A TWO-PERIOD PORTFOLIO PROBLEM IN A BALANCE SHEET CONTEXT

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Abstract

Previous work on discrete time portfolio selection models encompassed (a) transaction's costs, and (b) uncertainty about cash flows during the first (and only) period. This paper extends these models by considering uncertainty about asset yields in the second period and the optimal strategy for portfolio selection over a two-period horizon. Among the implications are i) the optimal initial portfolio is, in general, diversified and contains more short-term assets than the myopic investor's portfolio, and ii) the shape of the mean-variance locus ensures diversification for all (two-moment) types of investors, except certain forms of risk lovers. Other partial derivatives are investigated.

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In a recent journal article, Roger N. Waud [19] presents a model in which it is possible for risk-neutral investors, risk lovers, and plungers as well as risk averters to exhibit portfolio diversifying behavior. This result is achieved in a model in which the only source of uncertainty is the net outlay of the individual; and, from a methodological point of view, Waud's conclusions are drawn from a particular specification of his general model.[1/]

With respect to Waud's paper, this paper has two purposes. First of all, it is an extension of Waud's model in two directions: it extends the time horizon to two periods, and includes uncertainty about the return of the assets. Secondly, it employs a methodology similar to Waud's, but also utilizes Monte Carlo solutions in addition to numerical solutions. These generalizations permit a stronger statement of Waud's conclusions; namely, in the mean-variance space, the particular specification of the model that is analyzed produces portfolio diversification for <u>all</u> nonnegative sloping indifference curves, and makes diversification possible for downward sloping indifference curves.

Beyond its relation to Waud's model, the purpose of this paper is to extend a common form of single-period discrete time models into a

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[1/] This methodology is used because the complexity of the model makes closed form solutions (and standard techniques of analysis) virtually impossible, and has been used previously, e.g., Porter [13] and Supel [15].

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two-period setting.[2/] By this extension, transactions costs become even more important because the investor must consider the cost of adjusting to the portfolio that is optimal for the second period as well as the single-period cost of liquidation to cover shortfalls in cash flows. Also, the two-period framework permits the explicit consideration of the probability distribution of asset yields in the second period, and it is this feature that fundamentally alters the mean-variance opportunity locus vis-a-vis the single-period models.

Within a two-period context the concept of "riskyness" is even more ambiguous than it is in a single-period setting. Is a bond which matures with certainty at the end of the second period, but which has a highly uncertain capital value at the end of the first period riskier than a short-term security which matures with certainty at the end of the first period? Clearly the variance of the portfolio returns over both periods will depend on the probability distribution of net cash flows and short-term yields in the second period.

[2/] While a review of the extensive portfolio theory literature is beyond the scope of this paper, a few of the more directly relevant citations should be made. The classic works of Tobin [16] and Markowitz [6] utilize properties of the utility function to explain portfolio diversification when faced with uncertain asset returns. This model has been developed in enormous detail, and has been extended into a multi-period setting by, for example, Mossin [9]. Preceding Waud [19], Morrison [8], Poole [12], Porter [13], and Frost [3] developed single-period models where the distribution of net cash flows plays a key role in determining asset allocations. The Frost model is particularly interesting because of the way it deals with initial conditions. Multiperiod models have been developed by Wolf [20] and Daellenbach and Archer [2]. The Daellenbach and Archer model takes the probability of a cash shortage as exogenous, and net cash flow is the only source of uncertainty. The Wolf model contains no transactions costs for switching securities, and the results are independent of the distribution of cash flows--a result at odds with the conclusions of this paper. Finally, the paper by Miller and Orr [7] is representative of the literative dealing with the application of an inventory "policy of simple form" to explain transactions balances when cash flows are uncertain.

The model developed in this paper is basically an inventory type of model which considers yield uncertainty, cash flow uncertainty, and transaction costs. The combination of future rate uncertainty and transactions costs converts the portfolio problem into an interesting dynamic programming problem. With transactions costs in the model, it is possible to remove either future rate uncertainty or the uncertainty about cash flows and still explain diversification; but without transactions costs, the interesting multiperiod problems reduce to a sequence of one-period problems (except when the investor is certain that future short yields will be larger than future long yields).

Since the model encompasses rate uncertainty and transactions costs, it deals simultaneously with both transactions and speculative motives for holding money and other assets. It produces diversified portfolios, and deals explicitly with yield moments other than means without assuming declining marginal utility of income. In this paper only a two-period planning horizon is assumed. This is sufficient to allow comparison between the solution implied by dynamic programming for the two-period investor, and the result derived for the one-period, or myopic, maximizer. The investor with the two-period planning horizon tends to hold a more short-term assets than the myopic investor because of the potential transactions costs involved in adjusting the portfolio in the second period.

Section I below provides a detailed discussion of the model and its underlying assumptions; the next section describes the extent to which a solution of the model can be pursued without making specific assumptions about the stochastic structure of the problem; section III discusses portfolio implications of the model under the assumption of independent and uniformly distributed random variables; and the final section provides a review of the purpose and extent of the paper. Although the conceptual framework developed here has a number of applications, for purposes of discussion, the balance sheet studied is taken to be that of a single bank so that uncertainty about cash flows is synonymous with deposit variability. A three-part appendix is available upon request to the author which provides a detailed solution to the two-period problem, an analysis of the robustness of the solution to certain key assumptions, and the BASIC language computer program which generated the solutions used in the paper.

I. The Model

The model[3] developed in this paper takes as its starting point the following balance sheet that might be faced by a single bank at a particular point in time t_0:

t_0	
RR_0	D_0
R	E_0
L	

[3] This model has its roots firmly embedded in the work of Tobin. The balance sheet interpretation is taken largely from [17]; also see Tobin's discussion of the multiperiod investment problem in [18], pp. 37-47. The paper by Pierce [11] is also an excellent discussion of many of the issues explicitly considered here, and the papers by Gramley and Chase [4], and Kareken [5] emphasize the importance of the entire balance sheet to portfolio decisions.

where

RR_0 = required (legal) reserves

R = reserves or loans which mature in one period[4]

L = loans which mature in T period where T > 2

D_0 = deposits

E_0 = equity capital

The assets (other than required reserves) are distinguished primarily by their maturity dates. However, it is convenient to view the asset R as what Tobin[5] calls "defensive assets." These are assets which are often referred to as primary or secondary reserves or "liquid" assets, and include cash, excess reserves, federal funds loans, deposits in other banks, and securities such as Treasury bills which are acceptable collateral for discounting purposes. The L-assets may also be viewed as consols. This model does not attempt to segregate the total holdings of defensive assets into those held for transactions purposes and those held for investment purposes. Indeed, one of the advantages of the balance sheet model is that there is no need to make this distinction.

Since required reserves are always equal to kD_0,[6] where k is the legal reserve ratio, there are two decisions for the bank to make in establishing its initial balance sheet -- R and L. These variables are purposely left unsubscripted in order to emphasize the fact that they are the decision

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[4] Although a "period" is never explicitly defined, a working interpretation might be one reserve period.

[5] Tobin [17], p. 1.

[6] This formulation ignores the fact that required reserves are now established on a lagged basis.

variables. The problem of this bank is to decide on the optimal values of R and L subject to the constraints, definitions, and optimality criteria described below.

The modus operandus of this bank is to first decide on its balance sheet at t_0. For purposes of this analysis, it is assumed that there are no transactions at this bank between the discrete points in time.[7] At t_1 (the end of the first period and beginning of the second period), the bank obtains a new set of data, described below, and may adjust its balance sheet if it so desires. At t_2 (the end of the second period) it is assumed that all assets become worth their face value and could be converted to cash without transactions costs.[8]

The bank earns a (coupon) rate of r_{R1} on the one-period asset (R) during the first period, and the face value of each unit of this security is \$1. There is no default risk so that on n dollar investment in R at t_0 means that the bank will have cash of $\$(1+r_{R1})n$ from this investment at t_1 with certainty.

One-period assets may be purchased at t_1 for \$1 per unit, but the coupon attached to them now yields a rate of r_{R2}. This rate is revealed to the bank at t_1, but at t_0 it is unknown except for its cumulative distribution function (c.d.f.)Ω which is defined on the interval $[\omega_1, \omega_2]$ where $\omega_1 \geq 0$.

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[7]This assumption, of course, becomes less realistic as the length of the period increases. If a period was defined as one business day, the assumption would not be at all unrealistic. For then the bank would not need to be concerned with intra-period deposit depletions because it could simply draw overdrafts on its Federal Reserve account and cover them with end-of-the-period asset adjustments or borrowings.

[8]The critical part of this assumption is that, at t_0, the investor acts as if loans will have some fixed (nonrandom) value at t_2, which might be interpreted as an expected value. Using a value other than 1 simply alters the capital gains part of the formula for Z below, but does not alter the fundamental nature of the problem. Assuming T > 2 permits this broad interpretation, and would easily permit analysis of the model with respect to the expected price of L at t_2.

The T-period asset (L) also has a t_0 price of \$1 per unit, but it has T coupons attached each of which are worth $\$r_L$. T-period assets are distinguished from one-period assets primarily by the fact that the price (P_1) of the T-period asset at t_1 is uncertain at t_0. At t_0 the bank knows the c.d.f. of P_1 to be Φ which is defined on the interval $[\phi_1, \phi_2]$ where $\phi_1 > 0$.

Loans may also be purchased at t_1, but now each dollar will buy $1/P_1$ units of L. Since, by assumption, each unit will be worth \$1 at t_2, a capital gain (or loss) of $1-P_1$ is incurred during the second period so that the <u>rate</u> of return on L during the second period is

$$Z = \frac{r_L + (1-P_1)}{P_1}$$

Note that if P_1 is large enough, the rate of return on loans (Z) in the second period becomes negative, and there is no such constraint built into the model to prevent this from happening. Interpreting Z as a "real" rate might be one reason for omitting the constraint; but pragmatically, it is unlikely that the constraint would alter the implications of the model for once Z is below r_{R2} the size of Z is largely irrelevant because the investor would never buy loans at t_1.

Loans in this model are described by characteristics usually attributed to less "liquid" types of assets. These assets, in addition to price uncertainty, are generally considered to have a less well developed secondary market.[9] Consequently, it is assumed that a transaction cost of b per unit is incurred only when loans (L) are <u>sold</u> at t_1, where $0 < b < 1$. It is convenient to think of b as a bid-asked spread, so that the transactions cost involved in purchasing the asset becomes part of the (net) return measured in r_L.

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[9] This is relative to the "liquid" asset R. Cf. Tobin [18], p. 3.

The level of deposits (D_1) which the bank will realize at t_1 is also unknown at t_0 except for its c.d.f. Δ which is defined on $[\delta_1, \delta_2]$ where $\delta_1 > 0$.

The joint distribution of the three random variables in the model is initially assumed to be of the form

$$F(r_{R2}, P_1, D_1) = G(r_{R2}, P_1)\Delta(D_1). \tag{1}$$

That is, D_1 is assumed to be independent of r_{R2} and P_1 which are jointly distributed by G.

Profits in this model are defined as the change in equity over a given period,[10/] or

$$\begin{aligned} Q_1 &= E_1 - E_0 \\ Q_2 &= E_2 - E_1 \\ Q &= E_2 - E_0 = Q_1 + Q_2. \end{aligned} \tag{2}$$

It is assumed throughout this paper that the utility function of the individial bank is a linear function only of two-period profits (Q), and that maximization of expected utility is the optimality criterion.[11/] Thus, expected utility is maximized whenever expected profits are maximized. The case of the myopic investor who looks only at Q_1 is also investigated below.

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[10] We ignore all distinctions between capital gains and ordinary income that a bank might make for tax purposes -- Cf. Wolf [20]. Also, see Klein's argument that this distinction is not theoretically relevant in his comment on Tobin [18], p. 287.

[11] Although the use of a quadratic utility function seems to be more appropriate for portfolio analysis, it appears that the implications of this model are reasonably robust with respect to any risk aversion kind of utility function. This is because the portfolio which yields the highest mean is quantitatively close to the portfolio with the smallest variance -- at least for parameter sets in the neighborhood of those investigated in this study. Cf. Supel [15] appendix E.

Since profits in the second period enter the utility function, and since action taken at t_0 may affect second-period profits, it is necessary for the bank to establish an investment strategy for t_1 in order to establish the optimal balance sheet at t_0. Maximization of the expected value of Q in (2) is essentially a dynamic programming problem so that the solution technique for that type of problem can be applied to the problem of deriving the investment strategy at t_1. Use is made of the principle of optimality which states that "An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."[12]

Having chosen R and L at t_0, the new state of nature is determined by the values of the random variables D_1, r_{R2}, and F_1. The cash position of the bank at t_1, other than required reserves, consists of R (which matures by definition) plus the cash flow $r_{R1}R + r_L L$. Cash is needed at t_1 to cover deposit depletions other than those covered by required reserves, i.e., $K(D_0 - D_1)$ where $K = 1-k$. If cash holdings at t_1 are at least as great as cash needs, then the bank is not <u>forced</u> to do anything. That is, if

$$(3) \qquad D_1 \geq D_0 - \frac{1}{K}[R + r_{R1}R + r_L L] = \alpha$$

then the bank need not alter its portfolio for the second period unless it so desires.[13] The right hand side of inequality (3) is fixed at the time of the initial balance sheet decision and serves to define α which represents

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[12] Bellman [1], p. 24.

[13] α assumes economic meaning in this model via definition (3) as an index of the portfolio and by the fact that it establishes that level of deposits which must be achieved before unnecessary transactions costs are incurred. A further interpretation of its role in the model is given in section III.

the lower bound to which deposits may fall before the bank must sell loans in order to maintain its legal commitments. When $D_1 < \alpha$, the amount of cash that must be raised by the bank through sales of L is $K(\alpha - D_1)$.

At t_1, the state of nature is revealed to the bank, and consequently, only one asset (other than required reserves) is held during the second period -- except for the special case described below. The bank now holds R_1 in cash and L_1 in loans on which it can earn r_{R2} and Z respectively. Applying the principle of optimality, the bank will choose to hold R_1 in defensive assets if $r_{R2} \geq Z$, otherwise it will switch R_1 into loans. Because transactions costs are incurred in switching loans into defensive assets, r_{R2} must be greater than X for this to be a profitable move,[14] where

$$X = \frac{Z + b}{1 - b}$$

is the total opportunity cost of switching loans into reserves. When $Z \leq r_{R2} \leq X$, the bank will maintain the status quo portfolio of whatever it ended the first period with -- allowing, of course, for loan sales that are necessitated by deposit depletions. The entire strategy is summarized in Table I.

By following the sequence of balance sheets determined by the optimal strategy, the two-period profit function (Q) is derived and presented in Table II where the cases of Table II correspond to the cases of Table I. Each branch of the profit function may be interpreted generically as consisting of earnings on the initial balance sheet (W), plus earnings in the

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[14] If λ dollars worth of loans were transferred into defensive assets, the net increase in profits (ψ) would be

$$\psi = r_{R2}\,(\lambda - b\lambda) - Z\lambda - b\lambda.$$

Then $\frac{d\psi}{d\lambda} > 0$ if and only if $r_{R2} > \frac{Z + b}{1 - b} = X$.

Table I

Strategy at t_1

	Contingency		Action	Case
$D_1 \geq \alpha$,	$r_{R2} < Z$	:	switch R to L	I
	$r_{R2} > X$	:	switch L to R	II
	$Z \leq r_{R2} \leq X$	:	status quo	III
$D_1 < \alpha$,	$r_{R2} < Z$	:	hold only L	IV
	$r_{R2} > X$	:	switch L to R	V
	$Z \leq r_{R2} \leq X$	:	status quo	VI

Table II

Two-Period Profit ([illegible])

Case:

I: $W - K\alpha Z + KZD_1$

II: $W - \xi L - K\alpha r_{R2} + KD_1 r_{R2}$

III: $W - K\alpha r_{R2} + Kr_{R2}D_1$

IV: $W - K\alpha X + KXD_1$

V: $W - \xi L - K\alpha r_{R2} + Kr_{R2}D_1$

VI: $W - K\alpha X + KXD_1$

where $W = r_{R1}$ [illegible] $r_L L$

$\xi = r$ [illegible] $+ (1-P_1) +$ [illegible] $-$ [illegible] r_{R2}

second period on deposit deviations from α. The latter term, $K(\alpha - D_1)$, reflects the cash flow that must be allocated at t_1, and earnings on this term reflect gains or losses which accrue to the bank because of its decision on the magnitude of α. The term ξ reflects the equity adjustment which must be made because of capital gains and transactions costs which were actually incurred in switching from loans to defensive assets as well as the earnings on defensive assets in the second period.

From Tables I and II and equation (1) we may now write expected two-period profits ($\overline{Q}$) as

$$(4) \qquad \overline{Q} = \int_{\alpha}^{\delta_2} \int_{\phi_1}^{\phi_2} \int_{\omega_1}^{Z} (I)dF + \int_{\alpha}^{\delta_2} \int_{\phi_1}^{\phi_2} \int_{X}^{\omega_2} (II)dF + \int_{\alpha}^{\delta_2} \int_{\phi_1}^{\phi_2} \int_{Z}^{X} (III)dF + \int_{\delta_1}^{\alpha} \int_{\phi_1}^{\phi_2} \int_{\omega_1}^{Z} (IV)dF + \int_{\delta_1}^{\alpha} \int_{\phi_1}^{\phi_2} \int_{X}^{\omega_2} (V)dF + \int_{\delta_1}^{\alpha} \int_{\phi_1}^{\phi_2} \int_{Z}^{X} (VI)dF$$

were the integrands correspond to the cases of Table II. And the optimization problem of the bank may be stated as

$$(5) \qquad \max_{R,\ L,\ \alpha} \ \overline{Q}$$

$$\text{s.t.} \quad \text{(i)} \quad R + L = KD_0 + E_0$$

$$\text{(ii)} \quad K\alpha = KD_0 - r_L L - (1+r_{R1})R$$

$$\text{(iii)} \quad R,\ L \geq 0$$

where (i) and (iii) are the balance sheet constraints and (ii) is the definition of α. Because of the solution method adopted below it is convenient at this point to ignore (iii) and state the problem as

$$(6) \qquad \max_{R,\, L,\, \alpha} \quad \overline{Q}$$

$$\text{s.t.} \quad \text{(i)} \quad R = \frac{1}{v}[(1-r_L)KD_0 - r_L E_0 - K\alpha]$$

$$\text{(ii)} \quad L = \frac{1}{v}[r_{R1}KD_0 + (1+r_{R1})E_0 + K\alpha]$$

where $v = 1 + r_{R1} - r_L$.

In order to show that a two-period investor may hold a different portfolio than a one-period investor, it is necessary to explicitly consider the one-period or myopic profit function.[15] In the context of this model, a myopic investor who disregards decisions to be made in the future will base his balance sheet decision at t_0 only on the legal constraints that might prevail and ignore the portfolio allocation decisions at t_1. Thus the myopic investors' profit function (Q_1) is

$$(7) \qquad Q_1 = \begin{cases} I = r_{R1}R + r_L L - (1-P_1)L, & \text{if } D_1 \geq \alpha \\ II = r_{R1}R + r_L L - (1-P_1)L - \dfrac{b(\alpha-D_1)}{1-b}, & \text{if } D_1 < \alpha \end{cases}$$

where case I represents the profit outcome when cash flow is large enough to cover deposit depletions, and case II represents the profit outcome when securities are sold only in the amount necessary to cover reserve shortages.

II. The General Solution

To find the optimal balance sheet for the two-period problem, we first solve for the optimal α and then determine the balance sheet from the constraints of (6). Writing the expected profit function (4) in extensive form, and substituting in the constraints of (6) we have

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[15] Cf. Mossin [9].

$$(8)\qquad \overline{Q} = v_0 + v_1\alpha - KI_1\alpha + KI_2$$

$$- K\alpha \int_{\alpha}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{\omega_1}^{Z} Z dF + K \int_{\alpha}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{\omega_1}^{Z} ZD_1 dF$$

$$- K\alpha \int_{\alpha}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{Z}^{X} r_{R2} dF + K \int_{\alpha}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{Z}^{X} r_{R2}D_1 dF$$

$$- K\alpha \int_{\delta_1}^{\alpha}\int_{\phi_1}^{\phi_2}\int_{\omega_1}^{X} X dF + K \int_{\delta_1}^{\alpha}\int_{\phi_1}^{\phi_2}\int_{\omega_1}^{X} XD_1 dF$$

where

$$v_0 = \frac{r_{R1}KD_0}{v}(1+r_L-I_0) + \frac{E_0}{v}(2r_L+r_Lr_{R1}-(1+r_{R1})I_0)$$

$$v_1 = \frac{K}{v}(2r_L-r_{R1}-I_0)$$

$$I_0 = \int_{\delta_1}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{X}^{\omega_2} \xi dF$$

$$I_1 = \int_{\delta_1}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{X}^{\omega_2} r_{R2} dF$$

and

$$I_2 = \int_{\delta_1}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{X}^{\omega_2} r_{R2}D_1 dF.$$

The optimal value of α is then determined by setting the derivative of (8) equal to zero and solving for α. Thus, $\hat{\alpha}$ is that value of α which satisfies the equation

$$(9)\qquad \int_{\alpha}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{\omega_1}^{Z} Z dF + \int_{\alpha}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{Z}^{X} r_{R2} dF + \int_{\delta_1}^{\alpha}\int_{\phi_1}^{\phi_2}\int_{\omega_1}^{X} X dF$$

$$- \frac{1}{v}(2r_L-r_{R1} - \int_{\delta_1}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{X}^{\omega_2} \xi dF - v\int_{\delta_1}^{\delta_2}\int_{\phi_1}^{\phi_2}\int_{X}^{\omega_2} r_{R2} dF) = 0.$$

We assume throughout this paper that F has a continuous density f.

The derivation of $\hat{\alpha}$ as represented by equation (9) has required no particular assumption about the independence of D_1, P_1, and r_{R2}. However, the intractability of (9) can be alleviated to a large extent by assuming that D_1 is independent of r_{R2} and P_1, i.e. that F factors as in (1). With this assumption, $\hat{\alpha}$ is determined by

$$(10) \qquad b_1 - b_3\Delta(\hat{\alpha}) = 0$$

or

$$(11) \qquad \hat{\alpha} = \Delta^{-1}\left(\frac{b_1}{b_3}\right)$$

where

$$b_1 = \frac{1}{v}[2r_L - r_{R1} - h_2 - v(h_3 + h_1 + h_4)]$$

$$b_3 = h_5 + h_6 - h_1 - h_4$$

and

$$h_1 = \int_{\phi_1}^{\phi_2} \int_{\omega_1}^{Z} Z dG \qquad h_2 = \int_{\phi_1}^{\phi_2} \int_{X}^{\omega_2} \text{[illegible]} \, dG \qquad h_3 = \int_{\phi_1}^{\phi_2} \int_{X}^{\omega_2} r_{R2} dG$$

$$h_4 = \int_{\phi_1}^{\phi_2} \int_{Z}^{X} r_{R2} dG \qquad h_5 = \int_{\phi_1}^{\phi_2} \int_{\omega_1}^{Z} \text{[illegible]} \, dG \qquad h_6 = \int_{\phi_1}^{\phi_2} \int_{Z}^{X} X dG.$$

Solution (11) is, of course, well defined only when $0 < b_1 < b_3$, and there are insufficient constraints in the model to guarantee that such is the case. It is clear, however, that $b_3 > 0$ so that the second order conditions for a maximum are always satisfied. Thus, if an internal solution to (11), exists, it is a maximum. By an <u>internal solution</u> we mean a value of $\hat{\alpha}$ such that $\delta_1 < \hat{\alpha} < \delta_2$. If $b_1 \leq 0 \; (\geq b_3)$ then $\hat{\alpha}$ should be made as small (large) as possible subject to the balance sheet and nonnegativity constraints on R and L.

It should be noted that an internal solution for α is not a sufficient condition for a diversified portfolio in the sense that both $R > 0$ and $L > 0$, although it is a necessary condition.

Although solution (11) is much simpler than (9), the problem of tractability still remains because of the number of parameters in the model. In order to analyze the partial effects of the various parameters on the optimal portfolio it is further assumed that all the random variables are independent and uniformly distributed over their specified ranges.[16/] The following section presents an analysis of the model under this assumption, and the appendix describes the solution method.

III. Specific Solutions

In order to make inferences about the model, we use parametric deviations from the set S where

(12) $$S = \{D_0 = 100,\ K = .9,\ r_{R1} = .05,\ \delta_1 = 50,\ \delta_2 = 110,\ \phi_1 = .9,\ \phi_2 = 1.1,\ \omega_1 = .02,\ \omega_2 = .12,\ b = .1,\ E_0 = 10\}.$$

The convention $S' = S \sim z = z_0$ will be used to denote the parameter set S' which is identical to S except that $z = z_0$. Making general qualitative inferences from such a procedure as this may be questionable, but all of the experiments done so far indicate that this is an acceptable procedure. Except for the myopic case, all of the discussion in this section deals with two-period problems, and all of the changes discussed should be interpreted in the sense of partial derivatives.

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[16]For a discussion of this assumption see Supel [15] pp. 124-129. The assumption of complete independence between r_{R2} and P_1 as opposed to complete dependence in the sense that $Z \equiv r_{R2}$ appears to bias the portfolio in favor of the short-term asset; but it is not clear that the effects of changes in the various parameters should be seriously altered.

The loan rate (r_L) was purposely excluded from S because much of the analysis is in terms of the loan offer function (LOF), that is, the number of loans the bank would like to have in its portfolio given S and a particular loan rate. Given a particular LOF, the bank's demand for defensive assets is determined by the balance sheet constraint.

Some of the principle results derived from this model are as follows:

(a) Figure I shows the LOF generated from the parameter set S, and illustrates the basic properties of the model for both the one-period and two-period maximizer.[17] The myopic investor generally holds more loans than the two-period investor except in the case that they both hold non-diversified portfolios. In the extreme case, it is possible to show that for certain parameter sets the myopic investor will hold only loans while the two-period maximizer holds only defensive assets.

It may also be shown that when yields in the second period are known with certainty (say r^*_{R2}, Z^*, and X^*), then the two-period investor holds the same portfolio as the myopic investor -- except for the case when $Z^* < r^*_{R2}$. In other words, given transactions costs in the model, it is the uncertainty about future rates that generates the dynamic programming problem and not the uncertainty about cash flow.

(b) Transactions costs are critical to this model for if $b = 0$, the model is incapable of generating diversified portfolios. As b approaches zero, the LOF becomes more elastic. And as b increases, a higher loan rate

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[17] Although it is not obvious because of the scale of the diagram, the LOF is convex in the diversified range, but is essentially of the same form inferred by Pierce [11], p. 1099.

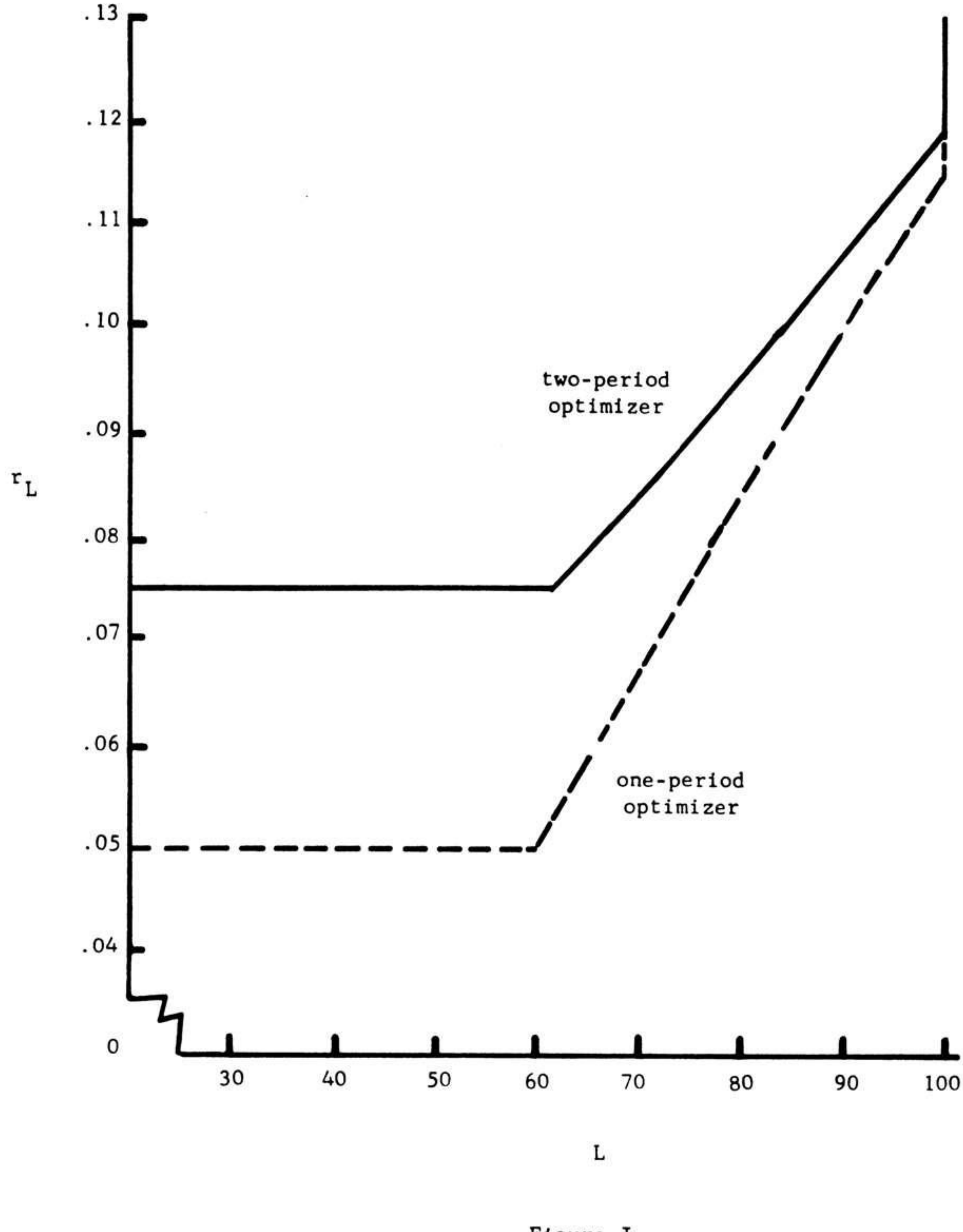

Figure I
LOF Given S

is necessary to induce the bank to hold a given quantity of loans because of the higher penalty incurred if there are deposit shortfalls.

(c) The impact of deposit variability on the optimal portfolio derived from this model is consistent with other models investigating this issue,[18] namely, the variance of the deposit distribution alone is insufficient information to infer whether the bank with a small variance will hold more loans than a bank with a large variance. The effect of a change in the variance of the deposit distribution on the portfolio can be seen from equation (11).[19] If Δ is uniform, this equation becomes

$$\hat{\alpha} = \delta_1 + b_4(\delta_2 - \delta_1)$$

where

$$b_4 = \frac{b_1}{b_3}.$$

A change in the variance of D_1, with the mean unchanged, means that δ_1 and δ_2 are changed by the same amount. Thus, if $b_4 = \frac{1}{2}$, then $\hat{\alpha}$ (and hence, $\hat{L}$) is unchanged. For $b_4 < \frac{1}{2}$, an increase in the variance of deposits results in fewer loans; but if $b_4 > \frac{1}{2}$, an increase in the variance of deposits will actually result in an increase in the desired holdings of loans by the bank. The reason for this phenomenon centers on the interpretation of the role of α in the model. Since the profit function (π) depends on α, the role of α is to position the profits function over the probability mass in such a way as to make the weighted average (expected value) of profits a maximum. According to (11), the optimal way is when the probability that deposits are less than $\hat{\alpha}$ equals b_4. But when $b_4 > \frac{1}{2}$, an increase in the variance of D_1

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[18] See Poole [12], and also Porter [13].

[19] The argument which follows can be applied directly to equation (11) without the necessity of the assumption of a uniform Δ. It also does not depend on the assumption of independence between Φ and Ω.

means that this probability has fallen; whence, $\hat{\alpha}$ and the optimum quantity of loans must increase to maintain this probability at b_4.

Figure II illustrates the effect of an increase in the variance of D_1 on the LOF. The two curves intersect at that loan rate (r_L^*) which makes $b_4 = \frac{1}{2}$ and also $\hat{\alpha} = \overline{D}_1$, the mean of D_1. If $\overline{D}_1$ is large as determined by

$$K\overline{D}_1 \geq KD_0 - r_L^*(KD_0 + E_0)$$

then the intersection point is to the right of the maximum quantity of loans permitted by the balance sheet constraint; and, starting from an internal solution, an increase in deposit variance would always result in a decline in the quantity of loans desired by the bank.

(d) Starting from an internal solution, the derivatives of the constraint equations in (6) show that a marginal increase in reserve requirements results in a decrease in holdings of both assets. The condition that $\hat{\alpha} \geq \frac{1}{2}D_0$ is sufficient to insure that loans decline more than defensive assets.

(e) An increase in the mean of P_1 shifts the LOF to the right, but there is also a clockwise rotation; an increase in the variance of P_1 shifts the LOF to the left also with a clockwise rotation. This simply says, when the coupon return is high, uncertainty about capital losses becomes less relevant to the bank.

(f) Increases in the mean and variance of r_{R2} each shift the LOF in the same way -- to the left with a clockwise rotation. The fact that holdings of long-term assets diminish as uncertainty about the yield on short-term assets in future periods increases is essentially consistent with the result of Scott [14] in explaining the Availability Doctrine. In the context of this model, an increase in the variance of

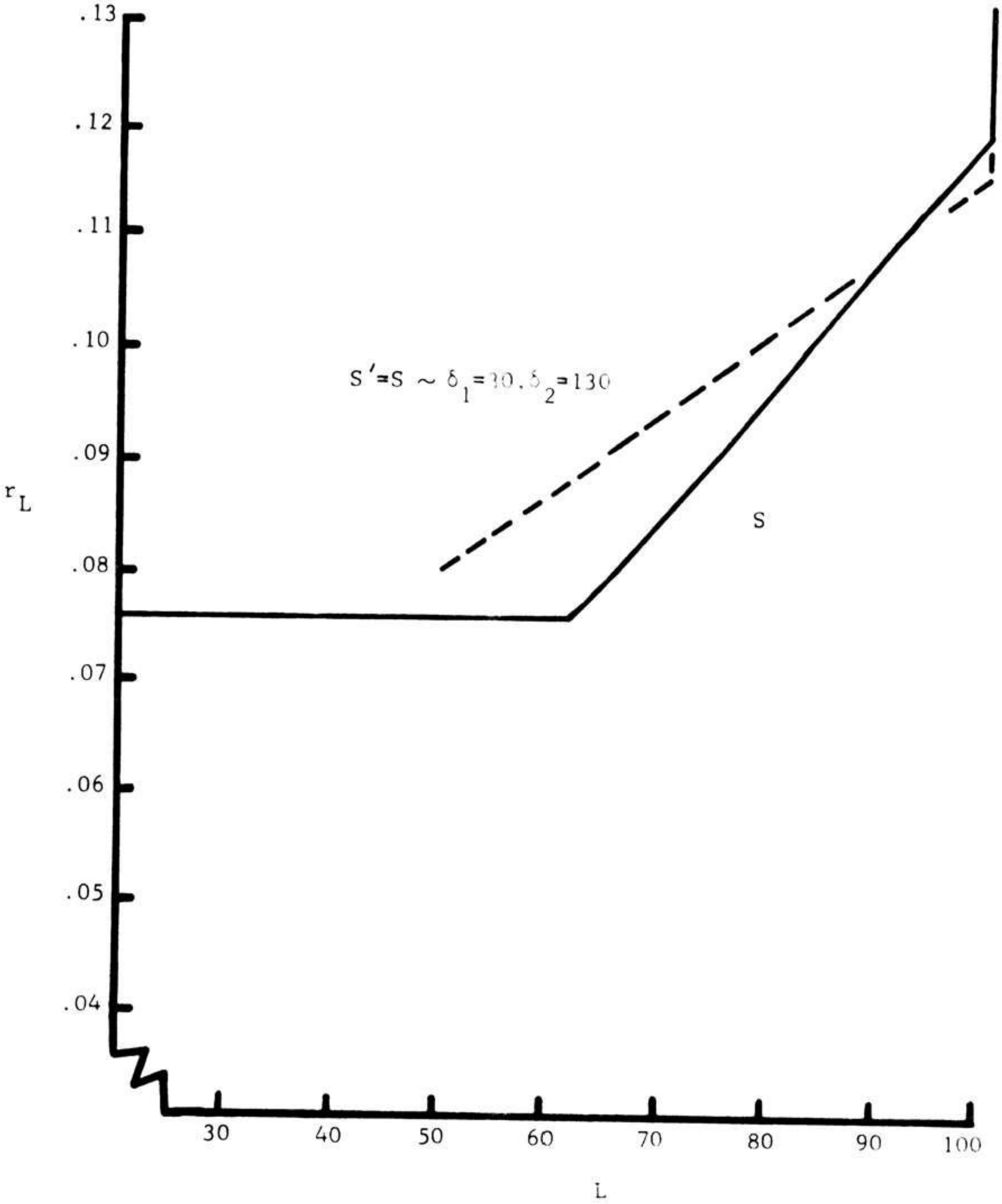

Figure II
Variance in D_1

r_{R2} means that more defensive assets should be held in the first period in order to be in a position to capitalize on the higher values of r_{R2} that now have positive probability. Of course, more lower values of r_{R2} also have positive probability; but this is largely irrelevant because for low values of r_{R2} the optimizing strategy requires that only loans are held in the second period.

(g) In examining scale effects on the optimal portfolio, the question arises as to the proper definition of scale. Since defensive assets are held largely to protect against the penalties of deposit depletions, marginal increases in capital (E_0) will go into loans; but marginal increases in the initial deposit level (D_0) will go mainly into defensive assets.[20]

An alternative way of examining scale effects is to let all the size parameters (D_0, δ_1, δ_2, E_0) increase by the same (percentage) amount. In this case, optimal quantity of loans increases by the same (percentage) amount so that the loan/deposit ratio (L/D_0) remains invariant. Also, the standard deviation grows proportionately so that the coefficient of variation remains invariant. The variance of deposits, however, increases more rapidly than the scale increase. This suggests that the proper "variability" variable in empirical estimation of LOF's is the standard deviation.[21]

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[20] These results are derived by differentiating the balance sheet constraints in (6) noting that α is independent of E_0 and D_0. The derivative $\frac{dR}{dE_0}$ is in fact negative because of the larger cash flow that results at t_1 from increasing loans.

[21] The evidence in Murphy [10] indicates that the coefficient of variation is independent of deposit size, and thus lends support to the idea of viewing scale in terms of a proportionate increase in all size variables. Further his method of computing variability in terms of deviations from a trend line seems to be more consistent with the concepts of this model than other methods which have been advanced.

(h) Finally, we note that this model implies that the proper interest rate specification in a LOF is to treat each interest rate individually and not as ratios or differences. In other words, two sets of r_{R1} and r_L will yield different optimal portfolios even though the difference or ratio may be the same between the two sets. This model also implies that when the rate structure is raised, i.e., r_{R1} and r_L raised proportionately, the optimal portfolio shifts in the direction of the longer maturity.

The mean-variance locus of two-period profits, shown in Figure III, is computed for the parameter set S and the loan rate fixed at $r_L = .09$. The means are computed exactly using the solution method described in Section II, but the variances are each estimated via Monte Carlo procedures from a sample of size 1000. With this sample size, a 95 percent confidence interval for the variance is a band of $\pm$ 0.5 around the estimated variance when the estimate is in the neighborhood of 5.0.

Figure III shows that only when loans are in the range of approximately 70 to 75 does the mean-variance curve have the properties customarily ascribed to the choice set available to the investor, vig 3. increasing variance along with increasing mean. This range appears to vary inversely with the variance of the return on the portfolio. For example, a ceteris paribus decrease in the variance of P_1 would extend the range of portfolios with directly related means and variances.

The shape of the locus shown in Figure III permits the following inferences: (1) A wide variety of attitudes toward risk can be consistent with portfolio diversification. To the extent that this model

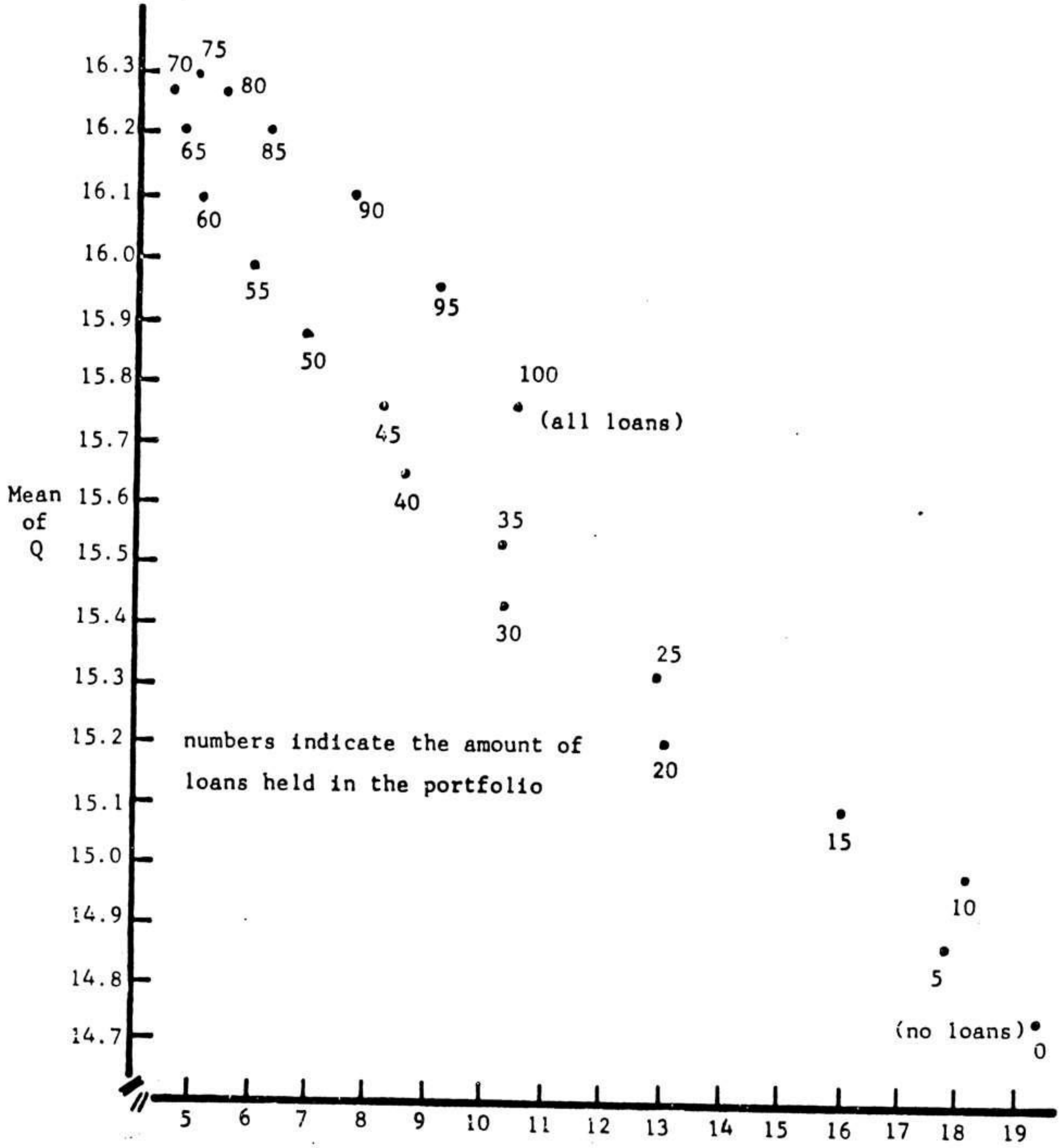

Figure III

Means and Variances of Two-Period Profits captures actual mean-variance possibilities, observing portfolio diversification alone, does not delimit investor attitudes toward risk in a significant way; (2) in this particular example, had we imposed any utility function with nonnegative risk aversion properties, the optimal portfolio would contain loans of at least 70, but not more than 75. This suggests that quantitatively similar portfolios are implied by both linear and nonlinear (but risk averse) utility functions, but unfortunately, this inference does not tell us anything about the comparative statics of the different utility function; (3) if the risk of the portfolio is measured by its variance of return over two-periods, then, for the example described in Figure III, the riskiest portfolio consists entirely of defensive assets. This is contrary to the intuitive notion that portfolios consisting of short-term assets (e.g., Treasury bills) are less risky than long-term assets (e.g., government bonds). Of course, the point here is that if the probability is small that two-period assets will have to be sold at the end of the first period, then the variance of return from holding this asset is likewise small.

IV. Summary

The purpose of this study was to examine the balance sheet of a bank in the context of a two-period uncertainty model. In this regard, the burden of portfolio diversification and adjustment was placed on the underlying economic and statistical structure of the balance sheet and not on the particular form of the utility function.

The model has served to delineate qualitative hypotheses concerning a bank's portfolio which might be used as the basis for empirical investigation. It has also served to identify such factors as length of planning horizon, size of capital accounts, and deposit variability as important considerations in comparing the customary loan/deposit ratios of banks. While many of these concepts have been investigated individually in previous studies, the balance sheet model has provided a unified framework in which they can be investigated simultaneously.

BIBLIOGRAPHY

[1] Bellman, Richard, "The Basic Concepts of Dynamic Programming," in Recent Developments in Information and Decision Processes, Robert E. Machol and Paul Gray (eds.), New York: Macmillan Co., 1962.

[2] Daellenbach, Hans G. and Archer, Stephan H., "The Optimal Bank Liquidity: A Multi-period Stochastic Model," The Journal of Financial and Quantitative Analysis, 4, No. 3, (September 1969), 329-343.

[3] Frost, Peter A., "Banks' Demand for Excess Reserves," Journal of Political Economy, 79, No. 4 (July/August 1971), 805-825.

[4] Gramley, Lyle E. and Chase, Samuel B., "Time Deposits in Monetary Analysis," The Federal Reserve Bulletin, 51, No. 10, (October 1965), 1380-1404.

[5] Kareken, John H., "Commercial Banks and the Supply of Money: A Market Determined Demand Deposit Rate," The Federal Reserve Bulletin, 53, No. 10, (October 1967), 1699-1712.

[6] Markowitz, Harry, Portfolio Selection, Cowles Foundation Monograph 16, New York: John Wiley and Sons, Inc., 1959.

[7] Miller, Merton H. and Orr, Daniel, "The Demand for Money by Firms: Extensions of Analytic Results," The Journal of Finance, XXIII, No. 5, (December 1968), 735-759.

[8] Morrison, George R., Liquidity Preference of Commercial Banks, Chicago: The University of Chicago Press, 1966.

[9] Mossin, Jan, "Optimal Multiperiod Portfolio Policies," The Journal of Business, 41, No. 2, (April 1968), 215-229.

[10] Murphy, Neil B., "A Cross-Section Analysis of Demand Deposit Variability," The Journal of Financial and Quantitative Analysis, III, No. 1, (March 1968), 87-95.

[11] Pierce, James L., "Commercial Bank Liquidity," The Federal Reserve Bulletin, 52, No. 8, (August 1966), 1090-1101.

[12] Poole, William, "Commercial Bank Reserve Management in a Stochastic Model: Implications for Monetary Policy," The Journal of Finance, XXIII, No. 5, (December 1968), 769-791.

[13] Porter, Richard C., "A Model of Bank Portfolio Selection," Yale Economic Essays, I, No. 2, (Fall 1961), 322-359.

[14] Scott, Ira O., Jr., "The Availability Doctrine: Theoretical Underpinings," The Review of Economic Studies, XXV, (October 1957), 41-48.

[15] Supel, Thomas M., "A Two-Period Balance Sheet Model for Banks," unpublished Ph.D. dissertation, University of Minnesota, 1969.

[16] Tobin, James, "Liquidity Preference as Behavior Towards Risk," The Review of Economic Studies, XXV (2) (February 1958), 65-86.

[17] Tobin, James, Chapter 8 of an unpublished manuscript, March 1964.

[18] Tobin, James, "The Theory of Portfolio Selection," Chapter 1 in The Theory of Interest Rates, F. H. Hahn and F. P. R. Brechling (eds.), London: Macmillan and Co., Ltd., 1965.

[19] Waud, Roger N., "Net Outlay Uncertainty and Liquidity Preference As Behavior Toward Risk," Journal of Money Credit and Banking, VII, No. 4, (November 1975), 499-506.

[20] Wolf, Charles R., "A Model for Selecting Commercial Bank Government Security Portfolios," The Review of Economics and Statistics, LI, No. 1, (February 1969), 40-52.

Appendix to: "A Two-Period Portfolio Problem in Balance Sheet Context

APPENDIX A

Because it was necessary to solve for first-period profits (Q_1) in order to analyze the myopic investor, the two-period profit problem was actually solved by computing second-period profits (Q_2) and then letting $Q = Q_1 + Q_2$. First-period profits are defined by (7), and second-period profits are defined by Table A.1.

From (7) and the uniformity assumption, the optimal α for the myopic investor is determined by

$$(A.1) \qquad e_1 - e_3\Delta(\alpha) = 0$$

or

$$(A.2) \qquad \hat{\alpha} = \delta_1 + \frac{e_1}{e_3}(\delta_2 - \delta_1)$$

where

$$e_1 = \frac{K}{v}\left[\frac{(\phi_1+\phi_2)}{2} - v\right]$$

and

$$e_3 = \frac{bK}{1-b}.$$

The same conditions apply to the e's as on the b's of (11) discussed in the text.

In order to compute algebraically expected second-period profits ($\overline{Q}_2$), it is necessary to account specifically for the regions of zero probability which are determined by the parameters of the probability distributions and the interest rates. Doing this in a systematic way requires the specification of every possible case that will be considered. Treating Z and X as functions of P_1, the computational problem arises because of the variety of ways that these curves can lie in the set determined by $[\phi_1, \phi_2]$ and $[\omega_1, \omega_2]$. In order to state the general limits of integration, we define

Table A.1

Second-Period Profits (E_2-E_1)

$$A_{11} = L - (1+r_{R1})R + KD_0 - LP_1 + \frac{(1+r_L)(1+r_{R1})R}{P_1}$$

$$+ \frac{r_L(1+r_L)L}{P_1} + \frac{K(1+r_L)D_1}{P_1} - KD_1 - \frac{K(1+r_L)D_0}{P_1}$$

$$A_{12} = (1+r_{R1})Rr_{R2} + r_L Lr_{R2} + KD_1 r_{R2} - KD_0 r_{R2} + LP_1 r_{R2}$$

$$- bLP_1 r_{R2} - bLP_1$$

$$A_{13} = r_L L + L - LP_1 + (1+r_{R1})Rr_{R2} + r_L Lr_{R2} + KD_1 r_{R2} - KD_0 r_{R2}$$

$$A_{21} = (1+r_L)L - \frac{(1+r_L)K\alpha}{(1-b)P_1} + \frac{(1+r_L)KD_1}{(1-b)P_1} + \frac{K\alpha}{1-b} - \frac{KD_1}{1-b} - LP_1$$

$$A_{22} = \frac{bK\alpha}{1-b} - \frac{bKD_1}{1-b} - bLP_1 + LP_1 r_{R2} - K\alpha r_{R2} + KD_1 r_{R2} - bLP_1 r_{R2}$$

where

$$Q_2 = E_2 - E_1 = \begin{cases} A_{11} & \text{if } D_1 \geq \alpha \text{ and } r_{R2} < Z \\ A_{12} & D_1 \geq \alpha \text{ and } r_{R2} > X \\ A_{13} & D_1 \geq \alpha \text{ and } Z \leq r_R \leq X \\ A_{21} & D_1 < \alpha \text{ and } r_{R2} \leq X \\ A_{22} & D_1 < \alpha \text{ and } r_{R2} > X \end{cases}$$

$$\phi_6 = Z^{-1}(\omega_2) = \frac{1 + r_2}{1 + \omega_2}$$

$$\phi_7 = Z^{-1}(\omega_1) = \frac{1 + r_2}{1 + \omega_1}$$

$$\phi_8 = X^{-1}(\omega_2) = \frac{1 + r_2}{(1-b)(1+\omega_2)}$$

$$\phi_9 = X^{-1}(\omega_1) = \frac{1 + r_2}{(1-b)(1+\omega_1)}.$$

Since $X > Z$, and assuming that we always have $\phi_1 < \phi_7$, there are 18 different arrangements of these parameters which require different limits of integration. These are listed in Table A.2 and the corresponding integration limits are shown in Table A.3. The general statement of expected second-period profits may now be stated as in Table A.4, and the maximization problem may be stated as in (6) of the text with $\overline{Q}_2$ substituted for $\overline{Q}$. Utilizing the integrals defined in Table A.5, the solution to the second-period problem may be written

(A.3) $$f_1 - f_3\Delta(\alpha) = 0$$

or

(A.4) $$\hat{\alpha} = \delta_1 + \frac{f_1}{f_3}(\delta_2-\delta_1)$$

where $$f_1 = \frac{K}{v}[v(M_2-(1+r_L)I_2+I_5-(1+r_L)I_7-I_8-M_9+M_9M_5- I_{18}-I_{21}$$
$$-g_3M_9M_8+g_3M_9M_7-g_4I_{27}) + (1+r_L)(M_7-I_{15}+I_{19}+g_3M_8$$
$$-g_3M_7+g_4I_{23}+I_5-g_4I_{24}) + (-I_1-I_6-I_{17}+I_{16}- I_{20}-g_3I_{22}$$
$$-g_4I_{25}+g_4I_{26}) + (1-b)(I_9+M_9I_{14}) - b(I_{10}-I_{12}+I_{14})]$$

Table A.2

Possible Cases of Limits on the Ω and Φ Integrals

Case:

I:	$\varphi_6 \le \varphi_1,\ \varphi_7 \le \varphi_2,\ \varphi_8 \le \varphi_1,\ \varphi_9 \le \varphi_2$
II:	$\varphi_6 \le \varphi_1,\ \varphi_7 \le \varphi_2,\ \varphi_8 \le \varphi_1,\ \varphi_9 > \varphi_2$
III:	$\varphi_6 \le \varphi_1,\ \varphi_7 > \varphi_2,\ \varphi_8 \le \varphi_1,\ \varphi_9 > \varphi_2$
IVa:	$\varphi_6 \le \varphi_1,\ \varphi_7 \le \varphi_2,\ \varphi_1 < \varphi_8 \le \varphi_7,\ \varphi_9 \le \varphi_2$
IVb:	$\varphi_6 \le \varphi_1,\ \varphi_7 \le \varphi_2,\ \varphi_7 < \varphi_8 < \varphi_2,\ \varphi_9 \le \varphi_2$
Va:	$\varphi_1 < \varphi_6 < \varphi_2,\ \varphi_7 \le \varphi_2,\ \varphi_6 < \varphi_8 \le \varphi_7,\ \varphi_9 \le \varphi_2$
Vb:	$\varphi_1 < \varphi_6 < \varphi_2,\ \varphi_7 \le \varphi_2,\ \varphi_7 < \varphi_8 \le \varphi_2,\ \varphi_9 \le \varphi_2$
VIa:	$\varphi_6 \le \varphi_1,\ \varphi_7 \le \varphi_2,\ \varphi_1 < \varphi_8 \le \varphi_7,\ \varphi_9 > \varphi_2$
VIb:	$\varphi_6 \le \varphi_1,\ \varphi_7 \le \varphi_2,\ \varphi_7 < \varphi_8 < \varphi_2,\ \varphi_9 > \varphi_2$
VII:	$\varphi_6 \le \varphi_1,\ \varphi_7 > \varphi_2,\ \varphi_1 < \varphi_8 \le \varphi_2,\ \varphi_9 > \varphi_2$
VIIIa:	$\varphi_1 < \varphi_6 < \varphi_2,\ \varphi_7 \le \varphi_2,\ \varphi_6 < \varphi_8 \le \varphi_7,\ \varphi_9 > \varphi_2$
VIIIb:	$\varphi_1 < \varphi_6 < \varphi_2,\ \varphi_7 \le \varphi_2,\ \varphi_7 < \varphi_8 \le \varphi_2,\ \varphi_9 > \varphi_2$
IX:	$\varphi_1 < \varphi_6 < \varphi_2,\ \varphi_7 > \varphi_2,\ \varphi_6 < \varphi_8 \le \varphi_2,\ \varphi_9 > \varphi_2$
X:	$\varphi_6 \le \varphi_1,\ \varphi_7 \le \varphi_2,\ \varphi_8 > \varphi_2,\ \varphi_9 > \varphi_2$
XI:	$\varphi_6 \le \varphi_1,\ \varphi_7 > \varphi_2,\ \varphi_8 > \varphi_2,\ \varphi_9 > \varphi_2$
XII:	$\varphi_1 < \varphi_6 < \varphi_2,\ \varphi_7 \le \varphi_2,\ \varphi_8 > \varphi_2,\ \varphi_9 > \varphi_2$
XIII:	$\varphi_1 < \varphi_6 < \varphi_2,\ \varphi_7 > \varphi_2,\ \varphi_8 > \varphi_2,\ \varphi_9 > \varphi_2$
XIV:	$\varphi_6 \ge \varphi_2$, other parameters irrelevant

Table A.3

Special Cases of the Expected Value of Q_2

Case	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	g_3	g_4
I	φ_1	φ_1	φ_7	φ_1	φ_9	φ_2	φ_1	φ_7	0	1
II	φ_1	φ_1	φ_7	φ_1	φ_2	φ_2	φ_1	φ_7	0	1
III	φ_1	φ_1	φ_2	φ_1	φ_2	φ_2	φ_1	φ_2	0	1
IVa	φ_1	φ_1	φ_7	φ_8	φ_9	φ_2	φ_8	φ_7	0	1
IVb	φ_1	φ_1	φ_7	φ_8	φ_9	φ_2	φ_7	φ_8	1	0
Va	φ_1	φ_6	φ_7	φ_8	φ_9	φ_2	φ_8	φ_7	0	1
Vb	φ_1	φ_6	φ_7	φ_8	φ_9	φ_2	φ_7	φ_8	1	0
VIa	φ_1	φ_1	φ_7	φ_8	φ_2	φ_2	φ_8	φ_7	0	1
VIb	φ_1	φ_1	φ_7	φ_8	φ_2	φ_2	φ_7	φ_8	1	0
VII	φ_1	φ_1	φ_2	φ_8	φ_2	φ_2	φ_8	φ_2	0	1
VIIIa	φ_1	φ_6	φ_7	φ_8	φ_2	φ_2	φ_8	φ_7	0	1
VIIIb	φ_1	φ_6	φ_7	φ_8	φ_2	φ_2	φ_7	φ_8	1	0
IX	φ_1	φ_6	φ_2	φ_8	φ_2	φ_2	φ_8	φ_2	0	1
X	φ_1	φ_1	φ_7	φ_2	φ_2	φ_2	φ_7	φ_2	1	0
XI	φ_1	φ_1	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	0	1
XII	φ_1	φ_6	φ_7	φ_2	φ_2	φ_2	φ_7	φ_2	1	0
XIII	φ_1	φ_6	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	0	1
XIV	φ_1	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	0	0

Note that: $x_7 = \min[x_3, x_4]$ and: $x_8 = \max[x_3, x_4]$

Also: $x_3 < x_4 \Rightarrow g_3 = 1$ and $g_4 = 0$

and: $x_4 \leq x_3 \Rightarrow g_3 = 0$ and $g_4 = 1$

except for Case XIV when $g_3 = g_4 = 0$.

Table A.4

Expected Second-Period Profits in the General Case

$$\bar{Q}_2 = \int_{\alpha}^{\delta_2}\int_{\varphi_1}^{x_2}\int_{\omega_1}^{\omega_2} A_{11}dF + \int_{\alpha}^{\delta_2}\int_{x_2}^{x_3}\int_{\omega_1}^{Z} A_{11}dF$$

$$+ \int_{\alpha}^{\delta_2}\int_{x_4}^{x_5}\int_{X}^{\omega_2} A_{12}dF + \int_{\alpha}^{\delta_2}\int_{x_5}^{\varphi_2}\int_{\omega_1}^{\omega_2} A_{12}dF$$

$$+ \int_{\alpha}^{\delta_2}\int_{x_2}^{x_7}\int_{Z}^{\omega_2} A_{13}dF + \int_{\alpha}^{\delta_2}\int_{x_8}^{x_5}\int_{\omega_1}^{X} A_{13}dF$$

$$+ g_3\int_{\alpha}^{\delta_2}\int_{x_7}^{x_8}\int_{\omega_1}^{\omega_2} A_{13}dF + g_4\int_{\alpha}^{\delta_2}\int_{x_7}^{x_8}\int_{Z}^{X} A_{13}dF$$

$$+ \int_{\delta_1}^{\alpha}\int_{\varphi_1}^{x_4}\int_{\omega_1}^{\omega_2} A_{21}dF + \int_{\delta_1}^{\alpha}\int_{x_4}^{x_5}\int_{\omega_1}^{X} A_{21}dF$$

$$+ \int_{\delta_1}^{\alpha}\int_{x_4}^{x_5}\int_{X}^{\omega_2} A_{22}dF + \int_{\delta_1}^{\alpha}\int_{x_5}^{\varphi_2}\int_{\omega_1}^{\omega_2} A_{22}dF$$

Table A.5

Integrals Needed to Compute $\overline{Q}_2$

$$I_1 = \int_{\varphi_1}^{x_2} P_1 d\Phi \quad I_2 = \int_{\varphi_1}^{x_2} \frac{1}{P_1} d\Phi \quad I_3 = \int_{\varphi_1}^{x_4} P_1 d\Phi \quad I_4 = \int_{\varphi_1}^{x_4} \frac{1}{P_1} d\Phi$$

$$I_5 = \int_{x_2}^{x_3} \Omega(Z) d\Phi \quad I_6 = \int_{x_2}^{x_3} P_1 \Omega(Z) d\Phi \quad I_7 = \int_{x_2}^{x_3} \frac{\Omega(Z)}{P_1} d\Phi$$

$$I_8 = \int_{x_4}^{x_5} \int_{X}^{\omega_2} r_{R2} d\Omega d\Phi \quad I_9 = \int_{x_4}^{x_5} \int_{X}^{\omega_2} P_1 r_{R2} d\Omega d\Phi \quad I_{10} = \int_{x_4}^{x_5} P_1 d\Phi$$

$$I_{11} = \int_{x_4}^{x_5} \Omega(X) d\Phi \quad I_{12} = \int_{x_4}^{x_5} P_1 \Omega(X) d\Phi \quad I_{13} = \int_{x_4}^{x_5} \frac{\Omega(X)}{P_1} d\Phi$$

$$I_{14} = \int_{x_5}^{\varphi_2} P_1 d\Phi \quad I_{15} = \int_{x_2}^{x_7} \Omega(Z) d\Phi \quad I_{16} = \int_{x_2}^{x_7} P_1 \Omega(Z) d\Phi$$

$$I_{17} = \int_{x_2}^{x_7} P_1 d\Phi \quad I_{18} = \int_{x_2}^{x_7} \int_{Z}^{\omega_2} r_{R2} d\Omega d\Phi \quad I_{19} = \int_{x_8}^{x_5} \Omega(X) d\Phi$$

$$I_{20} = \int_{x_8}^{x_5} P_1 \Omega(X) d\Phi \quad I_{21} = \int_{x_8}^{x_5} \int_{\omega_1}^{X} r_{R2} d\Omega d\Phi \quad I_{22} = \int_{x_7}^{x_8} P_1 d\Phi$$

$$I_{23} = \int_{x_7}^{x_8} \Omega(X) d\Phi \quad I_{24} = \int_{x_7}^{x_8} \Omega(Z) d\Phi \quad I_{25} = \int_{x_7}^{x_8} P_1 \Omega(X) d\Phi$$

$$I_{26} = \int_{x_7}^{x_8} P_1 \Omega(Z) d\Phi \quad I_{27} = \int_{x_7}^{x_8} \int_{Z}^{X} r_{R2} d\Omega d\Phi \quad M_1 = \int_{\varphi_1}^{\varphi_2} P_1 d\Phi$$

$$M_2 = \Phi(x_2) \quad M_4 = \Phi(x_4) \quad M_5 = \Phi(x_5) \quad M_7 = \Phi(x_7) \quad M_8 = \Phi(x_8)$$

$$M_9 = \int_{\omega_1}^{\omega_2} r_{R2} d\Omega$$

and

$$f_3 = K[\frac{1 + r_L}{1-b} I_4 + \frac{1 + r_L}{1-b} I_{13} - M_4 - I_{11} - \frac{b}{1-b}$$

$$- (1+r_L)I_2 + M_2 + I_5 - (1+r_L)I_7 - I_{18} - I_{21}$$

$$-g_3 M_9 M_8 + g_3 M_9 M_7 - g_4 I_{27}].$$

Note that solutions (10), (A.1), and (A.3) are related by

$$b_1 = e_1 + f_1$$

and

$$b_3 = e_3 + f_3$$

which is demanded by the definition of profits and the linearity of expectation.

APPENDIX B

The purpose of this appendix is to investigate the implications of the model when some of the more important assumptions are altered. We are interested in the robustness of the model from two points of view: (a) does the model still produce diversification, and (b) does the dynamic programming nature of the problem remain in tact, i.e., does the two-period investor hold a different portfolio than the myopic investor? Note that the myopic solution is given in equation (A.2).

B1. Certain Future Yields

Consider the situation where yields in the second period are known with certainty at t_0, i.e. assume

$$r_{R2} \equiv r^*_{R2} \equiv \omega_1 \equiv \omega_2$$

and

$$P_1 \equiv P^*_1 \equiv \phi_1 \equiv \phi_2.$$

Then

$$Z \equiv Z^* \equiv \frac{r_L + 1 - P^*_1}{P^*_1}$$

and

$$X \equiv X^* \equiv \frac{Z^* + b}{1 - b}.$$

Using the definitions of Table II and the constraints of equation (6), we have

$$\xi = r_L + 1 - P_1^* + bP_1^* - (1-b)P_1^* r_{R2}^*$$

and

$$W = \frac{1}{v}[(1+r_L)r_{R1}KD_0 + (2+r_{R1})r_L E_0] + \frac{(2r_L - r_{R1})K}{v}\alpha$$

$$= q + s\alpha$$

where q and s are defined by this equation.

There are three cases to consider. From Tables I and II we obtain the profit functions, and proceed to solve for the optimal portfolio in each case.

Case I: $r_{R2}^* < Z^*$

$$Q = \begin{cases} q + s\alpha - K\alpha Z^* + KZ^* D_1 & , D_1 \geq \alpha \\ q + s\alpha - K\alpha X^* + KX^* D_1 & , D_1 < \alpha \end{cases}$$

$$\overline{Q} = q + s\alpha - KZ^*\alpha - (KX^* - KZ^*)\alpha\Delta(\alpha) + KZ^*\overline{D}_1 + (KX^* - KZ^*)\int_{\delta_1}^{\alpha} D_1 d\Delta$$

$$\frac{d\overline{Q}}{d\alpha} = b_1 - b_3\Delta(\alpha)$$

where

$$b_1 = \frac{K(1+r_L)(P_1^* - v)}{vP_1^*} \text{ and } b_3 = K(X^* - Z^*)$$

Thus, the necessary conditions for diversification are

$$1 + r_{R1} - r_L < P_1^* < \frac{1 + r_{R1} - r_L}{1 - b}$$

Case II: $r^*_{R2} > X^*$

$$\overline{Q} = [q - \frac{\xi}{v}(r_{R1}KD_0 + (1+r_{R1})E_0)] + (s - \frac{\xi K}{v} - Kr^*_{R2})\alpha + Kr^*_{R2}\,\overline{D}_1$$

$$\frac{d\overline{Q}}{d\alpha} = b_1$$

where

$$b_1 = \frac{K}{v}[(P^*_1 - v - bP^*_1)(1+r^*_{R2})]$$

In this case there is no diversification, and only loans are held when

$$P^*_1 > \frac{1 + r_{R1} - r_L}{1 - b}$$

and only defensive assets are held when

$$P^*_1 < \frac{1 + r_{R1} - r_L}{1 - b}$$

Case III: $Z^* \le r^*_{R2} \le X^*$

$$Q = \begin{cases} q + s\alpha - Kr^*_{R2}\alpha + Kr^*_{R2}D_1 & , D_1 \ge \alpha \\ q + s\alpha - KX^*\alpha + KX^*D_1 & , D_1 < \alpha \end{cases}$$

$$\overline{Q} = (q + Kr^*_{R2}\overline{D}_1) + (s - Kr^*_{R2})\alpha - (KX^* - Kr^*_{R2})\alpha\Delta(\alpha) + (KX^* - Kr^*_{R2})\int_{\delta_1}^{\alpha} D_1 d\Delta$$

$$\frac{d\overline{Q}}{d\alpha} = b_1 - b_3\Delta(\alpha)$$

where $b_1 = \frac{K}{v}[(r_L - r^*_{R2}) + (r_L - r_{R1})(1+r^*_{R2})]$

$$b_3 = K(X^* - r^*_{R2})$$

In order that $b_1 > 0$, we must have

$$r^*_{R2} < \frac{2r_L - r_{R1}}{1 + r_{R1} - r_L}.$$

And for $\frac{b_1}{b_3} < 1$, we must have

$$X^* > \frac{2r_L - r_{R1}}{1 + r_{R1} - r_L},$$

or

$$Z^* > \frac{2r_L - r_{R1} - (1 + r_L)b}{1 + r_{R1} - r_L}.$$

These conditions on X^* and Z^* imply that

$$P_1^* < \frac{1 + r_{R1} - r_L}{1 - b},$$

and since $Z^* \leq r_{R2}^*$, the condition on Z^* implies that

$$P_1^* > 1 + r_{R1} - r_L.$$

Thus, the necessary conditions for diversification in Case III are the same as those of Case I. However, the end points of Case III deserve special consideration. When $r_{R2}^* = X^*$, then the conditions of Case II apply since $b_3 = 0$ and the b_1's of both cases are the same. When $r_{R2}^* = Z$, then the conditions of Case I apply since the b_i's are the same for both cases.

From these solutions to the certain yield case, it can be shown that only when $r_{R2}^* \leq Z^*$ will the two-period and myopic investors necessarily hold the same portfolio. In Case I, the necessary conditions for diversification are the same for both investors, and, further, the ratio of b_1 to b_3 is the same for each type of investor even though the b_i's are different. In Case II, the two-period investor never diversifies, and both investors need hold the same portfolio only if they both hold all loans. The two-period investor holding only defensive assets implies nothing about the myopic investor's portfolio. In Case III, the same necessary conditions for diversification apply to both investors; but they may hold different (diversified) portfolios, because the actual portfolio distribution for the two-period investor is a function of r^*_{R2} which is, by definition, ignored by the myopic investor.

Thus, when future yields are certain, the model (because of transactions costs) still generates diversified portfolios and also retains its two-period nature except when $r^*_{R2} \leq Z^*$. In other words, if an investor is both (a) certain about future rates, and (b) certain that short-term yields will be less than long-term yields, then, according to this model, he should act myopically.

B2. Certain Deposits

Under the assumption that deposits are independent of the other random variables in the model, the generic form of the expected profits equation may be written as

$$\text{(B.1)} \qquad \overline{Q} = c_0 + c_1\alpha - c_3\,[\alpha\Delta(\alpha) - \int_{\delta_1}^{\alpha} D_1 d\Delta].$$

While the coefficients for the two-period investor will be different from those of the myopic investor, the generic form of the equation remains the same for both problems. Now suppose deposits are certain, i.e., the deposit distribution is defined by

$$\Delta(\alpha) = \begin{cases} 0 \text{ if } \alpha < D_1^* \\ 1 \text{ if } \alpha \geq D_1^* \end{cases}$$

Under this assumption, expected profits are defined by two linear segments

$$\text{(B.2)} \qquad \overline{Q} = \begin{cases} c_0 + c_1\alpha & \text{if } \alpha < D_1^* \\ [c_0 + c_3 D_1^*] + [c_1 - c_3]\alpha & \text{if } \alpha \geq D_1^* \end{cases}$$

Equation (B.2) shows that if the necessary conditions ($0 < c_1 < c_3$) for diversification hold, then the expected profits function has the shape of an inverted V with a maximum at D_1^*. Thus $\hat{\alpha} = D_1^*$. Further, if the necessary conditions hold for <u>both</u> types of investors, then they will hold the same portfolio because $\hat{\alpha}$ and the constraint equations are the same for both investors. If it is not the case that $0 < c_1 < c_3$ for at least one of the investors, then a specific analysis of the particular portfolios must be made for comparison purposes. However, although this has not been proved rigorously for the general case, the general results still seem to be (i) if the two-period investor holds only loans, the myopic investor holds only loans, and (ii) if the two-period investor holds only defensive assets, nothing is implied about the portfolio of the myopic investor.

Thus the assumption of certain deposits still permits the two-period balance sheet model to generate diversified portfolios, but it destroys the interesting dynamic programming aspects of the original model in the sense that if both investors hold diversified portfolios, then those portfolios are the same.

B3. Zero Transactions Costs

A particular idiosyncrasy of this model is the result that when there are no transactions costs, it does not necessarily follow that the two-period investor will hold the same portfolio as the myopic investor even though they both hold nondiversified portfolios. The reason for this result is, essentially, that the random variable P_1 enters the profit equation in a ratio rather than a linear form -- and the expected value of a reciprocal is not equal to the reciprocal of the expected value.

To illustrate this problem, consider the case when there are no transactions costs (b=0), and second-period yields on defensive assets are certain ($r_{R2} \equiv r$). Since there are no transactions costs, the portfolio held in the second period is independent of the first period outcome and hence of D_1. Thus, utilizing Table II, the profit function may be written as

$$(B.3) \qquad Q = \begin{cases} W - KZ\alpha + KZD_1 & \text{if } P_1 \leq p \\ W - \xi L - Kr\alpha + KD_1 r & \text{if } P_1 > p \end{cases}$$

where

$$W = r_{R1}R + 2r_L L$$

$$\xi + (1+r_L) - (1+r)P_1$$

and

$$p = \frac{1 + r_L}{1 + r}.$$

Letting K = 1 and $E_0 = 0$, then the maximization problem is

$$\max_{\alpha} \overline{Q} = \int_{\delta_1}^{\delta_2}\int_{\phi_1}^{p} (W - KZ\alpha + KZD_1)d\Phi d\Delta + \int_{\delta_1}^{\delta_2}\int_{p}^{\phi_2}(W - \xi L - Kr\alpha + KrD_1)d\Phi d\Delta$$

$$\text{s.t. } W = \frac{(1+r_L)r_{R1}D_0}{v} + \frac{2r_L - r_{R1})}{v}\alpha$$

$$= q + s\alpha$$

and

$$L = \frac{r_{R1}D_0}{v} + \frac{1}{v}\alpha$$

where the constraints incorporate the balance sheet constraints on R and L in terms of α.

Substituting the constraints, integrating with respect to uniform distributions, and then differentiating yields the first order condition

(B.4) $$\frac{d\bar{Q}}{d\alpha} = s - \frac{(1+r_L)}{\phi_3}\ln\frac{P}{\phi_1} + \frac{(p-\phi_1)}{\phi_3} - \frac{(1+r_L)(\phi_2-p)}{\phi_3 v}$$

$$+ \frac{(1+r)(\phi_2^2-p^2)}{2\phi_3 v} - \frac{r(\phi_2-p)}{\phi_3}.$$

Assuming that the myopic investor holds only loans (i.e. $\frac{\phi_1+\phi_2}{2} > 1 + r_{R1} - r_L$), the point of this exercise is to construct an example such that the two-period investor holds only defensive assets (i.e. $\frac{d\bar{Q}}{d\alpha} < 0$). Suppose $p = \phi_2$. Then by appropriate manipulation of (B.4), it follows that

$$\frac{d\bar{Q}}{d\alpha} < 0 \text{ if}$$

(B.5) $$\frac{1}{\phi_3}\ln\frac{\phi_2}{\phi_1} > \frac{1}{1+r_{R1}-r_L}.$$

A set of values which satisfy inequality (B.5) is:

$$\phi_1 = 0.8 \qquad r_{R1} = 0.19$$

$$\phi_2 = 1.2 \qquad r = 0$$

$$r_L = 0.2$$

Thus, reducing transactions costs to zero eliminates diversification in this model, but it is possible that the two types of investors hold different (nondiversified) portfolios.

One way of avoiding this idiosyncrasy of the model would be to take P_1 as fixed (say $P_1 \equiv 1$) and to let r_L be random in the second period. This formulation might be reasonable if, for example, the L-asset were interpreted as a passbook savings account where the interest payment varied from period to period. Taking r_L as random rather than P_1 would also have the advantage of considerably simplifying the algebra of the problem.

APPENDIX C

The computer program given below is written in the BASIC language for a time sharing consol. It should run without major alteration on any machine accepting the BASIC language.

The notation of the program follows that of the text quite closely and mnemonic translations are made where possible. Specifically, the following translations from the text to BASIC are used:

D_0 = D0	ϕ_1 = F1
K = K	ϕ_2 = F2
r_{R1} = R1	ω_1 = W1
r_L = R2	ω_2 = W2
δ_1 = D_1	b = B
δ_2 = D2	E_0 = E0

The integrals of Table A.5 evaluated with uniform distributions, and the x_i's of Table A.3 translate directly into BASIC with the following exceptions:

I_i becomes $J_{(i-9)}$, $i = 10, \ldots, 18$

and

I_i becomes $K_{(i-18)}$, $i = 19, \ldots, 27$.

The program denotes the coefficients e_1 and e_3 as S1 and S3, the coefficients f_1 and f_3 as S5 and S6 respectively, and b_1 and b_3 as T1 and T3.

The bulk of the program is devoted to determining the values of the x_i's. After the coefficients are calculated, the balance sheet is computed and printed in subroutine 1500. As discussed in the text, profits in the second period are not computed for $\varphi_1 > \varphi_7$. In this case, "No Intersection" is printed for both Q_1 and Q profits even though this consideration does not apply to Q_1.

```
100 DO = 100
105 K = .9
110 R1 = .05
120 D1=50
125 D2=110
130 F1=.9
135 F2=1.1
140 W1=.02
145 W2=.12
150 B=.1
155 E0=10
200 FØR R2=.04 TØ .131 STEP .01
201 PRINT
202 PRINT"                RL=" R2
300 C = 1-B
305 V = 1+R1-R2
310 V2 = 1+R2
315 F3 = F2-F1
320 F4 = 1/F3
325 W3 = W2-W1
330 W6 = 1/W3
335 W7 = W6*F4
340 W8 = 1+W1
345 D3 = D2-D1
350 M1 = (F1+F2)/2
355 M9 = (W1+W2)/2
360 DEF FNF(X) = (X-F1)/F3
365 F6 = V2/(1+W2)
370 F7 = V2/W8
375 F8 = V2/((1+W2)*C)
380 F9 = V2/((1+W1)*C)
395 IF F6>F2 THEN 955
396 IF F1<=F7 THEN 400
397 PRINT "NØ INTERSECTIØN"
398 GØ TØ 5000
400 IF F6>F1 THEN 700
```

```
405 IF F7>F2 THEN 615
410 IF F8>F1 THEN 470
415 IF F9>F2 THEN 445
420 X2 = F1
425 X3 = F7
430 X4 = F1
435 X5 = F9
440 GØ TØ 900
445 X2 = F1
450 X3 = F7
455 X4 = F1
460 X5 = F2
465 GØ TØ 900
470 IF F8>F7 THEN 530
475 IF F9>F2 THEN 505
480 X2 = F1
485 X3 = F7
490 X4 = F8
495 X5 = F9
500 GØ TØ 900
505 X2 = F1
510 X3 = F7
515 X4 = F8
520 X5 = F2
525 GØ TØ 900
530 IF F8>F2 THEN 590
535 IF F9>F2 THEN 565
540 X2 = F1
545 X3 = F7
550 X4 = F8
555 X5 = F9
560 GØ TØ 900
565 X2 = F1
570 X3 = F7
575 X4 = F8
580 X5 = F2
585 GØ TØ 900
590 X2 = F1
595 X3 = F7
600 X4 = F2
605 X5 = F2
610 GØ TØ 900
615 IF F8>F1 THEN 645
620 X2 = F1
625 X3 = F2
630 X4 = F1
635 X5 = F2
640 GØ TØ 900
645 IF F8>F2 THEN 675
650 X2 = F1
```

```
655 X3 = F2
660 X4 = F8
665 X5 = F2
670 GØ TØ 900
675 X2 = F1
680 X3 = F2
685 X4 = F2
690 X5 = F2
695 GØ TØ 900
700 IF F7>F2 THEN 850
705 IF F8>F7 THEN 765
710 IF F9>F2 THEN 740
715 X2 = F6
720 X3 = F7
725 X4 = F8
730 X5 = F9
735 GØ TØ 900
740 X2 = F6
745 X3 = F7
750 X4 = F8
755 X5 = F2
760 GØ TØ 900
765 IF F8>F2 THEN 825
770 IF F9>F2 THEN 800
775 X2 = F6
780 X3 = F7
785 X4 = F8
790 X5 = F9
795 GØ TØ 900
800 X2 = F6
805 X3 = F7
810 X4 = F8
815 X5 = F2
820 GØ TØ 900
825 X2 = F6
830 X3 = F7
835 X4 = F2
840 X5 = F2
845 GØ TØ 900
850 IF F8>F2 THEN 880
855 X2 = F6
860 X3 = F2
865 X4 = F8
870 X5 = F2
875 GØ TØ 900
880 X2 = F6
885 X3 = F2
890 X4 = F2
895 X5 = F2
900 IF X3<X4 THEN 930
```

```
905 X7 = X4
910 X8 = X3
914 G3 = 0
920 G4 = 1
925 GØ TØ 1000
930 X7 = X3
935 X8 = X4
940 G3 = 1
945 G4 = 0
950 GØ TØ 1000
955 X2=F2
956 X3=F2
957 X4=F2
958 X5=F2
959 X7=F2
960 X8=F2
961 G3=0
962 G4=0
1000 I1 = (F4/2)*(X2↑2-F1↑2)
1005 I2= F4*(LØG(X2/F1))
1010 I3 = (F4/2)*(X4↑2-F1↑2)
1015 I4 = F4*(LØG(X4/F1))
1020 I5 = W7*(V2*(LØG(X3/X2))-W8*(X3-X2))
1025 I6 = W7*(V2*(X3-X2)-(W8/2)*(X3↑2-X2↑2))
1030 I7 = W7*(V2*(1/X2-1/X3)-W8*(LØG(X3/X2)))
1035 I8 = (W7/2)*((W2↑2-1)*(X5-X4)-(V2↑2/C↑2)*(1/X4-1/X5))
1036 I8 = I8+(W7/2)*(2*V2/C)*(LØG(X5/X4))
1040 I9=(W7/2)*((W2↑2-1)/2)*(X5↑2-X4↑2)
1041 I9=I9-(W7/2)*(V2↑2/C↑2)*(LØG(X5/X4))
1042 I9=I9+(W7/2)*(2*V2/C)*(X5-X4)
1045 J1 = (F4/2)*(X5↑2-X4↑2)
1050 J2 = W7*((V2/C)*(LØG(X5/X4))-W8*(X5-X4))
1055 J3 = W7*((V2/C)*(X5-X4)-(W8/2)*(X5↑2-X4↑2))
1060 J4 = W7*((V2/C)*(1/X4-1/X5)-W8*(LØG(X5/X4)))
1065 J5 = (F4/2)*(F2↑2-X5↑2)
1070 J6=W7*(V2*LØG(X7/X2)-W8*(X7-X2))
1075 J7 = W7*(V2*(X7-X2)-(W8/2)*(X7↑2-X2↑2))
1080 J8 = (F4/2)*(X7↑2-X2↑2)
1085 J9 = (W7/2)*((W2↑2-1)*(X7-X2)-(V2↑2)*(1/X2-1/X7))
1086 J9 = J9+(W7/2)*(2*V2)*(LØG(X7/X2))
1090 K1 = W7*((V2/C)*(LØG(X5/X8))-W8*(X5-X8))
1095 K2 = W7*((V2/C)*(X5-X8)-(W8/2)*(X5↑2-X8↑2))
1100 K3=(W7/2)*(V2↑2/C↑2)*(1/X8-1/X5)
1101 K3=K3-(W7/2)*(2*V2/C)*(LØG(X5/X8))
1102 K3=K3+(W7/2)*(1-W1↑2)*(X5-X8)
1105 K4 = (F4/2)*(X8↑2-X7↑2)
1110 K5 = W7*((V2/C)*(LØG(X8/X7))-W8*(X8-X7))
1115 K6 = W7*(V2*(LØG(X8/X7))-W8*(X8-X7))
1120 K7 = W7*((V2/C)*(X8-X7)-(W8/2)*(X8↑2-X7↑2))
1125 K8 = W7*(V2*(X8-X7)-(W8/2)*(X8↑2-X7↑2))
```

```
1130 K9 = (W7/2)*(((V2↑2-(C↑2)*(V2↑2))/C↑2)*(1/X7-1/X8))
1131 K9 = K9-(W7/2)*(2*(V2-C*V2)/C)*(LØG(X8/X7))
1135 M2 = FNF(X2)
1140 M4 = FNF(X4)
1145 M5 = FNF(X5)
1150 M7 = FNF(X7)
1155 M8 = FNF(X8)
1210 PRINT"Q1 : ";
1220 S1 = (K/V)*(M1-V)
1230 S3 = (K*B)/C
1240 B1 = S1
1250 B3 = S3
1260 GØ SUB 1500
1270 S5 = K*(M2-V2*I2+I5-V2*I7-I8-M9)
1280 S5 = S5+K*(M9*M5-J9-K3-G3*M9*M8+G3*M9*M7-G4*K9)
1290 S5 = S5+V2*(M7-J6+K1+G3*M8-G3*M7+G4*K5+I5-G4*K6)*(K/V)
1300 S5=S5+(K/V)*(-I1-I6-J8+J7-K2-G3*K4-G4*K7+G4*K8)
1310 S5 = S5+(K/V)*(C*(I9+M9*J5)-B*(J1-J3+J5))
1320 S6 = (V2/C)*(I4+J4)-M4-J2-(B/C)-V2*I2+M2+I5
1330 S6 = S6-V2*I7-J9-K3-G3*M9*M8+G3*M9*M7-G4*K9
1340 S6 = K*S6
1360 PRINT"Q  : ";
1370 T1 = S1+S5
1380 T3 = S3+S6
1390 B1 = T1
1400 B3 = T3
1410 GØ SUB 1500
1490 GØ TØ 5000
1500 C1=B1/B3
1520 IF B1<0 THEN 1610
1530 IF C1>1 THEN 1590
1540 A=D1+C1*D3
1550 L = (K*A+K*R1*D0+(1+R1)*E0)/V
1552 IF L>K*D0+E0 THEN 1590
1560 R = K*D0+E0-L
1565 L=(INT(10*L+.5))/10
1566 R=(INT(10*R+.5))/10
1570 PRINT "L = "L, "R = "R
1580 GØ TØ 1640
1590 L=K*D0+E0
1592 GØ TØ 1560
1610 R = K*D0+E0
1611 L=0
1612 GØ TØ 1570
1640 RETURN
5000 NEXT R2
6000 END
```