to m, and m is exogenous with respect to y. Though y and m are correlated, neither one helps predict the other. To see that m is exogenous with respect to y, notice that (A1) and (A2) can be rearranged in the triangular Wold representation

$$y_t = \lambda^{-1}(L)\, \gamma\, d_o\, \varepsilon_t + \lambda^{-1}(L)\, u_t \tag{A3}$$

$$m_t = d(L)\, \varepsilon_t \quad , \tag{A4}$$

the existence of which shows that m is exogenous with respect to y by virtue of Sims' Theorem 1. To see that y is exogenous with respect to m, observe directly from (A1) that

$$E_{t-1}\, y_t = - \sum_{i=1}^{\infty} \lambda_i\, y_{t-i} \quad ,$$

which shows that m doesn't help predict y once lagged y's are accounted for. Alternatively, rewrite (A1) - (A2) as

$$y_t = \lambda^{-1}(L)\, w_t \tag{A5}$$

$$m_t = d(L)\, \phi\, w_t + d(L)\, \xi_t \tag{A6}$$

where

$$w_t \equiv \gamma\, d_o\, \varepsilon_t + u_t$$

and ξ_t and ϕ obey

$$\varepsilon_t = \phi\, w_t + \xi_t , \qquad E[\xi_t | w_t] = 0$$

(ϕ is the regression coefficient of ε_t against w_t, ξ_t being the residual). Since ξ_t is orthogonal to w_t and since both are serially uncorrelated by construction, it follows that (A5) - (A6) is a Wold representation for the $(y_t - m_t)$ process. The existence of such a triangular Wold representation establishes that y is exogenous with respect to m, again by virtue of Sims' Theorem 1.

Inverting (A4) and substituting it into (A3) delivers

$$\lambda(L)\; y_t = \gamma\, d_o\; d(L)^{-1}\, m_t + u_t \quad ,$$

which is a y on m reduced form which is consistently estimated by least squares, since u_t is orthogonal to lagged y's and current and lagged m's. This is exactly the form of reduced form manipulated and assumed invariant under alternative policy rules in the proof that Friedman's rule is suboptimal.

If the reduced form (A1) is invariant under rules changes, then one deterministic rule is as good as any other.

Thus, there are alternative ways of deriving policy implications from empirical evidence generated by a "classical" model like (A1) - (A2). Depending on what sort of invariance assumption is imposed, drastically different inferences follow about the implications of different feedback rules.

Appendix B

OBSERVATIONAL EQUIVALENCE IN THE PRESENCE OF FEEDBACK FROM y TO m

Suppose that during the estimation period the y - m process possessed the vector autoregressive representation

(B1) $$y_t = a(L)\, y_{t-1} + b(L)\, m_{t-1} + \varepsilon_t$$

(B2) $$m_t = c(L)\, y_{t-1} + d(L)\, m_{t-1} + u_t$$

where $a(L) = \sum_{j=0}^{\infty} a_j L^j$, etc., and

ε_t and u_t are serially independent random variables with means zero and finite variance. In general, $E(\varepsilon_t\, u_t) \neq 0$ although it is easy to prove that $E(\varepsilon_t\, u_s) = 0$ for $t \neq s$. The last equality follows because, for example, u is the residual in a projection of m on lagged m's and y's and so is orthogonal to them. Since lagged u's and ε's are linear combinations of lagged y's and m's, it follows that current u is orthogonal to lagged u's and ε's. Here ε_t and u_t are the one-step-ahead prediction errors for y_t and m_t, respectively, both predictions being conditional on lagged m's and lagged y's. Consistent with usual usage, by a *y-on-m reduced form I mean a regression of* y on past y's and current and past m's. Analogously, by a m-on-y reduced form I mean a regression of m on past m's and current and past y's. For a system with probability distribution characterized by (B1) - (B2), the implied pair of y-on-m, m-on-y reduced forms is identifiable only if sufficient a priori information is imposed on the covariance between the disturbances in the two reduced forms and on the contemporaneous coefficient in either the y-on-m reduced form or the m-on-y reduced form. Here I will impose the restrictions that there is no contemporaneous feedback from y to m and that the reduced form disturbances are contemporaneously orthogonal. These restrictions serve to identify the $y - m$ feedback structure. In particular, (B2) gives the "feedback rule" that governs m in the sample period. To find the y-on-m reduced form in terms of the parameters of (B1) - (B2), first project ε_t on u_t to get the decomposition

$$\varepsilon_t = \rho u_t + \xi_t \ , \quad E[\xi_t \cdot u_t] = 0$$

$$\rho = \frac{E\ u_t \varepsilon_t}{E\ u_t^2} \quad .$$

Then subtract ρm_t from (B1) and rearrange to obtain

(B3) $$y_t = (a(L) - \rho c(L))\ y_{t-1} + \rho m_t + (b(L) - \rho d(L))\ m_{t-1} + \xi_t \quad .$$

Equation (B3) is a y-on-m reduced form with a disturbance ξ_t that is orthogonal to the regressors and also serially uncorrelated. Hence, the parameters of (B3) will be consistently estimated by least squares. Notice that (B2) is also consistently estimated by least squares.

Write (B3) more compactly as

(B3') $$y_t = h(L)\ y_{t-1} + g(L)\ m_t + \xi_t$$

where $h(L) = a(L) - \rho c(L)$, $g(L) = [\rho + L(b(L) - \rho d(L))]$.

Then solve (B2) for m_t in terms of lagged y's and current and lagged u's:

$$m_t = (1-d(L))^{-1}\ c(L)\ y_{t-1} + (1-d(L))^{-1}\ u_t \quad .$$

Substituting this for m_t in (B3') gives

$$y_t = \{h(L) + g(L)\ (1-d(L))^{-1}\ c(L)\}\ y_{t-1}$$
$$+\ g(L)\ (1-d(L))^{-1}\ u_t + \xi_t$$

or

(B4) $$y_t = i(L)\ y_{t-1} + j(L)\ (m_t - E_{t-1}\ m_t) + \xi_t$$

where $i(L) = \{h(L) + g(L)\ (1-d(L))^{-1}\ c(L)\}$,
$j(L) = g(L)\ (1-d(L))^{-1}$, and ξ_t is orthogonal to lagged y's and current and lagged u's (i.e., $(m_t - E_{t-1}\ m_t)$'s). So (B4) is consistently estimated by least squares.

Now (B3') and (B4) have identical residuals, and, thus, are observationally equivalent. The argument in the text thus goes through for this system.

Footnotes

1. The paper by Sims [8], especially his appendix, provides a useful summary of the statistical theory used here.

2. I will also assume that ε and η are mutually and serially independent, which facilitates the computations below but is not essential. Abandoning that assumption would require replacing mathematical expectations with linear least squares forecasts at several points below. With that replacement, my argument would go through.

3. I am assuming that $\beta(L)$ and σ_η^2 are so normalized that $\beta_o = 1$, which implies that the zero order coefficient of $\beta(L)^{-1}$ is also unity.

4. Notice that

$$c(L)(m_t - E_{t-1}\, m_t) = c_o(m_t - E_{t-1}\, m_t) + c_1(m_{t-1} - E_{t-2}\, m_{t-1}) + c_2(m_{t-2} - E_{t-3}\, m_{t-2}) + \ldots \quad .$$

5. For example, see Sargent and Wallace [6].

6. See Sargent [7].

7. Lucas' international comparisons [2] provide an excellent example of this approach.

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