Frequency Domain Characterizations of
Rational Expectations Equilibria with the
Method of Undetermined Coefficients

by

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1. INTRODUCTION

Saracoglu and Sargent [1977] gave a formula describing the covariation of the rate of inflation and the rate of money creation in a world where Cagan's portfolio balance schedule rules (Cagan [1956]). Specifically, they assume that the rate of money creation is an indeterministic stochastic process which is econometrically exogenous with respect to the rate of inflation. Under these assumptions they formulate the rational expectations equilibria in the frequency domain, utilizing Lucas' [1973] method of undetermined coefficients. As it turns out, the number of undetermined coefficients is one regardless of the autoregressive order of the rate of money creation, provided that the portfolio balance schedule involves only one period ahead forecasts of the rate of inflation.

However, the Cagan's portfolio balance schedule is restrictive in the sense that it involves only one period ahead forecasts of the rate of inflation. A more realistic approach would require that the demand function for money to depend upon the expected rates of inflation over a horizon that extends beyond one period. In particular, one would normally want to formulate the problem over an infinite horizon and consider finite horizon models as special cases. The generalization of Saracoglu and Sargent's method to handle infinite horizons, however, seemingly results in infinitely many undetermined coefficients. In particular, (one can conclude that) if the demand for money depends upon forecasts of the rate of inflation of T periods ahead, there will be T undetermined coefficients. If this conclusion were correct, than Saracoglu and Sargent's method will be applicable only to a limited number of problems.

It is the purpose of this paper to prove that the frequency domain characterization of rational expectations equilibria can be formulated such that even though there are infinitely many undetermined coefficients, only one of them help to characterize the covariation of the rate of inflation and the rate of money creation, and its value can be determined independently from other undetermined coefficients. Such a formulation not only results in a considerable simplification but also provides the necessary link between the parameters of the structural system formulated in terms of unobservable expectations and the parameters of estimable reduced form equations, thereby doing away with the cumbersome process of finding proxies for unobservable variables. The method developed in this paper is versatile enough to be applicable to a broad class of linear, rational expecations models with exogenous driving variables.

2. DERIVATION OF THE FORMULA

We start by considering a generalized version of Cagan's portfolio balance schedule[1/]

$$m_t - P_t = \sum_{j=1}^{\infty} \psi_j E_t x_{t+j} + u_t, \ \psi_j < 0 \tag{1}$$

where m_t is the log of money supply at time t, P_t is the log of price level at time t, x_{t+j} is the rate of inflation at time t+j which is defined as $x_{t+j}=P_{t+j}-P_{t+j-1}$, and $E_t x_{t+j}$ is the linear least squares estimate of x_{t+j} conditional on the information available at time t. It will be convenient to rewrite equation (1) in an alternative form

$$m_t - P_t = -\phi \sum_{j=1}^{\infty} \gamma_j E_t x_{t+j} + u_t \tag{1a}$$

where

$$-\phi = \sum_{j=1}^{\infty} \psi_j$$

and

$$\gamma_j = -\frac{\psi_j}{\phi} > 0.$$

Consequently,

$$\sum_{j=1}^{\infty} \gamma_j = 1.$$

If we first difference equation (1a), we get

$$\mu_t - x_t = -\phi(1-z) \sum_{j=1}^{\infty} \gamma_j E_t x_{t+j} + \varepsilon_t \tag{2}$$

with z being the lag operator and $\varepsilon_t (=u_t-u_{t-1})$ is the disturbance that is orthogonal to μ_t at all leads and lags.

Now suppose that the rate of money creation (μ_t) is a second-order stationary indeterministic stochastic process that possesses both an autoregressive representation and a moving-average representation:

$$A(z)\mu_t = \upsilon_t$$

or

$$\mu_t = B(z)\upsilon_t$$

where the generating functions A(z) and B(z) are related as

$$\sum_{i=0}^{j} a_i b_{j-i} = \begin{cases} 1 & j=0 \\ 0 & j>0 \end{cases} .$$

Strict econometric exogeneity of μ_t with respect to x_t implies that the projection of x_t on the entire μ_t process is one sided on the present and the past. Let $S_{x\mu}(z)$ be the cross-covariance generating function defined as

$$S_{x\mu}(z) = \sum_{j=-\infty}^{\infty} z^j E[(x_t - Ex_t)(\mu_{t-j} - E\mu_{t-j})].$$

Then the projection of x_t and on the entire μ_t process is given by[2/]

$$E[x_t | \mu_t, \mu_{t-1}, \ldots] = \frac{S_{x\mu}(z)}{B(z)B(z^{-1})} \mu_t = \frac{S_{x\mu}(z)}{S_\mu(z)} \mu_t. \qquad (3)$$

The econometric exogeneity of μ_t implies that the distributed lag generating function in equation (3) involves only nonnegative powers of z and therefore can be written as

$$\frac{S_{x\mu}(z)}{B(z)B(z^{-1})} = \frac{P(z)}{B(z)}$$

where both P(z) and B(z) are one sided, and P(z) is the generating function for the undetermined coefficients.

The linear least-squares prediction of x_{t+j} conditional on $(\mu_t, \mu_{t-1}, \ldots)$ is given by:

$$E[x_{t+j} | \mu_t, \mu_{t-1}, \ldots] = [z^{-j}P(z)]_+ \frac{1}{B(z)} \mu_t \tag{4}$$

and $[z^{-j}P(z)]_+$ means ignore all negative powers of z. As P(z) is one sided then we can write

$$z^{-j}P(z) = z^{-j}p_0 + z^{-j+1}p_1 + \ldots + p_j + zp_{j+1} + \ldots.$$

Hence,

$$[z^{-j}P(z)]_+ = z^{-j}P(z) - \sum_{i=0}^{j-1} p_i z^{i-j}.$$

Substituting this result into equation (4) yields

$$E_t x_{t+j} = [z^{-j}P(z) - \sum_{i=0}^{j-1} p_i z^{i-j}] \frac{1}{B(z)} \mu_t$$

$$E_t x_{t+j} = [z^{-j} \frac{S_{x\mu}(z)}{S_\mu(z)} - \frac{1}{B(z)} \sum_{i=0}^{j-1} p_i z^{i-j}] \mu_t. \tag{5}$$

Using equation (5) we can form

$$\sum_{j=1}^{\infty} \gamma_j E_t x_{t+j} = \{\frac{S_{x\mu}(z)}{S_\mu(z)} \sum_{j=1}^{\infty} \gamma_j z^{-j} - \frac{1}{B(z)} \sum_{j=1}^{\infty} z^{-j} \sum_{i=0}^{\infty} p_i \gamma_{i+j}\} \mu_t.$$

Substituting this result into (2) and solving for x_t gives

$$x_t = [1+\phi(1-z)\{\frac{S_{x\mu}(z)}{S_\mu(z)} \sum_{j=1}^{\infty} \gamma_j z^{-j} - \frac{1}{B(z)} \sum_{j=1}^{\infty} z^{-j} \sum_{i=0}^{\infty} p_i \gamma_{i+j}\}]\mu_t - \varepsilon_t. \tag{6}$$

If we multiply both sides of equation (6) by μ_{t-s} and take expectations we get

$$S_{x\mu}(z) = [1+\phi(1-z)\{\frac{S_{x\mu}(z)}{S_{\mu}(z)}\sum_{j=1}^{\infty}\gamma_j z^{-j} - \frac{1}{B(z)}\sum_{j=1}^{\infty} z^{-j}\sum_{i=0}^{\infty} p_i\gamma_{i+j}\}]S_{\mu}(z)$$

$$(1-\phi(1-z)\sum_{j=1}^{\infty}\gamma_j z^{-j})\frac{S_{x\mu}(z)}{S_{\mu}(z)} = 1 - \frac{\phi(1-z)}{B(z)}\sum_{j=1}^{\infty}\eta_j z^{-j} \qquad (7)$$

where

$$\eta_j = \sum_{i=0}^{\infty} p_i\gamma_{i+j} \ .$$

For ease of notation let us define the generating functions

$$R(z^{-1}) = r_0 + r_1 z^{-1} + r_2 z^{-2} + \ldots$$

where

$$r_j = \begin{cases} 1 + \phi\gamma_1 & j=0 \\ \phi(\gamma_{j+1}-\gamma_j) & j>0 \end{cases}$$

$$Q(z^{-1}) = q_0 + q_1 z^{-1} + q_2 z^{-2} + \ldots$$

where

$$q_j = \begin{cases} \phi\eta_1 & j=0 \\ \phi(\eta_{j+1}-\eta_j) & j>0 \end{cases}$$

and rewrite equation (7) as

$$R(z^{-1})\frac{S_{x\mu}(z)}{S_{\mu}(z)} = 1 + \frac{Q(z^{-1})}{B(z)} \ . \qquad (8)$$

Note that in equation (8) the undetermined coefficients are generated by $Q(z^{-1})$. Solving for $\frac{S_{x\mu}(z)}{S_{\mu}(z)}$ yields

$$\frac{S_{x\mu}(z)}{S_{\mu}(z)} = \frac{1}{R(z^{-1})} + \frac{Q(z^{-1})}{R(z^{-1})B(z)}$$

$$= T(z^{-1}) + \frac{T(z^{1})Q(z^{-1})}{B(z)} \qquad (9)$$

where

$$T(z^{-1}) = \frac{1}{R(z^{-1})} = \sum_{j=0}^{\infty} t_j z^j .$$

As the strict econometric exogeneity implies that the left-hand side of equation (9) is one sided on nonnegative powers of z, so must be the right-hand side for equality to be meaningful. This condition implies a solution for q_j's. In particular the undetermined coefficients, q_j's, should be chosen so that the coefficients in front of the negative powers of z are zero.

The first step in solving for q_j's is to note that

$$\frac{S_{x\mu}(z)}{S_{\mu}(z)} = \frac{P(z)}{B(z)} .$$

Multiplying both sides of equation (9) by B(z) gives

$$P(z) = B(z)T(z^{-1}) + Q(z^{-1})T(z^{-1}). \qquad (10)$$

The product $B(z)T(z^{-1})$ is clearly two sided and therefore can be written as

$$B(z)T(z^{-1}) = \lambda^{*}(z) + \lambda(z^{-1})$$

where

$$\lambda^{*}(z) = \sum_{j=0}^{\infty} \lambda_j^{*} z^j$$

$$\lambda_j^{*} = \sum_{i=0}^{\infty} t_i b_{i+j}$$

$$\lambda(z^{-1}) = \sum_{j=1}^{\infty} \lambda_j z^{-j}$$

$$\lambda_j = \sum_{i=0}^{\infty} b_i t_{i+j} .$$

Hence we can write equation (10) as

$$P(z) = \lambda^*(z) + \lambda(z^{-1}) + Q(z^{-1})T(z^{-1}). \qquad (11)$$

Suppose that we can choose $Q(z^{-1})$ so as to make the right-hand side of (11) one sided on nonnegative powers of z; then P(z) has the representation

$$P(z) = \lambda^*(z) + q_0 t_0. \qquad (12)$$

Note that a knowledge of P(z) is sufficient in terms of characterizing the covariance structure between x and μ. In fact, P(z) is the generating function of the original undetermined coefficients p_i's. Therefore, equation (12) constitutes a solution of the infinite dimensional problem of undetermined coefficients in terms of structural parameters and an additional undetermined coefficient q_0.

So far we have assumed that in equation (11) $Q(z^{-1})$ can be chosen so as to make the coefficients of negative powers to z equal to zero. Consequently, we now turn to prove that a solution actually exists. For (11) to be one sided we require that[3/]

$$\lambda(z^{-1}) + Q(z^{-1})T(z^{-1}) - q_0 t_0 = 0.$$

The solution for $Q(z^{-1})$ is given by

$$Q(z^{-1}) = \frac{1}{T(z^{-1})}(t_0 q_0 - \lambda(z^{-1}))$$

$$Q(z^{-1}) = R(z^{-1})(t_0 q_0 - \lambda(z^{-1})). \qquad (13)$$

Equation (13) expresses $Q(z^{-1})$ in terms of structural parameters and the unknown quantity q_0. To solve for q_0 evaluate (13) at z=1. Then

$$\sum_{i=0}^{\infty} q_i = (\sum_{i=0}^{\infty} r_i)(t_0 q_0 - \sum_{i=1}^{\infty} \lambda_i).$$

From the definition of q_i's and r_i's we get

$$\sum_{i=0}^{\infty} q_i = 0, \quad \sum_{i=0}^{\infty} r_i = 1.$$

Consequently,

$$t_0 q_0 = \sum_{i=1}^{\infty} \lambda_i = \lambda(1). \qquad (14)$$

Substituting this result into equation (12) gives the solution of the problem of undetermined coefficients in terms of structural parameters

$$P(z) = \lambda^*(z) + \lambda(1).$$

Note that the only undetermined coefficient that we are interested in is q_0 due to the fact that it is the only unknown quantity that helps to characterize the distributed lag coefficients of x_t on μ_t. Recalling the definition of q_0 gives

$$q_0 = \phi \eta_1 = \phi \sum_{i=0}^{\infty} p_i \gamma_{i+1}.$$

In the special case when $\gamma_1=1$ and $\gamma_j=0$ $j>1$ the undetermined coefficient reduces to

$$q_0 = \phi p_0$$

which is precisely the result that Saracoglu and Sargent got.

3. CHARACTERIZATIONS OF THE DISTRIBUTED LAG COEFFICIENTS

The results of the previous section indicate that the rational expectations equilibrium of the vector stochastic process (x_t, μ_t) is such that

$$x_t = k(z)\mu_t + e_t$$

where e_t is orthogonal to μ_t at all leads and lags and the distributed lag generating function, $k(z)$, is one sided on present and past; and is given by

$$k(z) = \frac{S_{x\mu}(z)}{S_\mu(z)} = \frac{P(z)}{B(z)} = A(z)P(z)$$

$$k(z) = A(z)\lambda^*(z) + \lambda(1)A(z). \qquad (15)$$

An alternative expression for distributed lag generating function can be obtained by substituting (13) and (14) into (9) to yield

$$\tilde{k}(z) = T(z^{-1}) + \frac{T(z^{-1})}{B(z)}R(z^{-1})(\lambda(1)-\lambda(z^{-1}))$$

$$\tilde{k}(z) = T(z^{-1}) + \lambda(1)A(z) - A(z)\lambda(z^{-1}). \qquad (16)$$

<u>Proposition A</u>: The distributed lag generating function as given by equation (16) is one sided on nonnegative powers of z.

<u>Proof</u>: Write (16) as follows:

$$\tilde{k}(z) = T(z^{-1}) + \lambda(1)A(z) - \sum_{j=0}^{\infty} z^j \sum_{i=1}^{\infty} \lambda_i a_{i+j} - \sum_{j=1}^{\infty} z^{-j} \sum_{i=0}^{\infty} a_i \lambda_{i+j}.$$

Consider the coefficient of z^{-h} $(h>0)$

$$t_h - \sum_{i=0}^{\infty} a_i \lambda_{i+h} = t_h - \sum_{i=0}^{\infty} a_i \sum_{j=0}^{\infty} b_j t_{j+i+h}.$$

The second expression on the right-hand side is equal to

$$\begin{array}{l} a_0 b_0 t_h + a_0 b_1 t_{h+1} \quad + \dots + \\ a_1 b_0 t_{h+1} + a_1 b_1 t_{h+2} \quad + \dots + \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_s b_0 t_{h+s} + a_s b_1 t_{h+s+1} + \dots + \\ \vdots \qquad\qquad\qquad\qquad \vdots \end{array}$$

$$= a_0 b_0 t_h + (a_0 b_1 + a_1 b_0) t_{h+1} + \dots + t_{h+s} \sum_{i=0}^{\infty} a_i b_{s-i} + \dots$$

$$= \sum_{n=0}^{\infty} t_{h+n} \sum_{i=0}^{n} a_i b_{n-i}.$$

But

$$\sum_{i=0}^{n} a_i b_{n-i} = \begin{cases} 1 & n=0 \\ 0 & n>0 \end{cases}$$

and therefore

$$\sum_{i=0}^{\infty} a_i \sum_{j=0}^{\infty} b_j t_{i+j+h} = \sum_{n=0}^{\infty} t_{h+n} \sum_{i=0}^{n} a_i b_{n-i} = t_h.$$

Consequently,

$$t_h - \sum_{i=0}^{\infty} a_i \lambda_{i+h} = t_h - t_h = 0. \qquad \text{Q.E.D.}$$

Proposition B: The distributed lag coefficients as implied by equations (15) and (16) are identical.

Proof: Subtract equation (16) from equation (15) to get

$$k(z) - \tilde{k}(z) = A(z)\lambda^*(z) + A(z)\lambda(z^{-1}) - T(z^{-1})$$

$$= A(z)(\lambda^*(z) + \lambda(z^{-1})) - T(z^{-1})$$

$$= A(z)B(z)T(z^{-1}) - T(z^{-1})$$

$$= T(z^{-1}) - T(z^{-1}) = 0.$$

This implies that $\tilde{k}(z)=k(z)$. Q.E.D.

As a result of propositions A and B, we have two characterizations of the distributed lag generating function k(z) which we repeat here for convenience.

$$k(z) = \lambda(1)A(z) + \lambda^{*}(z)A(z) \tag{17}$$

$$k(z) = t_0 + \lambda(1)A(z) - \sum_{j=0}^{\infty} z^j \sum_{i=1}^{\infty} \lambda_i a_{i+j}. \tag{18}$$

From equation (15) we see that the sum of distributed lag coefficients is one regardless of the parameters of the structural system. To obtain this result evaluate equation (15) at z=1

$$k(1) = A(1)\lambda^{*}(1) + \lambda(1)A(1)$$

$$= A(1)[\lambda(1)+\lambda^{*}(1)]$$

$$= A(1)B(1)T(1) = 1$$

This result is not surprising because it is implied by equation (2) and it serves as a check on our results.

Suppose that the rate of money creation is governed by an m'th order autoregressive process; i.e., $a_j=0$ for $j>m$. Then the projection of the rate of inflation on the rate of money creation will also be of order m, i.e., $k_j=0$ for $j>m$.[4/] As μ_t is econometrically exogenous with respect to x_t we can estimate $a_i(i=1, \ldots, m)$ with OLS. Call these estimates $\hat{a}_i$. Now if we regress x_t on $\mu_t, \mu_{t-1}, \ldots, \mu_{t-m}$ we will get the estimates of $k_i(i=0, \ldots, m)$. Call them $\hat{k}_i$. Note that equation (18) implies

$$k_m = \lambda(1)a_m.$$

Therefore, an estimate of $\lambda(1)$ is given by

$$\hat{\lambda}(1) = \hat{k}_m/\hat{a}_m.$$

Then using equation (17) we can solve for $\lambda^*(z)$ using the estimates of $k(z)$, $A(z)$, and $\lambda(1)$ yielding

$$\hat{\lambda}^*(z) = \frac{\hat{k}(z)}{\hat{A}(z)} - \frac{\hat{k}_m}{\hat{a}_m}$$

$$\hat{\lambda}^*(z) = \hat{B}(z)\hat{k}(z) - \hat{k}_m/\hat{a}_m.$$

In terms of the coefficients

$$\hat{\lambda}_0^* = \hat{k}_0 - \hat{k}_m/\hat{a}_m$$

$$\hat{\lambda}_1^* = \hat{b}_1\hat{k}_0 + \hat{k}_1$$

$$\vdots \qquad \vdots$$

$$\hat{\lambda}_j^* = \sum_{i=0}^{j} \hat{b}_i\hat{k}_{j-i} .$$

Using the definition λ^*'s we get

$$
\begin{aligned}
t_0b_0 + t_1b_1 + t_2b_2 + \cdots &= \hat{k}_0 - \hat{k}_m/\hat{a}_m \\
t_0b_1 + t_1b_2 + t_2b_3 \cdots &= \hat{b}_1\hat{k}_0 + \hat{k}_1 \qquad (19) \\
&\vdots \\
t_ob_j + t_1\,b_{j+1} + t_2b_{j+2} + \cdots &= \hat{b}_j\hat{k}_0 + \hat{b}_{j-1}\hat{k}_1 + \cdots + \hat{k}_j . \\
&\vdots
\end{aligned}
$$

Or in matrix notation

$$
\begin{bmatrix}
b_0 & b_1 & b_2 & \cdots \\
b_1 & b_2 & b_3 & \cdots \\
\vdots & & & \\
b_j & b_{j+1} & b_{j+2} & \cdots \\
\vdots & & &
\end{bmatrix}
\begin{bmatrix}
t_0 \\ t_1 \\ \vdots \\ t_j \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
\hat{k}_0 - \hat{k}_m/\hat{a}_m \\
\hat{b}_1\hat{k}_0 + \hat{k}_1 \\
\vdots \\
\hat{b}_j\hat{k}_0 + \hat{b}_{j-1}\hat{k}_1 + \cdots + \hat{k}_j \\
\vdots
\end{bmatrix} .
$$

Even though the above equation system is of infinite order, it has only m linearly independent columns. Therefore, we can use the first m rows and m columns to solve for $t_0, \ldots, t_{m-1}$. Suppose that we approximate t_{m+j} (j=0, 1, ...) by the geometric distribution as follows

$$t_{m+j} = s\delta^j$$

where s and δ are to be determined. Then

$$\sum_{j=0}^{\infty} t_{m+j} = \frac{s}{1-\delta} .$$

But we know that

$$\sum_{j=0}^{\infty} t_{m+j} + t_0 + t_1 + \ldots + t_{m-1} = 1.$$

Hence,

$$\frac{s}{1-\delta} = 1 - t_0 - t_1 - \ldots - t_{m-1}$$

$$\delta = 1 - \frac{s}{1-t_0-t_1-\ldots-t_{m-1}} .$$

Therefore, by choosing s appropriately we can make $|\delta|<1$ assuring the convergence of the sequence t_i's. Under this approximation we can write

$$\hat{T}(z^{-1}) = t_0 + t_1 z^{-1} + \ldots + t_{m-1} z^{-m+1} + \frac{s\delta^m z^{-m}}{1-\delta z^{-1}} . \quad (20)$$

The relationship between $R(z^{-1})$ and $T(z^{-1})$ then can be used to calculate the estimates of r_j's from which we can derive the structural parameters ϕ and γ_j's.

An alternative approximation can be obtained as follows: as the sequence $\{b_j\}$ converge to zero we can approximate B(z) by a finite (but large) order polynominal $\tilde{B}(z)$ where

$$\tilde{b}_j = \begin{cases} b_j & j \leq s \\ 0 & j > s \end{cases}$$

Then the equation system (20) can be written as

$$\begin{bmatrix} b_0 & b_1 \cdots b_s & 0 - \cdots \\ b_1 & \cdot^{\cdot^{\cdot}} & \\ \vdots & & \\ \cdot & b_2 \; 0 & \\ b_s & 0 & \\ 0 & & \\ \vdots & & \\ & & \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_s \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{k}_0 - \hat{k}_m/\hat{a}_m \\ \sum_{i=0}^{1} b_i \hat{k}_{1-i} \\ \\ \sum_{i=0}^{s} b_i \hat{k}_{s-i} \\ \\ \\ \end{bmatrix} \tag{22}$$

from which we can get the solution for t_0, ..., t_s, and the structural parameters. For expository purposes consider the following example. Let

$$\gamma_j = k\delta^j \qquad k = \frac{1-\delta}{\delta} \; .$$

Then

$$\begin{aligned} r_0 &= 1 + \phi k\delta \\ r_1 &= -\phi k(1-\delta)\delta \\ &\vdots \\ r_j &= -\phi k(1-\delta)\delta^j. \end{aligned}$$

The generating function for r_j's can be written as

$$\begin{aligned} R(z^{-1}) &= \frac{1+\phi k\delta-(\delta+\phi k\delta)z^{-1}}{1-\delta z^{-1}} \\ &= (1+\phi k\delta)\frac{(1-\theta z^{-1})}{(1-\delta z^{-1})} \end{aligned} \tag{23}$$

where

$$\theta = \frac{\delta+\phi k\delta}{1+\phi k\delta} < 1.$$

Then

$$T(z^{-1}) = \frac{1}{R(z^{-1})} = (1+\phi k\delta)^{-1}\frac{(1-\delta z^{-1})}{(1-\theta z^{-1})} . \qquad (24)$$

As η_j is defined as

$$\eta_j = \sum_{i=0}^{\infty} p_i \gamma_{i+j} = \sum_{i=0}^{\infty} p_i k\delta^{i+j} = k\delta^j \sum_{i=0}^{\infty} p_i \delta^i = kC\delta^j . \qquad (25)$$

Using equation (25) we can find that

$$q_0 = \phi\delta C, \quad q_j = -\phi(1-\delta)C\delta^j \qquad j > 0.$$

Hence,

$$Q(z^{-1}) = \frac{\phi C\delta(1-z^{-1})}{1-\delta z^{-1}} = \frac{q_0(1-z^{-1})}{1-\delta z^{-1}} . \qquad (26)$$

The definition of P(z) as given by equation (22) implies

$$P(z) = (1+\phi k\delta)^{-1}\left\{\frac{(1-\delta z^{-1})}{(1-\theta z^{1})}B(z) + \frac{q_0(1-z^{-1})}{(1-\theta z^{-1})}\right\} .$$

And consequently

$$k(z) = (1+\phi k\delta)^{-1}\left\{\frac{(1-\delta z^{-1})}{(1-\theta z^{-1})} + \frac{q_0(1-z^{-1})}{(1-\theta z^{-1})}A(z)\right\} . \qquad (27)$$

Now we note that

$$t_0 = (1+\phi\delta k)^{-1}$$

$$t_1 = (1+\phi\delta k)^{-1}(\theta-\delta)$$

$$t_2 = (1+\phi\delta k)^{-1}(\theta-\delta)\theta$$

$$\vdots$$

$$t_j = (1+\phi\delta k)^{-1}(\theta-\delta)\theta^{j-1}.$$

And the value of q_0 that makes k(z) one sided is given by

$$q_0 = (1+\phi\delta k)\sum_{j=1}^{\infty}\sum_{i=0}^{\infty} b_k t_{i+j}$$

$$= b_0(\theta-\delta) + b_1(\theta-\delta)\theta + -b_2(\theta-\delta)\theta^2 + \dots +$$

$$b_0(\theta-\delta)\theta + b_1(\theta-\delta)\theta^2 + b_2(\theta-\delta)\theta^3 + \dots +$$

$$\vdots$$

$$= (\theta-\delta)\left(\sum_{i=0}^{\infty} b_i\theta^i\right)/(1-\theta) = \frac{\theta-\delta}{1-\theta}\sum_{i=0}^{\infty} b_i\theta^i .$$

Substituting this in k(z) gives

$$k(z) = (1+\phi k\delta)^{-1}(1-\theta z^{-1})^{-1}\left[(1-\delta z^{-1})+\frac{(\theta-\delta)}{(1-\theta)}\left(\sum_{i=0}^{\infty} b_i\theta^i\right)(1-z^{-1})A(z)\right]$$

$$= \frac{1}{(1-\theta z^{-1})(1+\theta\delta k)}\left\{1-\delta z^{-1}+\frac{(\theta-\delta)}{(1-\theta)}\,\frac{(1-z^{-1})}{\left(\sum_{i=0}^{\infty} a_i\theta^i\right)}\right\}$$

due to the fact that

$$\sum_{i=0}^{\infty} b_i\theta^i = \frac{1}{\sum_{i=0}^{\infty} a_i\theta^i} .$$

If we assume that $a_j=0$, for $j>m$, then

$$k_m = \frac{C\delta\theta a_m}{1+\theta\delta k} = \frac{q_0 a_m}{1+\phi\delta k}$$

$$k_{m-i} = \theta k_{m-i+1} + \frac{a_{m-i}-a_{m-i+1}}{a_m}k_m \qquad m > i > 0$$

$$k_0 = \frac{1+(1-\theta)\phi k}{1+\phi k} + \theta k_1 + \frac{a_0-a_1}{a_m}k_m$$

From an econometric point of view the above system of equations imply m restrictions across the parameters k_0, ... k_m. Consequently, the following estimation procedure is warranted: suppose we know the values of δ and ϕ. Then the parameters k_0, ..., k_m can be computed from the formulas given above, and residuals calculated. Clearly these residuals will be functions of δ and ϕ. Hence, we express the sum of squared residuals as a function of δ and ϕ

$$s^2(\delta,\phi) = \sum_{t=1}^{T} (x_t - \sum_{j=0}^{m} k_j(\delta,\phi)\mu_{t-j})^2 .$$

Then the estimates of δ and ϕ are those values for which $s^2(\delta,\phi)$ is a minimum. Numerically this minimization can be done by using an appropriate search routine.

FOOTNOTES

1/ It is possible to obtain a version of the equation given by (1) through a term structure model where real rate of interest is constant through time.

2/ The standard reference on the properties of stochastic processes exploited in this paper is Whittle [1963].

3/ Basically we are trying to solve the following system of equations for q_i's.

$$\lambda_1 + q_0 t_1 + q_1 t_0 = 0$$

$$\lambda_2 + q_0 t_2 + q_1 t_1 + q_2 t_0 = 0$$

$$\vdots$$

$$\lambda_j + q_0 t_j + q_1 t_{j-1} + \dots + q_j t_0 = 0$$

4/ This result is obvious from equation (18). To see that this is also implied by equation (17) write $\lambda^*(z)A(z)$ as follows:

$$\sum_{n=0}^{\infty} z^n \sum_{i=0}^{n} \lambda_i^* a_{n-i}$$

Let $n=m+s$ $s>0$ and consider the coefficient of z^n

$$\sum_{i=0}^{m+s} \lambda_i^* a_{m+s-i}$$

$$a_{m+s}\lambda_0^* + a_{m+s-1}\lambda_1^* + \dots + a_0\lambda_{m+s}^* =$$

$$a_{m+s}(b_0t_0+b_1t_1+\dots) \quad +$$

$$a_{m+s-1}(b_1t_0+b_2t_1+\dots) +$$

$$\vdots$$

$$a_0(b_{m+s}t_0+b_{m+s+1}t_1+\dots) \ .$$

$$t_0\sum_{i=0}^{m+s} a_i b_{m+s-i} + t_1\sum_{i=0}^{m+s} a_i b_{m+s+1-i} + \dots .$$

Note that $a_j=0$ for $j>m$ implies that

$$\sum_{i=0}^{m+s} a_i b_{m+s+j-i} = \sum_{i=0}^{m+s+j} a_i b_{m+s+j-i} = 0$$

and consequently the coefficient of z^{m+s} will be zero for $s>0$.

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