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Dynamic Urn-Ball Discovery*

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Abstract

Under certain assumptions, monopolistic competition with CES preferences is efficient, as first discovered by Dixit and Stiglitz. One assumption, invariably left implicit, is that there are, at any given point in time, no bounds on the number of products that can be discovered. But square wheels do not work, and round wheels keep getting rediscovered. Giving away patents to entrepreneurs who happen to be the first to discover a product generates an inefficiently large amount of variety. The stock of undiscovered products is a commons that can attract too many discovery attempts. Perpetual patents can be efficient, but only when combined with just the right tax on patent-protected monopoly profits. Such a tax is, however, too crude an instrument in an economy with even the least amount of heterogeneity.

1 Introduction

Newtonian physics is called that by accident. Newton was undoubtedly a genius, but the physical reality is what it is, and it is hard to imagine that nobody would have come up with essentially the same insights had he not lived. Truly revolutionary insights and ideas can spread like wildfire. But large numbers of scientists, researchers, engineers, and tinkerers of all sorts are discovering more mundane insights all the time. And keeping track of what has been discovered takes time away from making attempts to discover something.[1] Repeated discovery of the same ideas is inevitable.

*This is work in progress. I thank seminar participants at UCSD and UCSB for helpful comments.

[1]As Muriel Rukeyser, biographer of J.W. Gibbs, put it, “Many of his formulas were re-discovered. It has been said that it is easier to re-discover Gibbs than to read him.” (Rukeyser [1942, p.4]). An early documentation of rediscovery in science is Ogburn and Thomas [1922]. Merton [1961, 1963] makes a strong case that

Suppose there are $E \in \mathbb{N}$ entrepreneurs and $K \in \mathbb{N}$ products waiting to be discovered. Think of these products as urns, and imagine every entrepreneur throwing a single ball that lands in a random urn, independently across entrepreneurs (as in the Butters [1977] model of advertising). Because of this lack of coordination, some products are discovered by several entrepreneurs, while others go undiscovered. For each product, the number of entrepreneurs discovering the product follows a binomial distribution.

In particular, the probability that a product is not discovered is $(1 - 1/K)^E$. When $E = \mathcal{E}H$ and $K = \mathcal{K}H$ and H becomes large, this is approximately equal to $e^{-\mathcal{E}/\mathcal{K}}$. Under these circumstances, the probability that a particular product is discovered by at least one entrepreneur is approximately $1 - e^{-\mathcal{E}/\mathcal{K}}$. Since there are $K = \mathcal{K}H$ products to be discovered, the mean number of products discovered by any entrepreneur at all is, per entrepreneur, equal to

$$\left(1 - \left(1 - \frac{1}{K}\right)^E\right)\frac{K}{E} \approx \left(1 - e^{-\mathcal{E}/\mathcal{K}}\right) \times \frac{\mathcal{K}}{\mathcal{E}}. \tag{1}$$

Adding up across entrepreneurs gives the familiar urn-ball matching function, which exhibits constant returns to scale in $\mathcal{K}$ and $\mathcal{E}$.

Suppose that for each product that is discovered, one of the entrepreneurs who discovered the product is randomly selected and assigned a patent. Then every entrepreneur expects to receive approximately $\left(1 - e^{-\mathcal{E}/\mathcal{K}}\right)/(\mathcal{E}/\mathcal{K})$ patents, a decreasing function of $\mathcal{E}/\mathcal{K}$. So, if more entrepreneurs are trying to discover one of the $\mathcal{K}H$ products waiting to be discovered, then, naturally, entrepreneurs expect to receive fewer patents. Since the number of products to be discovered is given, adding an entrepreneur hurts the other entrepreneurs. Someone deciding whether to be an entrepreneur or not does not take this into account. The only thing that matters for the potential entrepreneur is whether (1), evaluated at an equilibrium outcome $\mathcal{E}$, will be better than some outside option. Too many individuals will try to discover new products in this environment.

This paper describes a version of this argument in a dynamic economy that grows over time. In the model, individuals have CES preferences over differentiated products and can choose to be workers or entrepreneurs. Both occupations involve the supply of a certain amount of labor, but only the entrepreneurial occupation gives opportunities to discover new products. The set of products that can be discovered grows over time, in a manner that is taken to be exogenous. A natural explanation is the progress of sci-

rediscovery is the norm in science, a case that includes a revealed-preference argument: "The race to be first in reporting a discovery testifies to the assumption [of scientists] that if the one scientist does not soon make the discovery, another will." (Merton [1961, p.480]).

ence. Entrepreneurs are assumed not to be able to target their search based on what has already been discovered.[2] An unavoidable outcome is the rediscovery of products that others have already discovered. In a benchmark formulation, only entrepreneurs who have discovered a product have the knowledge required to direct workers producing it.

An entrepreneur who is the first to discover a product becomes a monopolist, and the CES utility function implies certain markup of price over marginal cost. When a second entrepreneur discovers the product, one possible equilibrium is Bertrand pricing. This eliminates monopoly profits in that market. Relative to a world in which everyone can immediately produce newly discovered products, the prospect of monopoly profits, even if transitory, draws more individuals into the entrepreneurial occupation, resulting in larger gains from variety. But transitory monopolies also result in markups that are positive in some markets and zero in other markets. This is a source of static misallocation across products that hurts welfare. The paper proves that the number of entrepreneurs enticed by temporary profits is lower than what a planner would recommend.

Another class of equilibria is one in which the incumbent monopolist and the entrepreneur who rediscovers a product collude and bargain over the share of the spoils of a continued monopoly. Of course, there will eventually be another entrepreneur who rediscovers the product again. The assumption made here is that every time this happens, the two parties can sign a binding non-compete agreement, with only one producer continuing as the monopoly producer of the product. In this class of equilibria, markups are the same in all markets and the static allocation of labor across products is efficient. But no matter what the bargaining share of the incumbent monopolist, these equilibria all generate too much variety. If all the bargaining power is with the incumbent monopolist, then the equilibrium is equivalent to a government handing out perpetual patents to whoever is the first to discover a product. So perpetual patents cannot be efficient either, even though they eliminate static misallocation. The difficulty is the commons problem: entrepreneurs overinvest in getting ahead in the enclosure process (the patenting) of the commons created by science.

In this world, the government can restore efficiency by taxing patent-protected wealth at a rate equal to the equilibrium value of $\mathcal{E}/\mathcal{K}$, the rate at which products are discovered or rediscovered. In other words, an incumbent monopolist is taxed at the same rate as the rate at which the monopolist would lose the monopoly when faced with Bertrand competition from entrepreneurs who rediscover the product. Equivalently, the government could shut down its patent office, rely on collusion, and simply tax monopoly wealth.

[2] This is an assumption that can be relaxed, for example by assuming that entrepreneurs are more likely to sample products made possible by the most recent science.

This could be viewed as a further generalization of the Dixit and Stiglitz [1977] efficiency result.[3] But it remains a fragile result: if the composite goods produce by different industries have different elasticities of substitution, then efficiency requires an additional layer of consumption taxes to eliminate inefficient markup variation.

One can be sure that $\mathcal{K} \in (0, \infty)$, but $\mathcal{K}$ is hard to measure directly, almost by definition. In a steady state, one possibility is to try to measure the number of products actually produced in the economy and combine this with a measurement of the fraction of entrepreneurial attempts to discover a product that actually result in a new product, rather than a rediscovery. For example, suppose 8 out of 9 of all possible products are actually produced at any point in time. If the population grows at 1% per annum, this means that the average delay between new science and the first discovery of a product made possible by that new science is $(1 - 8/9)/(0.01 \times 8/9) = 12.5$ years. Suppose further that 1% of the workforce is an entrepreneur, and that there are 100 workers per product. Then the number of possible products at any given point is $\mathcal{K} = 0.99/(100 \times 8/9) \approx 0.0111$ products per capita. On a workforce of 130 million (the US private sector workforce at the end of 2019) the implied number of possible products is 1.45 million, of which $160,875$ remain to be discovered.

A tentative calibration given in this paper suggests that a planner could then improve steady state consumption by the equivalent of about 15.7 years of growth relative to a free entry allocation in which all prices are equal to marginal cost (so that entrepreneurs are only rewarded for the labor they supply). The required tax on patent-protected wealth would be a massive 20% per annum. If there are no patents or taxes, and if entrepreneurs enjoy a temporary monopoly only until someone rediscovers a product, then the improvement in steady state consumption is equivalent to about 12.2 years of growth relative to free entry. This goes much of the way of the ideal outcome for the planner. And these numbers contrast sharply with the equivalent of 275 years of growth a planner could achieve if $\mathcal{K} = \infty$ and all other parameters are left unchanged.

Related Literature In part, this paper is a rediscovery of Tandon [1983] who uses a static finite urn-ball model to argue that the discovery process entails a commons problem that may lead to too many attempts at discovery. In Loury [1979] and Dasgupta and Stiglitz [1980], related versions of a "duplication of effort" argument can also lead to too much investment in R&D. Mortensen [1982] describes a patent race that generates too much investment and shows how a simple mechanism in which the winner of a patent race must

[3] See Theorem 2 of Judd [1985] for a dynamic version. There have been further extensions. For example, one can show that the equilibrium in the semi-endogenous growth model of Luttmer [2011] is efficient.

compensate the losers can be constructed to obtain the efficient outcome. Reinganum [1989] surveys these and related models of innovation and patent races. The twist here, as in Tandon [1983], is that researchers cannot direct their effort to particular goods or technologies to be discovered. The economy introduced here is dynamic and exhibits long-run growth.

The possibility of too much discovery is familiar from the quality ladder models of Grossman and Helpman [1991] and Aghion and Howitt [1992]. These are models in which the range of goods is fixed, while here growth is the result of gains in variety. In endogenous and semi-endogenous models growth arising from gains from variety (Romer [1991] and Jones [1994]), the standard assumption is that $\mathcal{K} = \infty$, not just over time, but also at a given point in time. With that assumption, the right-hand side of (1) equals 1, and the flow of new products becomes linear in $\mathcal{E}$. Here the flow of new products exhibits decreasing returns in $\mathcal{E}$. Moreover, the implicit fixed factor is not owned by anyone—it is a commons. Because there is no assignment of property rights in this fixed factor —the potential products implied by science— entrepreneurial incentives are too strong when first discoveries lead to permanent monopolies.

The $\mathcal{K} = \infty$ assumption is often implicit when the case is made that substantial monopoly profits are critical to promote discovery. A recent argument in this vein is Jones [2019]. Boldrin and Levine [2008] have made a strong case against this position. The argument against, put forth here, is that the social value of an entrepreneur attempting to discover a product is the value of accelerating the discovery of that product, not the value in perpetuity of the increase in variety resulting from its discovery.

Outline Section 2 describes occupational choice and the technology for product discovery. Section 3 characterizes the solution to the planner's problem. Section 4 considers a decentralized economy in which delays in rediscovery and delays in knowledge diffusion give rise to temporary monopolies. Section 5 assumes knowledge diffuses instantaneously and shows what happens when the government grants patents for free. Section 6 assumes no knowledge diffusion and shows how collusion can lead to too much variety. Section 7 describes the tax on patent-protected wealth that can be used to implement the efficient allocation. A tentative calibration is given in Section 8.

2 The Economy

There is a unit measure of dynastic households. Household sizes are $H_t = He^{\eta t}$, and the population growth rate η is taken to be positive. Household preferences over consumption trajectories $C = \{C_t\}_{t\geq 0}$ are

$$\mathcal{U}(C) = \int_0^\infty e^{-\rho t} H_t \ln(C_t/H_t)\, \mathrm{d}t,$$

where $\rho > \eta$. Consumption is a composite of differentiated products,

$$C_t = \left(\int_0^{J_t} C_{j,t}^{1-1/\varepsilon} \mathrm{d}j \right)^{1/(1-1/\varepsilon)},$$

where $\varepsilon \in (1, \infty)$ and $J_t \in (0, \infty)$. Given nominal prices $\{p_{j,t}\}_{j\in[0,J_t]}$, the demand curves are

$$c_{j,t} = \left(\frac{p_{j,t}}{P_t} \right)^{-\varepsilon} C_t, \quad j \in [0, J_t], \quad P_t = \left(\int_0^{J_t} p_{j,t}^{1-\varepsilon} \mathrm{d}j \right)^{1/(1-\varepsilon)}.$$

All products in $[0, J_t]$ can be produced using a linear labor-only technology. One unit of labor produces one unit of a differentiated product. The supply of labor at time t is denoted by L_t.

In a market economy with frictionless securities markets, the nominal interest rate i_t must satisfy

$$i_t \mathrm{d}t = \rho \mathrm{d}t + \frac{\mathrm{d}\left[P_t C_t / H_t\right]}{P_t C_t / H_t}. \tag{2}$$

Given household wealth, this Euler condition determines the trajectory for consumption.

2.1 Basic Research and New Product Discovery

At any point in time, there is a measure $K_t \in (0, \infty)$ of potential products that may or may not have been discovered. This measure is taken to evolve exogenously, according to $K_t = \mathcal{K} H_t$, for some parameter $\mathcal{K} \in (0, \infty)$.

One can imagine that an inelastically determined fraction of the population is engaged in basic research that gives rise a set of potential products. Suppose the technology for scientific discovery is $\mathrm{d}K_t = G(K_t, \mathcal{A} H_t)\mathrm{d}t$, for some constant returns to scale production function $G(\cdot, \cdot)$ and a positive parameter $\mathcal{A}$. So science is cumulative, in the sense that additions to the measure of possible products build on what is already possible. In a steady state, $\mathcal{K}$ is determined by $\eta \mathcal{K} = G(\mathcal{K}, \mathcal{A})$.

The measure of discovered products is $J_t \in [0, K_t]$. Over time, products can be discovered by entrepreneurs who randomly sample the population of potential products. If the flow of samples taken at time t is E_t, then the measure of discovered products evolves according to

$$\mathrm{d}J_t = \left(1 - \frac{J_t}{K_t}\right) E_t \mathrm{d}t. \tag{3}$$

A fraction $1 - J_t/K_t$ of all possible products is undiscovered. Only draws that hit these products result in new discoveries. The traditional assumption in models of growth and innovation is that $K_t = \infty$, and then this simplifies to $\mathrm{d}J_t = E_t\mathrm{d}t$. Here instead, sampling from the entire stock of potential products inevitably results in repeated discoveries, the more so when the measure of products that remain to be discovered, $K_t - J_t$, is small.[4]

Individual products in $[0, K_t]$ are sampled at the rate $\mu_t = E_t/K_t$. In a steady state, $\mu_t = \mu$, and then the number of times a particular product has been sampled will be a Poisson process with intensity μ. In particular, the fraction of products made possible by science at time t that remains undiscovered at time $s > t$ is $e^{-\mu(s-t)}$.

2.2 A Roy Model with Multiple Tasks per Occupation

At any point in time, individuals can choose to be either workers or entrepreneurs. Workers supply only labor. Entrepreneurs also supply labor, although perhaps not much. In addition, they sample the population of potential products to discover new products. Although referred to here as entrepreneurs, these individuals could well be employed in jobs with two tasks: to supply labor and search for new products.[5]

Individuals are endowed with ability vectors $(a, b) \in \mathbb{R}^2_{++}$, where a is a mean product sampling rate, and b is a quantity of labor. Individuals are heterogeneous, and the distribution of (a, b) in the population is described by a density $f(a, b)$ that is taken to have full support and finite mean.

Labor earns a wage w_t and v_t is the flow value of sampling products at a unit rate. An individual with ability vector (a, b) can earn $(w_t\xi + v_t)a$ as an entrepreneur, and w_tb as a worker, where $\xi > 0$ is a parameter. An individual of type (a, b) chooses to be an entrepreneur if $(w_t\xi + v_t)a > w_tb$, and a worker if $w_tb > (w_t\xi + v_t)a$. We can ignore indifference because the distribution of abilities has a density. Given a cutoff value $s_t = (w_t\xi + v_t)/w_t$ for the ratio b/a, define

$$\mathcal{E}(s_t) = \int_0^\infty \left(\int_0^{s_t a} af(a, b)\mathrm{d}b\right) \mathrm{d}a, \qquad \mathcal{L}(s_t) = \int_0^\infty \left(\int_{s_t a}^\infty bf(a, b)\mathrm{d}b\right) \mathrm{d}a.$$

So $\mathcal{E}(s_t)$ is the per-capita supply of entrepreneurial discovery attempts, and $\mathcal{L}(s_t)$ is the

[4]The picture of entrepreneurs standing on the shoulders of science only is too stark. Abstracting from the input of science, as many models of endogenous growth do, is also extreme.

[5]It is not difficult to make $\mathcal{A}$ endogenous by adding basic research to these occupational choices.

per-capita supply of labor of those who choose to be workers. But an entrepreneur who attempts discoveries at a unit rate also supplies $\xi > 0$ units of labor. The aggregate supplies of entrepreneurial discovery attempts and labor are therefore

$$E_t = \mathcal{E}(s_t)H_t, \quad L_t = \left(\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)\right)H_t.$$

Clearly, entrepreneurial discovery attempts are increasing in s_t, and the worker supply of labor is decreasing in s_t.

Importantly, the slopes of these supply curves are related. Since the occupational choices maximize $s_t a + b$, a version of the envelope condition says that the slope of $s_t\mathcal{E}(s_t) + \mathcal{L}(s_t)$ is $\mathcal{E}(s_t)$, and therefore

$$0 = s_t \mathrm{D}\mathcal{E}(s_t) + \mathrm{D}\mathcal{L}(s_t) \tag{4}$$

identically in s_t. One implication is that $\xi\mathrm{D}\mathcal{E}(s_t) + \mathrm{D}\mathcal{L}(s_t) < 0$ if $s_t > \xi$ and $\xi\mathrm{D}\mathcal{E}(s_t) + \mathrm{D}\mathcal{L}(s_t) > 0$ if $s_t < \xi$. This implies that $s = \xi$ solves $\max_s \{\xi\mathcal{E}(s) + \mathcal{L}(s)\}$. That is, the aggregate supply of labor is single-peaked at $s_t = \xi$. Also, for any $s_t \geq \xi$, the marginal cost in terms of labor of an increase in entrepreneurial discovery attempts is equal to $s_t - \xi$. Relative to $s_t = \xi$, setting $s_t > \xi$ increases the supply of labor by entrepreneurs but reduces the supply of labor by workers more, because the marginal worker supplies more labor as a worker than as an entrepreneur. That is, at $s_t = (w_t\xi + v_t)/w_t > \xi$, the marginal individuals are of the types (a, b) defined by $(w_t\xi + v_t)a = w_t b$, and for them $b > \xi a$.

3 The Planner's Problem

The planner can elicit any feasible combination of factor supplies $E_t = \mathcal{E}(s_t)H_t$ and $L_t = \left(\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)\right)H_t$ by picking a cutoff $s_t \in [0, \infty)$.[6] The planner is assumed to have access to the technology for producing products discovered by entrepreneurs. Given the symmetry of preferences over differentiated products, it is clear that the planner should allocate labor evenly across the J_t commodities known at time t. Per-capita consumption is then equal to

$$\frac{C_t}{H_t} = J_t^{1/(\varepsilon-1)}\left(\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)\right). \tag{5}$$

[6] The set of feasible (E_t, L_t) implied by the ability distribution is compact and convex. The cutoff s_t can be used to trace out its frontier.

The entrepreneurial flow of discovery attempts $E_t = \mathcal{E}(s_t)H_t$ implies a sampling rate $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$ for individual products. From (3),

$$\mathrm{d}J_t = \left(1 - \frac{J_t/H_t}{\mathcal{K}}\right)\mathcal{E}(s_t)H_t\mathrm{d}t.$$

Since flow utility is logarithmic, and since the size of a dynastic household is H_t, the Hamiltonian for the planner must be

$$\mathcal{H}_t(J,\lambda) = \max_s \left\{ H_t\left(\frac{\ln(J)}{\varepsilon - 1} + \ln\left(\xi\mathcal{E}(s) + \mathcal{L}(s)\right)\right) + \lambda\mathcal{E}(s)H_t\left(1 - \frac{J/H_t}{\mathcal{K}}\right)\right\}.$$

At (J_t, λ_t), the first-order condition for s_t is

$$0 = H_t \times \frac{\xi\mathrm{D}\mathcal{E}(s_t) + \mathrm{D}\mathcal{L}(s_t)}{\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)} + \lambda_t\mathrm{D}\mathcal{E}(s_t)H_t\left(1 - \frac{J_t/H_t}{\mathcal{K}}\right).$$

Because the Roy model satisfies $0 = s_t\mathrm{D}\mathcal{E}(s_t) + \mathrm{D}\mathcal{L}(s_t)$, this can also be written as

$$\frac{s_t - \xi}{\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)} = \left(1 - \frac{J_t/H_t}{\mathcal{K}}\right)\lambda_t. \tag{6}$$

This equates the marginal utility weighted marginal cost of an additional draw from $K_t = H_t\mathcal{K}$ to its marginal benefit. The left-hand side of (6) is strictly increasing in s_t when $s_t \geq \xi$, and it varies continuously throughout $[0,\infty)$. So (6) has a unique solution for s_t in terms of J_t/H_t and λ_t, as long as $J_t/H_t \in [0,\mathcal{K}]$ and $\lambda_t \geq 0$. The solution for s_t is strictly decreasing in J_t/H_t and converges to ξ from above as J_t/H_t approaches $\mathcal{K}$ from below. When the stock of products that remain to be discovered is negligible, it is optimal to maximize the per-capita supply of labor by setting $s_t = \xi$.

The shadow price λ_t evolves according to $\mathrm{d}\lambda_t = \rho\lambda_t\mathrm{d}t - \mathrm{D}_1\mathcal{H}_t(J_t,\lambda_t)\mathrm{d}t$. This yields

$$\mathrm{d}\lambda_t = \left(\rho + \frac{\mathcal{E}(s_t)}{\mathcal{K}}\right)\lambda_t\mathrm{d}t - \frac{1}{\varepsilon - 1}\frac{\mathrm{d}t}{J_t/H_t}. \tag{7}$$

The effective discount rate for the marginal utilities $(J_t/H_t)^{-1}/(\varepsilon - 1)$ is $\rho + \mu_t$, with $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$ reflecting the fact that the stock of products that remain to be discovered is being depleted when new products are discovered. Using $\mathrm{d}H_t = \eta H_t\mathrm{d}t$ and $\mathrm{d}J_t = \mathcal{E}(s_t)\left(H_t - J_t/\mathcal{K}\right)\mathrm{d}t$, the dynamics for J_t/H_t can be written as

$$\mathrm{d}\left(\frac{J_t}{H_t}\right) = -\eta\left(\frac{J_t}{H_t}\right)\mathrm{d}t + \mathcal{E}(s_t)\left(1 - \frac{J_t/H_t}{\mathcal{K}}\right)\mathrm{d}t. \tag{8}$$

The conditions (6), (7) and (8) define a time-invariant differential equation for $(J_t/H_t, \lambda_t)$. The initial value J_0/H_0 is given and taken to be in $(0, \mathcal{K})$. Optimality also requires the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t J_t/H_t = 0$.

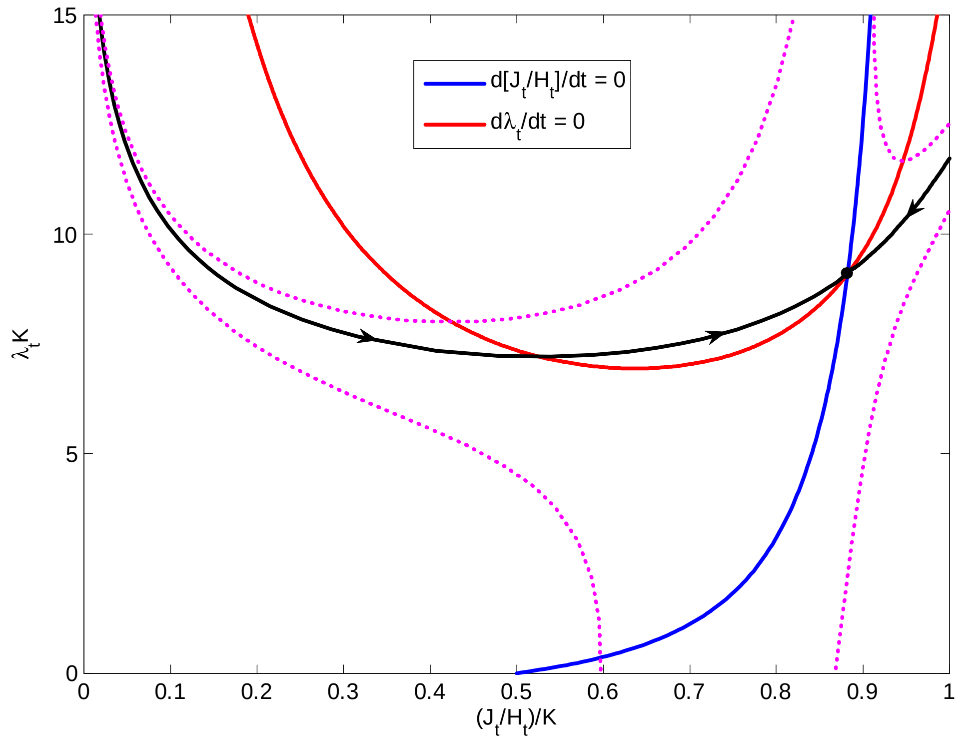

FIGURE 0 The Planner's Problem

As will be argued in the next section, the solution to the planner's problem has a unique steady state. Figure 0 shows the optimal trajectory for $[(J_t/H_t)/\mathcal{K}, \lambda_t\mathcal{K}]$, together with a few trajectories that do not converge to the steady state. Since λ_t is the slope of the value function of the planner, the optimal trajectory implies that the value function is concave for low values of $(J_t/H_t)/\mathcal{K}$ and convex for all higher values of $(J_t/H_t)/\mathcal{K}$, including at the steady state.[7] As $(J_t/H_t)/\mathcal{K}$ approaches a steady state that is sufficiently close to 1 from below, the increasing shadow price counteracts the strongly negative effect of a small and shrinking $1-(J_t/H_t)/\mathcal{K}$ on the incentives to discover new products.

[7]Taking $\mathcal{K}$ to be large enough removes the convexity. The underlying reason for the possible non-concavity of the value function is that $\mathrm{d}J_t = (K_t - J_t)\,(E_t/K_t)\mathrm{d}t$ is not jointly concave in (J_t, E_t). Specifically, $-J_tE_t$ is a saddle.

3.1 Balanced Growth

The planner's problem has a stationary solution that produces balanced growth. Setting $\mathrm{d}[J_t/H_t] = 0$ and $\mathrm{d}\lambda_t = 0$ and dropping time-subscripts in (7)-(8) gives

$$\frac{J}{H} = \frac{\mathcal{K} \times \mathcal{E}(s)/\eta}{\mathcal{K} + \mathcal{E}(s)/\eta}, \qquad \lambda = \frac{1}{\varepsilon - 1}\frac{1}{\rho + \mathcal{E}(s)/\mathcal{K}}\frac{1}{J/H}. \tag{9}$$

The steady state value of J/H given in (9) is the outcome of a matching process between the per-capita stock of possible products $\mathcal{K}$ and the per-capita flow $\mathcal{E}(s)$ of entrepreneurial attempts to discover them. The mapping from $(\mathcal{K}, \mathcal{E}(s)/\eta)$ into J/H can be interpreted as a steady state matching function. As (9) shows, this matching function turns out to be a constant elasticity of substitution function of $\mathcal{K}$ and $\mathcal{E}(s)/\eta$ with an elasticity of substitution equal to $1/2$.[8] The implied complementarity is natural: even in the $\mathcal{K} \to \infty$ limit, the measure of products discovered $J/H = \mathcal{E}(s)/\eta$ is finite, limited by how fast entrepreneurs can find them. Since per-capita consumption scales with $J_t^{1/(\varepsilon-1)}$, and given that $H_t = He^{\eta t}$, the growth rate of this economy is $\eta/(\varepsilon - 1)$. Only the level of the balanced growth path depends on the cutoff s.

3.1.1 The Optimal Cutoff s

Using (9) to eliminate J/H and λ from (6) yields

$$\frac{(s-\xi)\,\mathcal{E}(s)}{\xi\mathcal{E}(s) + \mathcal{L}(s)} = \frac{1}{\varepsilon - 1}\frac{\eta}{\rho + \mu}, \qquad \mu = \frac{\mathcal{E}(s)}{\mathcal{K}}. \tag{10}$$

The unitless expression on the left-hand side of (10) is the ratio of the entrepreneurial discovery attempts, evaluated at the shadow price $s - \xi$, over labor. It is increasing in s and ranges throughout $(0, \infty)$ as s traverses (ξ, ∞). The right-hand side is decreasing in s, a consequence of the fact that both the success probability $1 - (J/H)/\mathcal{K}$ and the shadow value λ of discovered products are decreasing functions of $\mathcal{E}(s)$. The following proposition describes some basic implications of (10).

Proposition 1 *The solution to the planner's problem has a unique steady state defined by (10). The resulting s is increasing in $\mathcal{K}$. It converges to ξ as $\mathcal{K} \downarrow 0$, and to a finite $s_\infty > \xi$ as $\mathcal{K} \to \infty$.*

[8] Observe that the steady state condition $(J/H)/\mathcal{K} = E/(\eta\mathcal{K})/(1 + E/(\eta\mathcal{K}))$ is strictly concave in E. The feasible set of (E, L) is convex. Although the dynamic problem of the planner has a non-convexity, finding the best steady state is a convex programming problem.

3.1.2 Consumption Implications

Write $C(s|\mathcal{K})$ for steady state consumption at any $s \geq \xi$ when the measure of products waiting to be discovered is $\mathcal{K}$. Then the scope for improvement in consumption relative to $s = \xi$ can be measured by

$$\frac{C(s|\mathcal{K})}{C(\xi|\mathcal{K})} = \left(\frac{\frac{\mathcal{E}(s)/\eta}{\mathcal{K}+\mathcal{E}(s)/\eta}}{\frac{\mathcal{E}(\xi)/\eta}{\mathcal{K}+\mathcal{E}(\xi)/\eta}} \right)^{1/(\varepsilon-1)} \frac{\xi\mathcal{E}(s)+\mathcal{L}(s)}{\xi\mathcal{E}(\xi)+\mathcal{L}(\xi)},$$

which has an elasticity equal to

$$\frac{\mathrm{D}C(s|\mathcal{K})\,s}{C(s|\mathcal{K})} = \left(\frac{1}{\varepsilon-1}\frac{\mathcal{K}}{\mathcal{K}+\mathcal{E}(s)/\eta} - \frac{(s-\xi)\,\mathcal{E}(s)}{\xi\mathcal{E}(s)+\mathcal{L}(s)} \right) \frac{\mathrm{D}\mathcal{E}(s)\,s}{\mathcal{E}(s)}. \tag{11}$$

At $s = \xi$, this elasticity is large if $\varepsilon - 1 > 0$ is small, if the fraction of all possible products that remains to be discovered is large, and if the elasticity $\mathrm{D}\mathcal{E}(s)\,s/\mathcal{E}(s)$ is large. Setting the right-hand side of (11) equal to zero gives the golden rule for this economy. This is equivalent to replacing ρ by η in (10). The fact that $\rho > \eta$ implies, as usual, that the golden rule value of s exceeds what the planner would do.

The ratios $C(s|\mathcal{K})/C(\xi|\mathcal{K})$ are shown in Figure 1 for alternative values of $\mathcal{K}$, ranging from the $\mathcal{K} \downarrow 0$ limit to the $\mathcal{K} \to \infty$ limit. For fixed s, the ratios $C(s|\mathcal{K})/C(\xi|\mathcal{K})$ are increasing in $\mathcal{K}$ and bounded. Also indicated in Figure 1 are the $C(s|\mathcal{K})/C(\xi|\mathcal{K})$ chosen by the planner, as well as ratios implied by alternative allocations that will be discussed in later sections. Actual consumption $C(s|\mathcal{K})$ goes to zero as $\mathcal{K}$ goes to zero. But the $\mathcal{K} \downarrow 0$ limit of the consumption ratio $C(s|\mathcal{K})/C(\xi|\mathcal{K})$ recovers the labor supply curve $[\xi\mathcal{E}(s)+\mathcal{L}(s)]/[\xi\mathcal{E}(\xi)+\mathcal{L}(\xi)]$, expressed relative to the maximum supply of labor. Figure 1 also shows its $s \to \infty$ asymptote.

As noted in Proposition 1, s converges from above to ξ as $\mathcal{K}$ becomes small. And then the fraction of products discovered at any point in time, $J/(\mathcal{K}H) = (\mathcal{E}(s)/\eta)\,/\,(\mathcal{K}+\mathcal{E}(s)/\eta)$, converges to 1. The scope for improving the steady state level of consumption implied by setting $s = \xi$ vanishes as $\mathcal{K}$ becomes small. At the same time, per-capita consumption grows at the positive rate $\eta/(\varepsilon-1)$, for any $\mathcal{K} > 0$ and $\varepsilon > 1$. When $\mathcal{K} > 0$ is small, it may not be possible to raise the balanced growth path much beyond what it is at $s = \xi$. But gains from variety do drive long-run growth, and at a very fast pace if $\varepsilon > 1$ is close to 1.

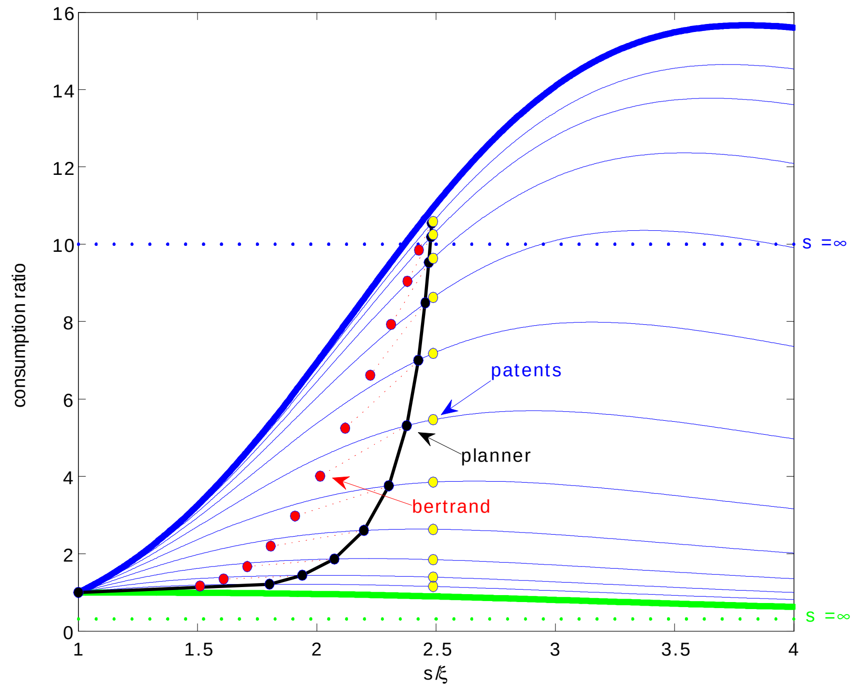

FIGURE 1 The Ratios $C(s|\mathcal{K})/C(\xi|\mathcal{K})$ for $\mathcal{K} \in \{0, 2^{-6}, \ldots, 2^4, \infty\}$

As $\mathcal{K}$ becomes large, the limiting measure of products discovered is $J/H = \mathcal{E}(s_\infty)/\eta$, which is a negligible fraction of the measure of products that could have been discovered. As illustrated by Figure 1, the large-$\mathcal{K}$ cutoff s_∞ may well be much lower than the golden rule.

3.2 The Free Entry Allocation

For this economy, the simplest decentralized allocation is one in which entrepreneurs discover products and everyone can immediately enter to produce these products. This assumes the required knowledge diffuses instantaneously.

Prices are $p_t = w_t$ for every product, and there are no profits. Entrepreneurs earn wages for the labor they supply and $s_t = \xi$. Individuals simply choose their occupation based on whether they can supply more labor as an entrepreneur or as a worker. This maximizes the supply of labor. It also implies a product discovery rate $\mu_t = \mathcal{E}(\xi)/\mathcal{K}$ that is too low relative to what the planner would choose. The equilibrium does not generate enough product variety. As is apparent from Figure 1, the extent to which this reduces consumption along the balanced growth path very much depends on $\mathcal{K}$.

A government could implement the efficient outcome by taxing workers. In this economy, there are no effort choices to worry about.

4 Laissez-Faire with Bertrand Competition

In a decentralized economy, the instantaneous knowledge diffusion that would lead to free entry is unrealistic. Suppose instead that the knowledge to produce a product that has been discovered for the first time only becomes public after an exponentially distributed delay with a mean $1/\delta$. In the meantime, only entrepreneurs who actually discover the product can enable workers to produce it.

4.1 Product Markets

The entrepreneur who is the first to discover a product can set any price and, at a wage w_t, hire the workers needed to produce what is demanded. Since there is a continuum of products, and since the individual demand curves have a constant elasticity $\varepsilon > 1$, the producer will set a price equal to the Lerner price $w_t/(1 - 1/\varepsilon)$. This monopoly position can end because the knowledge needed to produce the product becomes public. That will reduce the price to w_t. Also, another entrepreneur may independently rediscover the same product. In that case, assume the two potential producers engage in Bertrand-style price competition. Again, this reduces the price to w_t.

Write $I_t \in [0, J_t]$ for the measure of products that are produced by a monopoly producer. Then

$$p_{j,t} = w_t \times \begin{cases} \frac{1}{1-1/\varepsilon} & j \in [0, I_t), \\ 1 & j \in [I_t, J_t]. \end{cases}$$

Plugging these prices into the price index P_t and solving for the real wage gives

$$\frac{w_t}{P_t} = \left(\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon-1} I_t + J_t - I_t \right)^{1/(\varepsilon-1)}. \tag{12}$$

Producer revenues are $p_{j,t}c_{j,t} = (p_{j,t}/P_t)^{1-\varepsilon} P_t C_t$. Employment is $l_{j,t} = c_{j,t}$ and the monopoly producers earn variable profits $u_{j,t} = p_{j,t}c_{j,t}/\varepsilon$. This yields

$$\begin{bmatrix} w_t l_{j,t} \\ u_{j,t} \end{bmatrix} = \begin{bmatrix} 1 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix} \frac{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon-1} P_t C_t}{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon-1} I_t + J_t - I_t}$$

for $j \in [0, I_t]$ and

$$w_t l_{j,t} = \frac{P_t C_t}{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon-1} I_t + J_t - I_t}$$

for $j \in [I_t, J_t]$. Given a labor supply L_t, clearing the labor market gives

$$l_{j,t} = \frac{L_t}{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon} I_t + J_t - I_t} \times \begin{cases} \left(1-\frac{1}{\varepsilon}\right)^{\varepsilon} & j \in [0, I_t), \\ 1 & j \in [I_t, J_t]. \end{cases}$$

This is increasing in I_t/J_t, holding fixed L_t and J_t. This is because, measured by employment, competitive markets are larger than monopolized markets. For given L_t and J_t, an increase in the fraction I_t/J_t shifts employment from larger to smaller markets and must therefore expands all markets.

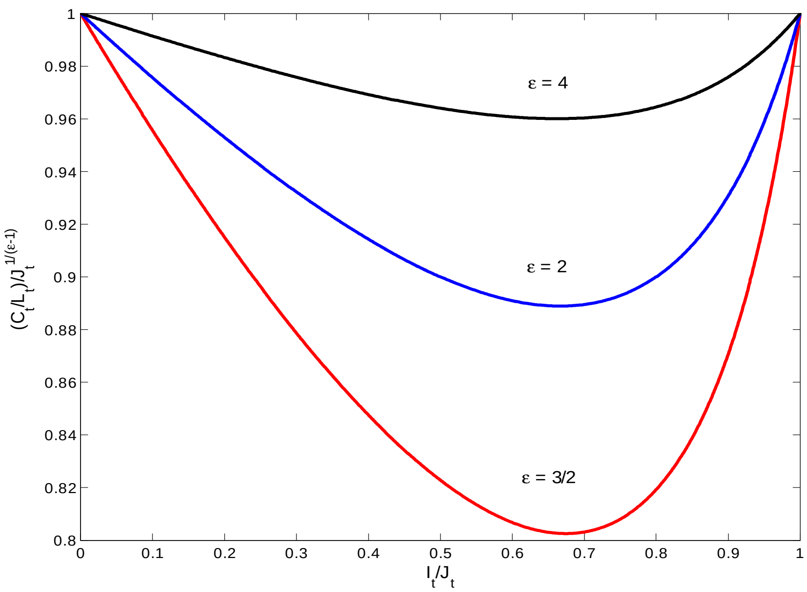

FIGURE 2 The Static Inefficiency of Monopoly

Plugging the labor allocation into the expression for the CES composite good gives

$$\frac{C_t}{H_t} = J_t^{1/(\varepsilon-1)} \left(\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon-1} \frac{I_t}{J_t} + 1 - \frac{I_t}{J_t} \right)^{1/(\varepsilon-1)} \frac{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon-1} \frac{I_t}{J_t} + 1 - \frac{I_t}{J_t}}{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon} \frac{I_t}{J_t} + 1 - \frac{I_t}{J_t}} \frac{L_t}{H_t}. \tag{13}$$

This is a convex function of $I_t/J_t \in [0,1]$ that simplifies to $J_t^{1/(\varepsilon-1)} L_t/H_t$ if and only $I_t/J_t \in \{0,1\}$. The variation in prices that results when some, but not all, markets are monopolized lowers per-capita consumption relative to $J_t^{1/(\varepsilon-1)} L_t/H_t$. This static inefficiency of monopoly is shown in Figure 2 for a few benchmark values of $\varepsilon > 1$.

4.2 Dynamics of the State

As before, the measure of discovered products evolves according to $\mathrm{d}J_t = \mu_t\,(K_t - J_t)\,\mathrm{d}t$, with $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$. But now we also need to keep track of the measure I_t of products that have been discovered precisely once and for which the production technology has not yet become public. Since every product is discovered and rediscovered at the rate μ_t, the dynamics for I_t is given by $\mathrm{d}I_t = [\mu_t(K_t - J_t) - (\delta + \mu_t)I_t]\,\mathrm{d}t$. In per-capita terms, this becomes

$$\mathrm{d}\left(\frac{J_t}{H_t}\right) = \mu_t\left(\mathcal{K} - \frac{J_t}{H_t}\right)\mathrm{d}t - \eta \times \frac{J_t\mathrm{d}t}{H_t}, \tag{14}$$

$$\mathrm{d}\left(\frac{I_t}{H_t}\right) = \mu_t\left(\mathcal{K} - \frac{J_t}{H_t}\right)\mathrm{d}t - (\eta + \delta + \mu_t) \times \frac{I_t\mathrm{d}t}{H_t}, \tag{15}$$

where $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$. The initial values are taken to be $J_0/H \in [0, \mathcal{K})$ and $I_0/H \in [0, J_0/H)$.

4.3 The Value of a Monopoly

Bertrand competition and free entry imply zero profits. But entrepreneurs who discover a product for the first time do earn profits, for a while. Write $\widetilde{q}_t$ for the present value of these monopoly profits. The rate at which potential products are discovered is μ_t, whether the product has been discovered before or not. So the rate at which the monopoly producer of a product loses the monopoly must be $\delta + \mu_t$.

Since $u_{j,t} = w_t l_{j,t}/(\varepsilon - 1)$, the solution for $l_{j,t}$ implies that the price of a monopoly must satisfy the asset-pricing equation

$$i_t\widetilde{q}_t\mathrm{d}t = \frac{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon}}{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon} I_t + J_t - I_t}\frac{w_t L_t \mathrm{d}t}{\varepsilon - 1} + \mathrm{d}\widetilde{q}_t - (\delta + \mu_t)\,\widetilde{q}_t\mathrm{d}t, \tag{16}$$

where $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$. Holding fixed J_t and $w_t L_t$, increasing in I_t/J_t raises profits, a consequence of the fact that market size is increasing in I_t/J_t given aggregate employment. It remains to determine s_t.

4.4 Entrepreneurial Earnings and Equilibrium

Recall that s_t represents the earnings, measured in units of labor, of an entrepreneur who samples products at a unit rate. The flow ξ of labor supplied by the entrepreneur accounts for part of these earnings. In addition, every time the entrepreneur makes an attempt, the probability of discovering a new product is $1 - (J_t/H_t)/\mathcal{K}$, and the discovery of a new product leads to a capital gain of $\widetilde{q}_t/w_t$, in units of labor. The entrepreneurial earnings are therefore

$$s_t = \xi + \left(1 - \frac{J_t/H_t}{\mathcal{K}}\right)\frac{\widetilde{q}_t}{w_t}. \tag{17}$$

To simplify the equilibrium conditions, it is useful to eliminate the interest rate from the asset pricing equation (16) for $\widetilde{q}_t$. To this end, consider the marginal utility weighted price

$$q_t = \frac{\widetilde{q}_t}{P_t C_t/H_t}.$$

Notice, from the real wage (12) and per-capita consumption (13), that the consumption-sector labor share is

$$\frac{w_t L_t}{P_t C_t} = \frac{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon} I_t + J_t - I_t}{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon-1} I_t + J_t - I_t}.$$

This labor share, together with the asset pricing equation (16), the definition of q_t, and the Euler equation (2), imply

$$\mathrm{d}q_t = \left(\rho + \delta + \frac{\mathcal{E}(s_t)}{\mathcal{K}}\right) q_t \mathrm{d}t - \frac{1}{\varepsilon - 1}\frac{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon}\mathrm{d}t}{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon-1}\frac{I_t}{H_t} + \frac{J_t}{H_t} - \frac{I_t}{H_t}}. \tag{18}$$

In terms of q_t, the entrepreneurial earnings (17) can be written as

$$\frac{s_t - \xi}{\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)} = \left(1 - \frac{J_t/H_t}{\mathcal{K}}\right) q_t \times \frac{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon-1}\frac{I_t}{H_t} + \frac{J_t}{H_t} - \frac{I_t}{H_t}}{\left(1-\frac{1}{\varepsilon}\right)^{\varepsilon}\frac{I_t}{H_t} + \frac{J_t}{H_t} - \frac{I_t}{H_t}}. \tag{19}$$

Given the state variables J_t/H_t and I_t/H_t, and given the marginal utility weighted price q_t, equation (19) pins down s_t, and then the trajectory for q_t follows from (18). In turn, the state variables J_t/H_t and I_t/H_t evolve according to (14)-(15), with $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$. The initial values of the state are given, and there is a transversality condition that supplies the other boundary condition needed to pin down the equilibrium.

4.5 Balanced Growth

Imposing $\mathrm{d}\left[J_t/H_t\right] = 0$ and $\mathrm{d}\left[I_t/H_t\right] = 0$ in (14) and (15) gives

$$\frac{J/H}{\mathcal{K}} = \frac{\mu}{\eta + \mu}, \quad \frac{I/H}{\mathcal{K}} = \frac{\mu}{\eta + \mu}\frac{\eta}{\eta + \delta + \mu},$$

where $\mu = \mathcal{E}(s)/\mathcal{K}$. As expected, the fraction I/J of markets that are monopolies is a decreasing function function of $\delta + \mu$. Imposing $\mathrm{d}q_t = 0$ in (18) gives $q_t = q$, where

$$q = \frac{1}{\varepsilon - 1} \frac{1}{\rho + \delta + \mu} \frac{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon}}{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon - 1} \frac{I}{H} + \frac{J}{H} - \frac{I}{H}}.$$

Using this result to eliminate q_t from (19), and then applying the steady state solutions for J/H and I/H gives

$$\frac{(s - \xi)\,\mathcal{E}(s)}{\xi \mathcal{E}(s) + \mathcal{L}(s)} = \frac{1}{\varepsilon - 1} \frac{\eta}{\rho + \delta + \mu} \frac{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon}}{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon} \frac{\eta}{\eta + \delta + \mu} + \frac{\delta + \mu}{\eta + \delta + \mu}}, \quad \mu = \frac{\mathcal{E}(s)}{\mathcal{K}}. \tag{20}$$

This equilibrium condition for s equates the marginal cost of a discovery attempt to the expected capital gain of becoming a producer with a temporary monopoly. It is the analog of the condition (10) for the planner.

4.5.1 Comparison with Free Entry and the Planner

The right-hand side of (20) is decreasing in $\delta + \mu$ and depends on μ only via the term $\delta + \mu$.[9] It follows that there is a unique solution for s. The temporary monopoly ensures that $s > \xi$. There is more variety than under free entry.

To compare this with the s chosen by the planner, observe that the last factor on the right-hand side of (20) is less than 1 for any $\delta + \mu > 0$. It declines from 1 at $\delta + \mu = 0$ to $(1 - 1/\varepsilon)^{\varepsilon}$ as $\delta + \mu$ becomes large. If $\delta = 0$, then the first two factors on the right-hand side of (20) match the right-hand side of the optimality condition (10) for the planner. It then follows, for any $\delta \geq 0$, that the s that solves (20) is lower than the efficient s implied by (10). Although entrepreneurs are compensated beyond just the labor they supply, laissez-faire with Bertrand competition provides too few incentives for entrepreneurs to discover the efficient number of products.

The magnitude of the problem depends very much on $\mathcal{K}$. In particular, note that the right-hand side of (20) goes to zero as $\mu = \mathcal{E}(s)/\mathcal{K}$ becomes large. This implies that $s \downarrow \xi$ as $\mathcal{K} \downarrow 0$, just as in the solution to the planner's problem. And most markets will be competitive. Alternatively, if $\mathcal{K} \to \infty$, then the solution for s is determined by setting $\mu = 0$ in (20). If $\delta = 0$, this again matches the planner's solution. The ratio I/J converges to 1 in this case, and so most markets will be monopolies. Figure 1 provides an example of

[9] This is true even though J/H does depend on μ alone. The fact that utility is logarithmic implies $q\mathcal{E}(s) \propto \mathcal{E}(s)/(J/H)$, with a constant of proportionality that only depends on $\delta + \mu$. The entrepreneurial success probability is $1 - (J/H)/\mathcal{K}$, and this satisfies the steady state condition $(1 - (J/H)/\mathcal{K})\mathcal{E}(s)/(J/H) = \eta$.

this phenomenon. On the other hand, if δ is positive, then the ratio I/J remains bounded away from 1 even when $\mathcal{K}$ becomes large. The resulting static misallocation of labor then rules out approximate efficiency. Proposition 2 summarizes these results.

Proposition 2 *Laissez-faire with Bertrand competition implies a unique s, determined by (20), that is lower than the efficient solution implied by (10). Both solutions for s converge to ξ from above as the measure $\mathcal{K}$ of products that can be discovered goes to zero. If $\delta = 0$, then laissez-faire with Bertrand competition also approximates the s chosen by the planner if $\mathcal{K}$ is large. But if $\delta > 0$, then the large-$\mathcal{K}$ limit remains bounded away from efficiency.*

4.5.2 More on the Role of Knowledge Diffusion

Holding fixed μ, taking $\delta \to \infty$ drives the right-hand side of (20) down to zero. It follows that laissez-faire with Bertrand competition converges to free entry as δ grows without bound. More generally, the effect of an increase in the knowledge diffusion rate δ is to lower the equilibrium value of s. The result is a lower product discovery rate $\mu = \mathcal{E}(s)/\mathcal{K}$ and, at the same time, a higher overall rate $\delta + \mu$ at which monopolies disappear. So an increase in δ implies less variety, and monopolies that last for a shorter period of time.

But the effect of faster knowledge diffusion on consumption is ambiguous. On the one hand, the reduction in variety has a negative effect on steady state consumption. On the other hand, there will also be an increase in the supply of labor that tends to increase consumption. Furthermore, recall from (13) that consumption is a convex function of I/J, as shown in Figure 2. From a baseline with I/J relatively low, a reduction in I/J reduces the product market distortions of monopoly, which tends to raise consumption.

One can use (13) to show that a small enough increase in $1/\delta$ relative to $\delta = \infty$ does increase steady state consumption.[10] A little bit of sand in the wheels of knowledge diffusion cannot hurt.

5 Giving Away Patents

If knowledge does not diffuse immediately, then being the first entrepreneur to discover a product yields monopoly profits. Suppose instead that knowledge diffusion is essentially instantaneous. Will patents improve the allocation relative to free entry?

Specifically, suppose the government grants patents that expire randomly at some rate σ, resulting in temporary monopolies. Because knowledge diffuses instantaneously,

[10] At $s = \xi$, the supply of labor attains its global maximum. And at $I/J = 0$, the elasticity with respect to I/J of the static cost of monopoly shown in Figure 2 is zero.

everyone can enter once a patent expires. The apparatus of Section 4 applies with only two modifications. The depreciation rate $\delta + \mu_t$ in the dynamics (15) for I_t/H_t must be replaced by σ. And similarly, the depreciation rate $\delta + \mu_t$ in the asset pricing equation (18) for q_t must be replaced by σ.

The balanced growth path is then determined by

$$\frac{J/H}{\mathcal{K}} = \frac{\mu}{\eta + \mu}, \qquad \frac{I/H}{\mathcal{K}} = \frac{\mu}{\eta + \mu}\frac{\eta}{\eta + \sigma},$$

where $\mu = \mathcal{E}(s)/\mathcal{K}$, and s is determined by

$$\frac{(s - \xi)\,\mathcal{E}(s)}{\xi\mathcal{E}(s) + \mathcal{L}(s)} = \frac{1}{\varepsilon - 1}\frac{\eta}{\rho + \sigma}\frac{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon}}{\left(1 - \frac{1}{\varepsilon}\right)^{\varepsilon}\frac{\eta}{\eta+\sigma} + \frac{\sigma}{\eta+\sigma}}. \tag{21}$$

Because the left-hand side of (21) is strictly increasing and ranges throughout $(0, \infty)$, the economy has a unique steady state. Since the right-hand side of (21) is decreasing in σ, lengthening the duration of a patent will raise s. In sharp contrast to the optimality condition for the planner (10) and the laissez-faire condition (20), the right-hand side of (21) does not depend on $\mu = \mathcal{E}(s)/\mathcal{K}$. For a given patent duration, the equilibrium value of s does not depend on how many products can be discovered.

Although the laissez-faire economy of Section 4 and the economy with patents described here are different, there is a close connection that follows from the fact that (20) and the associated fraction I/J of monopolized products only depend on $\delta + \mu$ and the fact that (21) and the associated I/J only depend on the patent expiration rate σ.

Proposition 3 *An economy with delayed diffusion at the rate δ, no patents, and Bertrand competition, is equivalent to an economy with instantaneous diffusion and patents expiring at the rate $\sigma = \mu + \delta$, where μ is the common equilibrium product discovery rate.*

To show this, simply take the s and $\mu = \mathcal{E}(s)/\mathcal{K}$ that solve (20), and then use (21) to construct the implicit patent expiration rate σ.

It is clear that a patent regime with $\sigma \in (0, \infty)$ cannot be efficient. As in the laissez-faire economy, labor will be misallocated across products because some markets are monopolies and others are competitive. This misallocation disappears in the extreme cases $\sigma \in \{0, \infty\}$. Free entry and perpetual patents imply $I/J \in \{0, 1\}$. We have seen that free entry is approximately efficient when $\mathcal{K}$ is small. A perpetual patent implies that the right-hand side of (21) is equal to $(\eta/\rho)/(\varepsilon - 1) > (\eta/(\rho + \mu))/(\varepsilon - 1)$. Together with (10), this immediately implies that perpetual patents result in an s that is too large. Since

$\mu = \mathcal{E}(s)/\mathcal{K}$ goes to zero when $\mathcal{K} \to \infty$, this inefficiency disappears when the measure of products that can be discovered is unbounded.

Proposition 4 *A perpetual patent generates more variety than is optimal for the planner. The Dixit-Stiglitz efficiency result for monopolistic competition emerges only in the large-$\mathcal{K}$ limit.*

So the equilibrium cutoffs for free entry and for an economy with perpetual patents bracket the cutoff s that is optimal for the planner. Which of these two regimes better approximates the optimal s completely depends on whether $\mathcal{K}$ is large or small.

6 Laissez-Faire with Collusion

Now simplify, to the other extreme, by assuming that the knowledge required to produce a product never becomes public.

So the first entrepreneur to discover a product only faces potential competition, eventually, when other entrepreneurs rediscover the product. Bertrand competition is only one possible outcome when that happens. Alternatively, it could well be that the incumbent monopolist and the entrepreneur, as people of the same trade with an opportunity to conspire against the public, collude to perpetuate the monopoly profits. Concretely, the incumbent and entrepreneur might sign a non-compete agreement requiring the incumbent to pay the entrepreneur a fraction $\beta \in [0, 1]$ of the value of the monopoly in exchange for a binding commitment by the entrepreneur to never produce the product. Nash bargaining could determine β.

6.1 Product Markets

If this happens in all markets, then all products are produced by monopoly producers who set the Lerner price $p_{j,t} = w_t/(1 - 1/\varepsilon)$. The price index then implies a real wage given by

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right) J_t^{1/(\varepsilon-1)}.$$

Labor costs and variable profits are

$$\begin{bmatrix} w_t l_{j,t} \\ u_{j,t} \end{bmatrix} = \begin{bmatrix} 1 - 1/\varepsilon \\ 1/\varepsilon \end{bmatrix} \frac{P_t C_t}{J_t}, \qquad j \in [0, J_t].$$

This implies $l_{j,t} = L_t/J_t$ for all $j \in [0, J_t]$, and $u_{j,t}/w_t = l_{j,t}/(\varepsilon - 1)$. Given a supply of L_t units of labor, per-capita consumption is then $C_t/H_t = J_t^{1/(\varepsilon-1)} L_t/H_t$, as it is for a planner who uses the same amount of labor.

6.2 The Equilibrium

In this economy, there is no need to keep track of how often a product has been discovered. The state is simply J_t and this evolves according to $\mathrm{d}J_t = \mu_t(K_t - J_t)\mathrm{d}t$, as before. Let $\widetilde{q}_t$ again be the present value of monopoly profits, taking into account that entrepreneurs who rediscover the product will have to be paid $\beta\widetilde{q}_t$. The asset pricing equation (16) for $\widetilde{q}_t$ must be modified by replacing the expected capital loss $\mu_t\widetilde{q}_t$ implied by Bertrand competition with $\mu_t \times \beta\widetilde{q}_t$, where $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$. Also, entrepreneurs trying to discover new products now earn

$$s_t = \xi + \left(1 - \frac{J_t/H_t}{\mathcal{K}} + \beta \times \frac{J_t/H_t}{\mathcal{K}}\right) \frac{\widetilde{q}_t}{w_t}$$

in units of labor. On each draw, they discover a new product with probability $1-(J_t/H_t)/\mathcal{K}$, and this results in a capital gain of $\widetilde{q}_t/w_t$. But now the rediscovery of a product is also rewarded, by capital gain equal to $\beta\widetilde{q}_t/w_t$. The resulting equilibrium conditions can be obtained from (18) and (19) by setting $I_t = J_t$, noting that the effective depreciation rate of a monopoly declines from $\rho + \mathcal{E}(s_t)/\mathcal{K}$ to $\rho + \beta\mathcal{E}(s_t)/\mathcal{K}$, and including the transfer from incumbents in the earnings of entrepreneurs. Using the fact that the labor share is now $w_t L_t/(P_t C_t) = 1 - 1/\varepsilon$, this yields

$$\mathrm{d}\left(\frac{q_t}{1-1/\varepsilon}\right) = \left(\rho + \beta \times \frac{\mathcal{E}(s_t)}{\mathcal{K}}\right) \frac{q_t \mathrm{d}t}{1-1/\varepsilon} - \frac{1}{\varepsilon - 1} \frac{\mathrm{d}t}{J_t/H_t}, \tag{22}$$

where s_t is determined by

$$\frac{s_t - \xi}{\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)} = \left(1 - \frac{J_t/H_t}{\mathcal{K}} + \beta \times \frac{J_t/H_t}{\mathcal{K}}\right) \frac{q_t}{1-1/\varepsilon}. \tag{23}$$

Together with $\mathrm{d}[J_t/H_t] = [\mu_t(\mathcal{K} - J_t/H_t) - \eta J_t/H_t]\,\mathrm{d}t$ and $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$, the conditions (22)-(23) describe the possible equilibrium trajectories for $(J_t/H_t, q_t)$. The initial value of J_0 is given and there is a transversality condition that can be used to pin down the equilibrium.

The monopoly outcome in all markets eliminates the misallocation of labor across products that happens in the economy with Bertrand competition. It is clear that $\beta = 0$ corresponds to an economy with perpetual patents. Monopolies live forever and their owners never have to buy out potential competitors. In other words, perpetual patents are redundant when monopoly rents go to incumbents rather than potential competitors. By

Proposition 4, then, $\beta = 0$ generates too much variety in the steady state. A comparison of (22)-(23) with the optimality conditions (6)-(7) for the planner shows that there can be no $\beta \in [0, 1]$ for which laissez-faire with collusion is efficient.

6.3 Balanced Growth

As usual, $\mathrm{d}[J_t/H_t] = 0$ implies $(J/H)/\mathcal{K} = \mu/(\eta+\mu)$. Setting $\mathrm{d}q_t = 0$ in (22) and using (23) then gives

$$\frac{(s-\xi)\,\mathcal{E}(s)}{\xi\mathcal{E}(s)+\mathcal{L}(s)} = \frac{1}{\varepsilon-1}\frac{\eta+\beta\mu}{\rho+\beta\mu}, \qquad \mu = \frac{\mathcal{E}(s)}{\mathcal{K}}. \tag{24}$$

Recall that $\rho > \eta$, and so the right-hand side of this condition is increasing in $\beta\mu$, varying throughout $(\eta/\rho, 1)/(\varepsilon - 1)$. Dividing the left-hand side by $\mathcal{E}(s)$ and the right-hand side by $\mu\mathcal{K} = \mathcal{E}(s)$ makes the right-hand side of (24) decreasing in μ. As before, this ensures a unique solution for s. When $\beta \in (0, 1]$, the fact that $\rho > \eta$ implies that this solution is decreasing in $\mathcal{K}$. This contrasts with fact that the s chosen by the planner is increasing in $\mathcal{K}$. The two solutions meet in the large-$\mathcal{K}$ limit.

Because $\rho > \eta$, the right-hand side of (24) is strictly increasing in $\beta\mu$. It follows that the solution for s is increasing in $\beta \in [0, 1]$. Holding fixed the price q of a monopoly, an increase in the entrepreneurial bargaining share β has a positive effect on the earnings of entrepreneurs. But the higher effective depreciation rate $\rho + \beta\mu$ lowers q. The fact that $\rho > \eta$ ensures that the net effect of an increase in β on entrepreneurial incentives is still positive. Entrepreneurs can gain from being more successful at shaking down incumbent monopolists, even though entrepreneurs will die by the same sword when they themselves become monopoly producers.

This reasoning can be summarized as follows.

Proposition 5 *The s that solves (24) exceeds the efficient s, and the gap increases with β. The solution to (24) is decreasing in $\mathcal{K}$ and matches the efficient s only in the large-$\mathcal{K}$ limit.*

In other words, laissez-faire with collusion can be particularly harmful when $\mathcal{K}$ is relatively small, and lowering the bargaining power of entrepreneurs reduces the harm.

7 Taxing Patent-Protected Monopoly Wealth

Temporary patents cannot be efficient because of the implied misallocation of labor across products. When patents are forever, or when the entrepreneurs who discover the same product collude, there is no misallocation of labor across products. Instead, the problem is too much variety. The commons that is products waiting to be discovered attracts too many attempts at discovery.

In this simple economy, the obvious solution is to somehow tax discovery attempts. One way to do so is to tax the monopoly profits that are made possible by perpetual patents. Let τ_t be the rate at which monopoly profits are taxed. The equilibrium conditions for this economy are then

$$\mathrm{d}\left(\frac{q_t}{1-1/\varepsilon}\right) = \rho \times \frac{q_t \mathrm{d}t}{1-1/\varepsilon} - \frac{1}{\varepsilon - 1}\frac{(1-\tau_t)\,\mathrm{d}t}{J_t/H_t}, \tag{25}$$

$$\frac{s_t - \xi}{\xi\mathcal{E}(s_t) + \mathcal{L}(s_t)} = \left(1 - \frac{J_t/H_t}{\mathcal{K}}\right)\frac{q_t}{1-1/\varepsilon}, \tag{26}$$

together with $\mathrm{d}[J_t/H_t] = [\mu_t(\mathcal{K} - J_t/H_t) - \eta J_t/H_t]\,\mathrm{d}t$, $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$, an initial value for J_0, and a transversality condition.

Observe that the equilibrium condition (26) for s_t exactly matches the optimality condition (6) for the planner if $q_t/(1-1/\varepsilon) = \lambda_t$. But the effective discount rate in the asset pricing equation (25) is ρ, while the planner (7) uses $\rho + \mu_t = \rho + \mathcal{E}(s_t)/\mathcal{K}$ to discount the flows $(1/(\varepsilon-1))/(J_t/H_t)$. The tax rate τ_t that lowers the value of a monopoly by just the right amount must therefore satisfy

$$\frac{1}{\varepsilon - 1}\frac{\tau_t}{J_t/H_t} = \frac{\mathcal{E}(s_t)}{\mathcal{K}}\frac{q_t}{1-1/\varepsilon}. \tag{27}$$

Translating the result back into nominal prices using $q_t = \widetilde{q}_t/(P_tC_t/H_t)$ gives

$$\tau_t \times \frac{1}{\varepsilon}\frac{P_tC_t}{J_t} = \mu_t\widetilde{q}_t,$$

where $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$. Since monopoly profits are $(P_tC_t/J_t)/\varepsilon$, this simply says that the tax paid on flow profits should be equal to the expected capital loss that would result from losing a monopoly to Bertrand competition against an entrepreneur who also discovers the product. By construction, individual entrepreneurs will be indifferent between applying for a patent in exchange for taxation at the rate τ_t, and risking Bertrand competition without a patent. An alternative way to collect this tax would be to tax wealth invested in patent-protected monopolies at the rate $\mu_t = \mathcal{E}(s_t)/\mathcal{K}$.[11]

Along the balanced growth path, the optimal tax on monopoly profits implied by (25)-(27) is $\tau = \mu/(\rho + \mu)$ where $\mu = \mathcal{E}(s)/\mathcal{K}$ and s solves (10). The usual assumption is $\mathcal{K} = \infty$

[11] A comparison of (22)-(23) with (25)-(27) shows that laissez-faire with collusion and $\beta = 1$ imposes the same "tax" on incumbent monopoly producers. But the opportunity to shake down incumbents draws in too many entrepreneurs.

and therefore $\tau = 0$. But the tax rate is positive as soon as $\mathcal{K} < \infty$. The planner lets $\mu \uparrow \infty$ as $\mathcal{K} \downarrow 0$, and so then $\tau \uparrow 1$. The monopoly profit tax becomes almost confiscatory when the measure of possible products is very small.

Along the balanced growth path, yet another simple tax that can be used together with perpetual patents to implement the efficient allocation is a tax the initial capital gains of entrepreneurs who discover a new product, at the rate $\tau = \mu/(\rho + \mu)$. Under that policy, the price of a monopoly, unencumbered by profit taxes, will exceed $(1 - 1/\varepsilon)\lambda$ by a factor $1/(1 - \tau) = (\rho + \mu)/\rho$. And then the initial capital gains tax results in an aftertax gain equal to $(1 - 1/\varepsilon)\lambda$.

7.1 Multi-Sector Complications

The idea that a combination of perpetual patents with taxes on patent-protected monopoly profits can implement the solution to the planner's problem is fragile. For a concrete example, suppose consumption is a Cobb-Douglas aggregator of N CES composite goods with expenditure shares $\{\beta_n\}_{n=1}^{N}$ and elasticities of substitution $\{\varepsilon_n\}_{n=1}^{N}$. In sector n, there is a measure of products $K_{n,t} = \mathcal{K}_n H_t$ of products that can be discovered. At any point in time, entrepreneurs can choose which sectors to search for new products. The solution to the planner's problem for this economy has a steady state in which the number of products grows at the same rate in all sectors. Because of the different elasticities $\varepsilon_n > 1$, the gains from variety differ across sectors, and so price indices and consumption will grow at different rates across sectors. If $\mathcal{K}_k = \kappa_k \mathcal{K}$ for fixed parameters $\kappa_n > 0$, then $\mathcal{K} \downarrow 0$ implies $s \downarrow \xi$, as in the one-sector economy. That is, free entry is close to what the planner would do.

But perpetual patents with taxes on monopoly profits cannot implement the planner's optimal allocation, for the simple reason that distinct markups in different sectors distort relative prices across sectors. Another layer of taxes could solve that problem. Taxing consumption in sector n at a rate θ_n that satisfies $(1 - \theta_n)(1 - 1/\varepsilon_n) = \phi$ for all n, for some ϕ that ensures $\theta_n \in (0, 1)$, would remove the relative price distortions implied by monopoly. For example, one could take $\phi = 1 - 1/\min_n \{\varepsilon_n\}$ so that the consumption tax in the sector with the lowest elasticity of substitution is zero. Sectors with higher elasticities have lower markups, and these consumption taxes raise consumer prices in those sectors to what they are in the sector with the highest markup. Lower $\phi > 0$ would work as well.

With consumption taxes removing the relative price distortions, the same profit taxes as before can be used to implement the solution to the planner's problem. Specifically, let

$\mu_{n,t}$ be the equilibrium rate at which products in sector n are sampled, and write $\widetilde{q}_{n,t}$ for the price of a monopoly in that sector. Given the results for a one-sector economy, it is now straightforward to show that sector-specific profit taxes $\tau_{n,t}$ of the form

$$\tau_{n,t} \times \frac{1}{\varepsilon_n} \frac{\beta_n P_t C_t}{J_{n,t}} = \mu_{n,t} \widetilde{q}_{n,t}$$

implement the planner's allocation. It turns out that $\widetilde{q}_{n,t} = \phi \lambda_{n,t}$, where $\lambda_{n,t}$ is the planner's shadow price of products waiting to be discovered in sector n. So alternative choices of the scale parameter ϕ will be reflected in the profit tax rates. As in the one-sector economy, the corresponding tax rate for patent-protected wealth is simply $\mu_{n,t}$.

A government trying to implement these taxes would need to have the same intricate information about the structure of the economy as it would need for an industrial policy that subsidizes discovery while enforcing free entry. Knowledge of the elasticities $\{\varepsilon_n\}_{n=1}^N$ is enough to calculate the consumption taxes. And it is true that the discovery rates $\{\mu_{n,t}\}_{n=1}^N$ could be observed in real time. But these discovery rates depend on equilibrium beliefs about future policy.

8 Tentative Calibrations

The empirical importance of geography and of intermediate goods, as well as the likelihood that there is substantial heterogeneity across industries, means that the calibration of a one-industry economy with a naive input-output structure can only be viewed as a suggestive first pass.

8.1 Factor Supplies

Suppose the abilities (a, b) are drawn from two independent Fréchet distributions $\exp(-T_E a^{-\theta})$ and $\exp(-T_L b^{-\theta})$, where T_E and T_L are positive, and $\theta > 1$ to ensure that aggregates are finite.[12] Then the occupational choice probabilities are

$$\begin{bmatrix} \mathcal{P}_E(s) \\ \mathcal{P}_L(s) \end{bmatrix} = \frac{1}{T_E s^\theta + T_L} \begin{bmatrix} T_E s^\theta \\ T_L \end{bmatrix}.$$

[12]The use of Fréchet distributions in models of discrete choice was pioneered by McFadden [1973]. Eaton and Kortum [2002] showed its power in the context of Ricardian models of trade. Luttmer [2011, pp.1048] and Lagakos and Waugh [2013] discovered that it also delivers a very tractable version of the Roy model.

The elasticity of $\mathcal{P}_E(s)$ with respect to s is equal to $(1 - \mathcal{P}_E(s))\theta$. The resulting factor supply curves are

$$\begin{bmatrix} s\mathcal{E}(s) \\ \mathcal{L}(s) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_E(s) \\ \mathcal{P}_L(s) \end{bmatrix} \left(T_E s^\theta + T_L\right)^{1/\theta} \Gamma_\theta,$$

where Γ_θ is the gamma function evaluated at $1 - 1/\theta$. As is well known, a convenient but very strong restriction implied by the independent Fréchet assumption is that the earnings distributions in the two occupations are the same. In particular, average earnings are the same. A corollary is that the average worker and the average entrepreneur supply the same amount of labor at $s = \xi$.

8.2 Consumption Possibilities

The upper ($\mathcal{K} \to \infty$) and lower ($\mathcal{K} \downarrow 0$) envelopes of the mappings $s/\xi \mapsto C\left(s|\mathcal{K}\right)/C\left(\xi|\mathcal{K}\right)$ shown in Figure 1 depend on only three parameters: the CES elasticity ε, the factor supply elasticity parameter θ, and the composite parameter $T_E\xi^\theta/T_L = \mathcal{P}_E(\xi)/[1 - \mathcal{P}_E(\xi)]$ that governs the occupational choice probabilities at $s/\xi = 1$. In particular, holding fixed θ and $\mathcal{P}_E(\xi)$, it does not matter how much labor entrepreneurs supply. The large-s asymptotes shown in Figure 1 are $\lim_{s\to\infty} C\left(s|0\right)/C\left(\xi|0\right) = \left[\mathcal{P}_E(\xi)\right]^{1/\theta}$ and $\lim_{s\to\infty} C\left(s|\infty\right)/C\left(\xi|\infty\right) = (\left[\mathcal{P}_E(\xi)\right]^{1/\theta})^{(\varepsilon-\theta)/(\varepsilon-1)} \in \left(\mathcal{P}_E(\xi), 1/\left[\mathcal{P}_E(\xi)\right]^{1/(\varepsilon-1)}\right)$. Holding fixed $\mathcal{P}_E(\xi)$, more elastic factor supplies provide greater opportunities for increases in consumption relative to $s/\xi = 1$.

In Figure 1 and below, the assumed factor supply elasticity parameter is $\theta = 4$, which implies that the top decile of all individuals in the economy earns about 20% of aggregate earnings.[13] Figure 1 also assumes $\mathcal{P}_E(\xi) = 0.01$. That is, at $s/\xi = 1$, 1% of the population would be an entrepreneur of some sort. The elasticity of substitution across products is taken to be $\varepsilon = 2$. Assuming a population growth rate equal to $\eta = 0.01$, this implies a per-capita consumption growth rate of 1% per annum. Figure 1 shows that for these parameters, $\mathcal{K} = \infty$ makes possible an almost 16-fold increase in consumption relative to free entry—equivalent to just over 275 years of growth.

The difficult question is: what value of $\mathcal{K} \in (0, \infty)$ is reasonable? And, related, what is $1/\xi$, the number of discovery attempts entrepreneurs make for every unit of labor they supply? To sketch the possibilities, it is easiest to use the free entry economy as a benchmark.

[13] Note well that the earnings risk of entrepreneurs is shared perfectly: a type-(a, b) individual attempts to discover new products at the *average* rate a. The before-insurance dispersion in entrepreneurial earnings will be greater than suggested by the tail index θ.

8.3 Parameters Based on a Free Entry Baseline

A convenient choice of units for labor is obtained by imposing $\mathcal{L}(\xi)/\mathcal{P}_L(\xi) = 1$. So the average worker supplies one unit of labor in the free entry economy. The independent Fréchet assumption implies that the average entrepreneur supplies the same amount of labor and generates $\mathcal{E}(\xi)/\mathcal{P}_E(\xi) = 1/\xi$ discovery attempts per annum, or one discovery attempt every ξ years. So the per-capita flow of discovery attempts is simply $\mathcal{E}(\xi) = \mathcal{P}_E(\xi)/\xi$.

The value of $\mathcal{K}$ can be inferred from ideal data on the number of entrepreneurs, how many discovery attempts they make on average, and how successful they tend to be. Write $\varphi = (J/H)/\mathcal{K}$ for the steady state fraction of products discovered at a point in time. So the per-capita flow of new products is $(1-\varphi)\mathcal{E}(\xi)$. Data on discovery attempts and successful new product discoveries from a sample of entrepreneurs would reveal

$$\frac{\mathcal{E}(\xi)}{\mathcal{P}_E(\xi)} = \frac{1}{\xi}, \qquad \frac{(1-\varphi)\,\mathcal{E}(\xi)}{\mathcal{P}_E(\xi)} = \frac{1-\varphi}{\xi}.$$

Such data would therefore pin down ξ and φ. The steady state restriction $\varphi = \mu/(\eta+\mu)$ then delivers the rate $\mu = \eta\varphi/(1-\varphi)$ at which possible products are sampled. A random sample of all individuals in the economy could be used to estimate $\mathcal{P}_E(\xi)$. This reveals $\mathcal{E}(\xi) = \mathcal{P}_E(\xi)/\xi$ and hence $\mathcal{K} = \mathcal{E}(\xi)/\mu$.

Discovery attempts are probably hard to measure. Continue to suppose it is possible to measure $\mathcal{P}_E(\xi)$, and suppose further that it is also possible to measure $\Lambda = \mathcal{P}_L(\xi)/(J/H)$, the number of workers per product. Since $\varphi = (J/H)/\mathcal{K}$, the definition of Λ implies $\mathcal{K} = (1-\mathcal{P}_E(\xi))/(\varphi\Lambda)$. Also, the steady state restriction implies $\mathcal{E}(\xi)/\mathcal{K} = \eta\varphi/(1-\varphi)$, and therefore $1/\xi = \mathcal{E}(\xi)/\mathcal{P}_E(\xi) = (\mathcal{K}/\mathcal{P}_E(\xi)) \times \eta\varphi/(1-\varphi)$. Given a conjecture for φ, one can then infer

$$\frac{1}{\xi} = \frac{\eta}{1-\varphi}\frac{1-\mathcal{P}_E(\xi)}{\mathcal{P}_E(\xi)\Lambda}, \qquad \mathcal{K} = \frac{1-\mathcal{P}_E(\xi)}{\varphi\Lambda}.$$

Holding fixed $\mathcal{P}_E(\xi)$ and Λ, these inferences are intuitive. If φ is close to 1, then entrepreneurs must have been very productive at generating attempts to find new products. On the other hand, if φ is close zero, then the number of products waiting to be discovered must be very large.

To judge the plausibility of alternative possibilities for φ, note that $1/(\mathcal{E}(\xi)/\mathcal{K}) = (1-\varphi)/(\eta\varphi)$ is the average time it takes for a product to be discovered for the first time, after it has been added to the set of possible products by science. Holding fixed the population growth rate η, this delay only depends on φ. At $\varphi = 8/9$, the average delay between new science and the first discovery of an actual new product is 12.5 years. This increases to a lengthy 50 years at $\varphi = 2/3$. These two examples will be used as benchmarks in what follows. For some salient products it may be possible to measure this delay retrospectively.

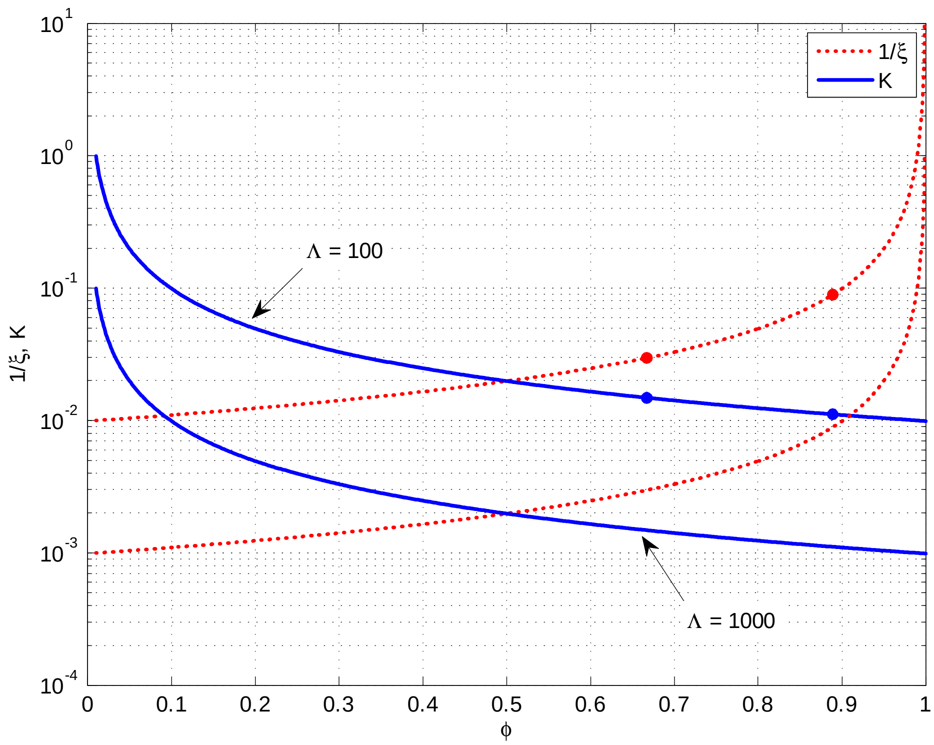

FIGURE 3 The implied $1/\xi$ and $\mathcal{K}$ at $\eta = 0.01$ and $\mathcal{P}_E(\xi) = 0.01$

Figure 3 shows these inferences for two conjectures about the number of workers per product. At $\Lambda = 100$ (since the average US firm has about 25 employees, this would be 4 average firms per product), an entrepreneurial success rate equal to $1 - \varphi = 1/9$ implies that individual entrepreneurs attempt to discover a new product on average about once every 11.2 years.[14] And the number of products waiting to be discovered (or rediscovered) is $\mathcal{K} = 0.0111 \approx 9/800$ per capita. On a base of 130 million private sector employees (US, December 2019), this would mean 1.45 million possible products, of which $160,875$ remain to be discovered. At $\varphi = 2/3$, $\mathcal{K} = 0.0149 \approx 3/200$, and this implies about $643,500$ products that remain to be discovered.

8.4 Comparing Allocations

Figure 4 shows the consumption implications for $\mathcal{K} \in \{0.0111, 0.0149\}$ and alternative assumptions about equilibrium. In the context of Figure 1, these are small values of $\mathcal{K}$.

[14] Keep in mind that the Fréchet distribution implies that some entrepreneurs are much more productive than others.

Statistics for these economies are reported in Tables 1 and 2.

TABLE 1	free entry	Bertrand	planner	perpetual patents
$(J/H)/\mathcal{K}$	0.8889	0.9544	0.9707	0.9898
$\mathcal{P}_E(s)$	0.0100	0.0360	0.0666	0.2781
$\mathcal{E}(s)/\mathcal{P}_E(s)$	0.0891	0.0647	0.0555	0.0388
$\mathcal{E}(s)/\mathcal{K}$	0.0800	0.2092	0.3015	0.9688
years ahead	0	5.6322	7.6709	0.4729

TABLE 2	free entry	Bertrand	planner	perpetual patents
$(J/H)/\mathcal{K}$	0.6667	0.8829	0.9256	0.9603
$\mathcal{P}_E(s)$	0.0100	0.0587	0.1144	0.2781
$\mathcal{E}(s)/\mathcal{P}_E(s)$	0.0297	0.0191	0.0161	0.0129
$\mathcal{E}(s)/\mathcal{K}$	0.0200	0.0754	0.1244	0.2422
years ahead	0	24.31	30.06	26.22

Both panels of Figure 4 also show curves representing the consumption levels implied by (5) and (13). For both curves, $(J/H)/\mathcal{K} = \mathcal{E}(s)/[\eta\mathcal{K} + \mathcal{E}(s)]$. The curve representing (13) has $I/J = (\eta\mathcal{K})/[\eta\mathcal{K} + \mathcal{E}(s)]$ and $L/H = \xi\mathcal{E}(s) + \mathcal{L}(s)$. Increasing s along the (13) curve reduces the fraction I/J of markets that are monopolized, which explains why the curves (5) and (13) merge for large s. Connecting the outcomes for free entry, laissez-faire with Bertrand competition, and laissez-faire with collusion and $\beta = 0$, is a curve that describes the combinations of s/ξ and consumption that emerge as one varies the mean duration $1/\sigma$ of a patent from 0 to infinity. This curve illustrates Proposition 3 and the role of static misallocation. Relative to the delay associated with the allocation for laissez-faire with Bertrand competition, a lengthening of the patent duration $1/\sigma$ increases s and $I/J = \eta/(\eta + \sigma)$ at the same time. While this increases $(J/H)/\mathcal{K}$, it also increases, initially, the static cost of misallocation. When φ is close to 1, implying that the scope for more variety is limited, the shape of this curve is mostly determined by dependence of $I/J = \eta/(\eta + \sigma)$ on σ and the static inefficiencies of monopoly shown in Figure 2. For example, at $\varphi = 0.995$ the static cost of monopoly dominates to such an extent that even the outcome of laissez-faire with Bertrand competition is worse than free entry.

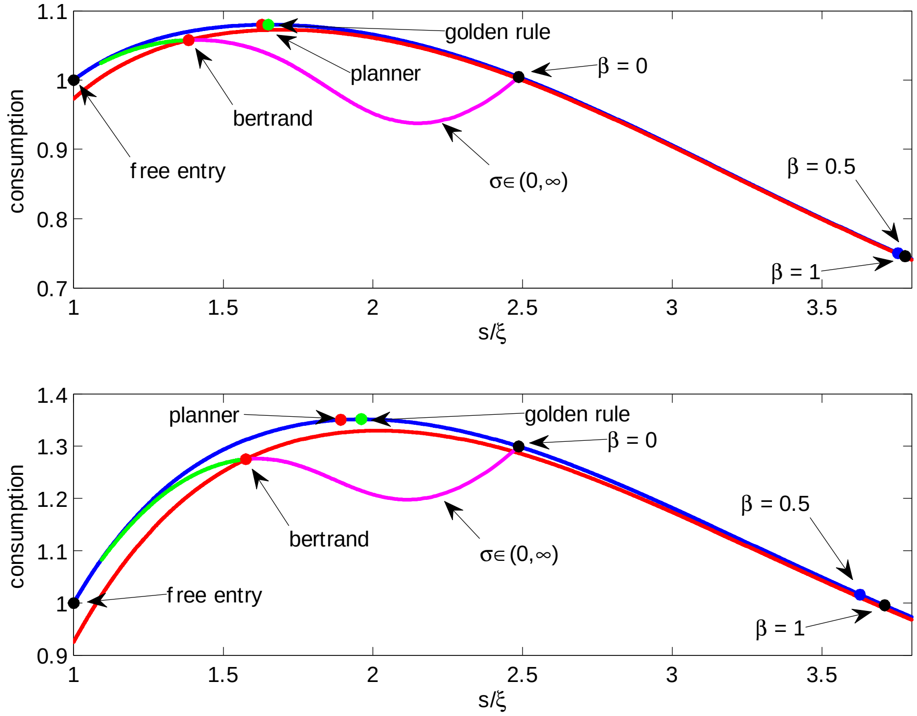

FIGURE 4 Consumption for $\varphi = 8/9$ (top) and $\varphi = 2/3$ (bottom)

The planner would set s/ξ at about 1.6 in the $\varphi = 8/9$ economy and at about 1.9 in the $\varphi = 2/3$ economy. Relative to free entry, the resulting increases in steady state consumption are about 8% and 35%, respectively. Laissez-faire with Bertrand competition lags behind the planner by only about 2 and 6 years in these two economies. Giving away patents (or, equivalently, laissez-faire with collusion and $\beta = 0$) is worse than Bertrand competition when $\varphi = 8/9$, and barely half a year ahead of free entry. As a result, a planner would want to combine patents with a tax on patent-protected wealth equal to just over 30% per annum.[15] The planner would still tax patent-protected wealth at about 12.5% per annum in the $\varphi = 2/3$ economy. The equivalent taxes on patent-protected profits are approximately 86% and 71%, respectively. This significantly lowers the rewards to being an entrepreneur and cuts the numbers of entrepreneurs from the very high 28% enticed by perpetual patents in both scenarios. Starting from such a large number of entrepreneurs, the increase in labor supply resulting from this tax more than offsets the negative effect of a decrease in variety.

[15]Pre-tax rates of return on patent-protected wealth would be about 35% per annum.

9 Conclusions

Giving away perpetual patents to provide incentives for discovery attracts too many entrepreneurs when rediscovery is a possibility. And it shifts the income distribution towards those who have a comparative advantage in occupations that come with opportunities to discover new products. Patents of limited duration can make things worse by creating variation in markups over marginal cost. It is possible to remedy the consequences of perpetual patents with a tax on patent-protected wealth. But in a multi-sector world, the tax rate has to vary by industry, and it also needs to be complemented with consumption taxes to eliminate across-industry variation in markups.

The tentative calibration given in this paper suggests that it may well be a reasonable alternative to rely on the first-mover advantages of entrepreneurs who discover products first, provided that this is combined with a policy that prevents collusion between entrepreneurs who, over time, discover the same product. The lack of fine-tuned taxes and patents only delays the discovery of new products that are bound to be discovered eventually. A crucial statistic needed to assess the cost of this delay is the number of products that has been discovered as a fraction of the number that could have been discovered given the current state of scientific knowledge. This will vary across industries, and a much more detailed model than could be given here would be needed to make a reliable assessment.

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